Oil Prices & Dynamic Games under Stochastic Demand

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Abstract

Oil prices remained relatively low but volatile in the 2015-17 period, largely due to declining and uncertain demand from China. This follows a prolonged decline from around \$110 per barrel in June 2014 to below \$30 in January 2016, due in large part to increased supply of shale oil in the US, which was spurred by the development of fracking technology. Most dynamic Cournot models focus on supply-side factors, such as increased shale oil, and random discoveries. However, uncertain demand is a major factor driving oil price volatility. This motivates the study of Cournot games in a stochastic demand environment. We present analytic and numerical results, as well as a modified Hotelling's rule for games with stochastic demand. We highlight how lower demand forces out higher cost producers from producing, and how such changing market structure can induce price volatility.

1 Introduction

Dramatic oil prices fluctuations since 2002 have been not just in response to global economic or political events, but also reactions to more traditional supply competitions in the face of uncertain demand. As shown in Figure 1, the West Texas Intermediate (WTI) spot price has ranged from under \$20 per barrel to over \$140 per barrel in the last fifteen years. Recently, the spot price has collapsed from \$110 per barrel in mid-2014 to below \$30 per barrel in April 2016. Oil price returns volatility, which is also plotted in Figure 1, has been high, particularly from early 2015 to mid-2016. Until recently, high volatility was largely attributed to the supply side of the market. More recent research, however, has emphasized the importance of demand uncertainty, and has shown that oil price movements are still not well understood. This, combined with decreasing levels of easily accessible oil reserves and the environmental damage caused by burning fossil fuels, makes studying game-theoretic models of energy and oil markets especially relevant.

Oil prices increased in the early and mid-2000s, as Chinese demand grew and increases in global crude oil production slowed. According to data from the World Bank, Chinese real GDP grew by almost 6 times between 2003 and 2013 and China accounted for 45% of the total growth in world oil demand. Total world crude oil production increased from 62 million barrels per day

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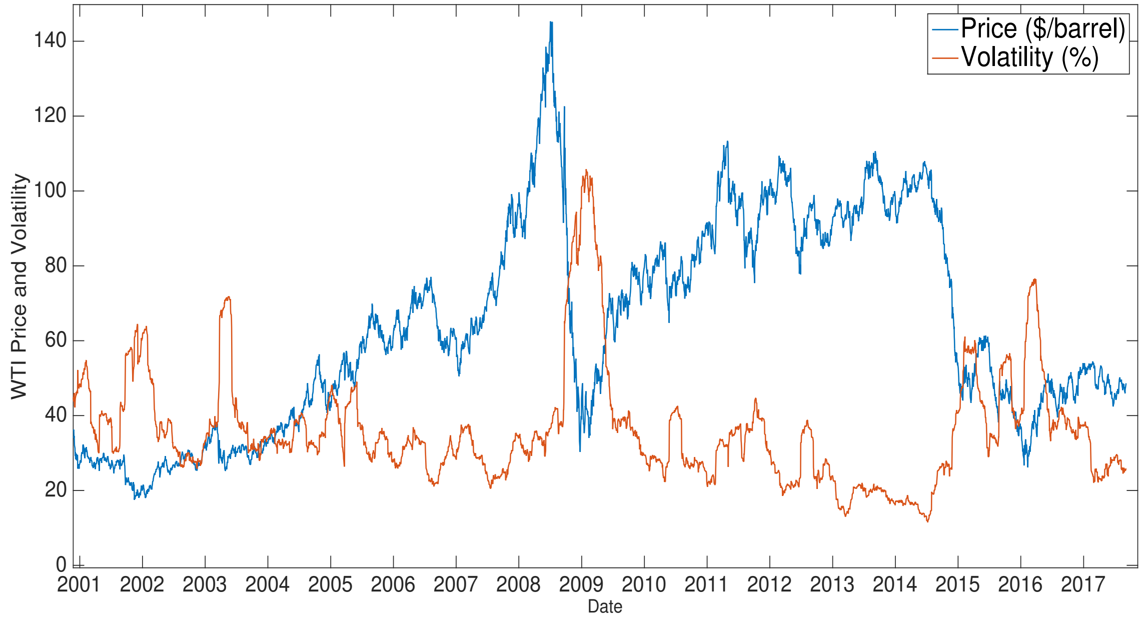

Figure 1: Historical oil price & volatility.

1995 to 73 million barrels per day in 2004. Between 2004 and 2011, however, growth stalled and production remained between 72 and 75 million barrels per day, as measured by the U.S. Energy Information Administration. Increasing world demand for energy, together with tapering supply, led to high prices.

The 2008 recession caused a global drop in the demand for oil, sending prices plummeting, but prices rebounded quickly, as demand from developing countries continued to expand. This price rally ended in 2014, as oil production began to accelerate again and growth in Chinese demand began to taper. The high prices in the mid-2000s led new producers to enter the market, utilizing updated technologies, such as fracking and slant drilling. These technologies expanded total energy reserves, making use of energy stores that were previously inaccessible, but were more expensive than traditional oil fields. The United States' oil output, which was predicted to have peaked in 2008, has almost doubled since then. At some times in recent years, the United States has had over 500 million barrels in inventory, which is a month's worth of supply for the country.

This global growth in supply was strengthened by the strategic decision of the Organization of the Petroleum Exporting Countries (OPEC) to maintain (until late 2016) high levels of oil production and the increase in supply from Iran due to the lifting of sanctions after the 2015 nuclear deal. Total world oil production rose to 80 million barrels per day in 2015 from its plateau of 75 million barrels between 2004 and 2011.

On the demand side, Chinese GDP grew by 7.3% in 2015, which was its slowest growth rate since 1990. Expectations of future demand for oil, especially from China, decreased dramatically during 2015 due to continued signs of global economic weakness and new regulations of carbon emissions. Since the initial decline in prices, total world oil production has grown by about 2 million barrels per day, while world demand has grown by less than 1.5 million barrels.

Understanding supply and demand dynamics of oil markets is key, as the price of oil and its volatility have significant effects on the economy. Consumers spend large portions of their budgets on energy, so high oil prices lead to a reduction in demand for other goods and services. James Hamilton [Ham83, Ham85] noticed in the 1980s that business cycle peaks were correlated with oil price increases and that most of these increases arose from supply shocks outside the United States economy. This led to a focus on researching supply shocks, based on a long-held belief that supply shocks had a significant negative effect on the economy. More recent research, such as that by Kilian and Park [KP09], however, suggests that the effects of oil prices shocks are primarily transmitted through precautionary demand, driven by changes in expectations of future supply or demand. For example, political disturbances in the Middle East often cause price increases, even though these events rarely lead to immediate decreases in supply, as growing uncertainty about future oil supply shortfalls drives up demand and prices. Using our models, we look at the relative effects of changes in supply and demand on oil prices and oil price volatility.

The recent price drop indicates that we still are not able to anticipate even broadly either swings in oil prices or longer-term price trends. In mid-2014, Hamilton [Ham14] predicted that $100 oil was here to stay. More than three years later, a common view is that oil prices will take many years to reach $100 again. Ed Morse, the global head of commodities at Citi Research, made an optimistic prediction in late 2015 that oil would return to $70 a barrel by the end of 2016, while Mohamed El-Erian, Chief Economic Adviser at Allianz, said that $40 oil is here to stay. The International Energy Agency (IEA) does not expect crude oil to rebound to $80 a barrel until after 2020, predicting instead that oil demand growth will be extremely slow over the next 25 years, while supply will remain strong. Such widely-varying expert predictions highlight the vast stochasticity underlying oil price dynamics.

Dynamic Cournot game-theoretic models have been developed with energy production as an application, initiated (in continuous time) by Hotelling's model of a monopolist with exhaustible resources [Hot31]. Recent multi-player versions, describing the competition between energy producers of various (heterogeneous) sources such as oil and solar include [HHS10, LS12] and [LS11b], and a survey article is [LS15]. Mean field game models, where there is a continuum of producers, in this context are studied in [GLL10, CS17]. However, in all of these, the inverse demand (or pricing function) function is fixed. The uncertain nature of energy demand, as described above, motivates us to analyze these dynamic Cournot games in a random demand environment.

Here we present analytical and numerical results on Cournot games under stochastic demand. A monopoly model with Markov chain stochastic demand was studied by numerical and asymptotic methods in [LY14]. Motivation for models with stochastic volatility of demand is discussed in the book [Pir12]. In addition we present some new results on deterministic games which are by-products of the stochastic analysis. Section 2 introduces the Cournot models and gives results for existence and uniqueness of Nash equilibrium in the static game. Then we study the monopolist's exhaustible resources problem under stochastic demand in Section 3, giving a stochastic Hotelling's rule among other results. In Section 4, we present new extensions of the classical Hotelling's rule, for deterministic demand, and for both monopolies and games. We return to games with stochastic demand in Section 5, with numerical solutions of the HJB PDEs. Finally in Section 6, we try to address the short term volatility of oil by removing the exhaustibility constraint, but allowing for a large number of players (individual producers) who enter and exit the market as demand fluctuates.

2 Price Function & Static Games

In a Cournot market with $N \geq 1$ producers, the price of a good is determined by aggregate quantity produced and brought to market. We summarize some results of a static (one-period) competition, whose notation will be used later in the dynamic models. Each player i chooses a quantity $q_i \geq 0$ to produce and sell, and has fixed per-unit cost of production $c_i \geq 0$. The market price is determined by the aggregate quantity produced $Q = \sum_{i=1}^{N} q_i$, and a given pricing (or inverse-demand) function $P(Q, Y)$. Here Y is a demand factor which will be a stochastic process in later sections, but for now is simply a constant.

We will use the notations $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

Assumption 2.1. *The price function $P : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ is smooth in both arguments (Q, Y), decreasing in quantity Q and increasing in demand level Y.*

Each producer faces a trade-off between producing more to sell and lowering the market price received by doing so. The highest possible price (for a fixed Y) is when Q approaches zero, and is known as the *choke price*. We will distinguish two types of price functions, those whose choke price is finite: $P(0, Y) < \infty$, and those with infinite choke price: $P(0^+, Y) = \infty$.

In the static competition, if each player produces $q_i \geq 0$, his profit, adjusting for costs, is $q_i(P(Q, Y) - c_i)$ where by convention we define $0 \times P(0, Y) = 0$, so if all players produce nothing and the choke price is infinite, they gain zero revenue. A Nash equilibrium for the static game is a vector $(q_1^*, q_2^*, \cdots, q_N^*) \in \mathbb{R}_+^N$ such that, for all i,

$$q_i^* \in \arg\max_{q_i \geq 0} q_i \left(P(q_i + Q_{-i}^*, Y) - c_i\right), \qquad Q_{-i}^* = \sum_{j \neq i} q_j^*.$$

That is, each player's equilibrium quantity q_i^* maximizes his profit when all the other players play their equilibrium quantities.

We shall assume that the players are, without loss of generality, labeled in order of increasing costs

$$0 \leq c_1 \leq c_2 \leq \cdots \leq c_N < \infty.$$

As discussed in [HHS10], it is convenient to define the relative prudence

$$\rho(Q, Y) = -Q\frac{P_{QQ}}{P_Q}, \qquad Q, Y > 0, \tag{1}$$

where subscripts denote partial derivatives, and, for fixed Y,

$$\bar{\rho} = \sup_{Q>0} \rho(Q, Y).$$

The following existence and uniqueness result is given in [HHS10].

Theorem 2.2. *For pricing functions with $\bar{\rho} < 2$, there is a unique Nash equilibrium to the static game, which can be constructed as follows. There is a unique $Q_n^* > 0$ that satisfies*

$$Q_n^* P_Q(Q_n^*, Y) + nP(Q_n^*, Y) = C_n, \quad \textit{where} \quad C_n = \sum_{i=1}^{n} c_i,$$

for each $n = 1, \cdots, N$*. Let* $\bar{Q}^* = \max\{Q_n^* \mid 1 \le n \le N\}$*. Then the unique Nash equilibrium is given by*

$$q_i^*(c, Y) = \max\left(\frac{P(\bar{Q}^*, Y) - c_i}{-P_Q(\bar{Q}^*, Y)}, 0\right), \quad 1 \le i \le N, \tag{2}$$

where $c = (c_1, c_2, \cdots, c_N)$ *is the vector of costs. In particular, players* $i = 1, \cdots, n^*$ *are active with* $q_i^* > 0$*, while the remaining players* $i = n^* + 1, \cdots, N$ *are* blockaded *with* $q_i^* = 0$*, where* $n^* = \min\{n \mid Q_n^* = \bar{Q}^*\}$*. The equilibrium profits are given by*

$$\Pi_i(c, Y) = q_i^*(c, Y)\left(P(\bar{Q}^*, Y) - c_i\right), \quad 1 \le i \le N. \tag{3}$$

In Theorem 2.2, the sufficient condition for existence of a unique Nash equilibrium is that (for fixed Y) the prudence $\rho(Q, Y)$ never exceeds two. There is a convenient family of power-type pricing functions for which this condition can be weakened considerably, depending on the number of players. In this family the prudence is constant: $\rho(Q, Y) \equiv \rho$.

Definition 2.3. Pricing functions of power-type are defined by

$$P(Q, Y) = \begin{cases} \frac{Y}{1-\rho}\left(1 - \left(\frac{Q}{Y}\right)^{1-\rho}\right), & \rho \neq 1 \\ Y\left(\log Y - \log Q\right), & \rho = 1, \end{cases} \tag{4}$$

where ρ is the constant prudence.

For $\rho < 1$, there is a finite choke price $P(0, Y) = Y/(1 - \rho)$, and the choke price is infinite for $\rho \ge 1$. Within this family, there is a unique Nash equilibrium as long as the number of players $N > \rho - 1$, as shown in the following result from [HHS10] (given there for $Y = 1$). We use $\lfloor \rho \rfloor$ to denote the largest integer less than or equal to ρ.

Theorem 2.4. *Let* $n_\rho = \max(1, \lfloor \rho \rfloor)$*, and suppose that* $\rho < N + 1$*. Then there is a unique Nash equilibrium in the Cournot game with power type pricing function with constant prudence* ρ *given as follows. Let*

$$\bar{P} = \min\{P_n \mid n_\rho \le n \le N\}, \quad \text{where} \quad P_n = \frac{Y + C_n}{n + 1 - \rho}, \quad n_\rho \le n \le N, \tag{5}$$

and

$$\bar{Q} = \begin{cases} Y\left(1 - (1-\rho)\frac{\bar{P}}{Y}\right)^{\frac{1}{1-\rho}} & \text{if } \rho \neq 1, \\ Y \exp\left(-\bar{P}/Y\right) & \text{if } \rho = 1. \end{cases}$$

Then the unique Nash equilibrium is given by

$$q_i^*(c, Y) = \bar{Q}^\rho \max\{\bar{P} - c_i, 0\}, \quad 1 \le i \le N,$$

with corresponding profits

$$\Pi_i(c, Y) = q_i^*(c, Y)(\bar{P} - c_i), \quad 1 \le i \le N.$$

The number of active players is

$$n^* = \min\{n \mid n_\rho \le n \le N, P_n = \bar{P}\}.$$

Note that, depending on the costs c, the number of players who are active ($q_i^* > 0$) is guaranteed to be at least n_ρ, which is at least bigger than one for $\rho \geq 1$, in which case an infinite choke price guarantees at least player 1 will be active at any (finite) cost c_1.

Throughout the paper, we will assume the conditions that guarantee existence of a unique Nash equilibrium in the static Cournot N-player game:

Assumption 2.5. *The price function P has, for all $Y > 0$, maximal relative prudence $\bar{\rho} < 2$; or P is of power type, as given in Definition 2.3, with constant relative prudence $\rho < N + 1$.*

3 Dynamic Monopolies with Exhaustible Resource

We first consider a single-player market for a commodity, and the effect of a stochastic demand factor on production and price in the face of an exhaustible resource. Let X_t denote the amount of resource left at time t, from which the monopolist extracts at rate $q_t \geq 0$, so its depletion is described by

$$dX_t = -q_t \mathbb{1}_{\{X_t>0\}}\, dt.$$

The pricing function $P(Q, Y)$ is driven by a continuous non-negative stochastic demand factor $(Y_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by Y. We assume, for simplicity, that the producer has zero cost of production, and at time zero, observing $X_0 = x > 0$ and $Y_0 = y > 0$, is faced with the stochastic control problem

$$v = \sup_{q \in \mathcal{A}(x,y)} \mathbb{E} \int_0^{\tau_x} e^{-rt} q_t P(q_t, Y_t)\, dt, \tag{6}$$

where $r > 0$ is a constant discount rate, and τ_x is the time at which the monopolist exhausts his resources, which may be infinite:

$$\tau_x = \inf\{t > 0 \mid X_t = 0\}.$$

A strategy q is in the set of admissible strategies $\mathcal{A}(x, y)$ if it is continuous, non-negative, adapted to $(\mathcal{F}_t)$, and

$$\int_0^\infty q_t\, dt \leq x, \quad \text{a.s.}$$

We recall that P is a pricing function satisfying Assumption 2.1, and with maximal prudence $\bar{\rho} < 2$ for all $Y > 0$ (Assumption 2.5). For the monopoly problem, when $N = 1$, we will denote the static 'game' optimal strategy in (2) and profit in (3) simply by q^* and Π respectively.

3.1 Finiteness of Monopolist's Value

We give two conditions on the demand factor that guarantee a finite monopoly value.

Proposition 3.1. *If the demand factor Y satisfies*

$$\mathbb{E}\{\Pi(0, Y_t)\} \leq K_y e^{\beta t}, \quad \forall t > 0, \tag{7}$$

for some constants K_y and $\beta < r$, then the value v in (6) is finite and bounded in x.

Proof. For any admissible strategy $q \in \mathcal{A}(x, y)$, we have

$$\begin{aligned}
\mathbb{E}\int_0^\infty e^{-rt} q_t P(q_t, Y_t)\,dt &\leq \mathbb{E}\int_0^\infty e^{-rt} \sup_{q_t \geq 0} q_t P(q_t, Y_t)\,dt \\
&= \int_0^\infty e^{-rt}\mathbb{E}\{\Pi(0, Y_t)\}\,dt \\
&\leq \int_0^\infty K_y e^{-(r-\beta)t} dt = \frac{K_y}{r-\beta} < \infty.
\end{aligned}$$

Hence the value v is bounded independently of x. □

Example 3.2. If the demand factor is a geometric Brownian motion driven by a Brownian motion W:

$$dY_t = \mu Y_t\,dt + \sigma Y_t\,dW_t, \tag{8}$$

and if P is a power-type pricing function as in Definition 2.3 with $\rho < 2$, then a straightforward calculation shows that $\Pi(0, Y_t) = kY_t^2$ for some constant k (depending on ρ), and so condition (7) is equivalent to

$$2\mu + \sigma^2 < r.$$

So a geometrically growing stochastic demand factor still has a finite monopoly value as long as μ and σ are not too large compared to r.

Proposition 3.3. *Suppose that the pricing function P has finite choke price $P(0, Y)$ for all $Y > 0$, and that the demand factor satisfies*

$$\mathbb{E}\{e^{-r\tau} P(0, Y_\tau)\} \leq K_y, \tag{9}$$

for any stopping time τ and for some constant K_y. Then the value v in (6) *is finite.*

Proof. For any admissible strategy $q \in \mathcal{A}(x, y)$, we have

$$\int_0^\infty e^{-rt} q_t P(q_t, Y_t)\,dt \leq \int_0^\infty e^{-rt} q_t P(0, Y_t) = \int_0^\infty e^{-rt} q_t P(0, Y_t)\,dQ(t),$$

where we define $Q(t) = \int_0^t q_s\,ds$.We have that $\lim_{t\to\infty} Q(t) = x$, and that, for any $q \geq 0$, $Q^{-1}(q) = \tau(q)$ is a stopping time defined by $\tau(q) = \inf\{t \geq 0 \mid Q(t) \geq q\}$. Therefore, we have

$$\mathbb{E}\int_0^\infty e^{-rt} q_t P(q_t, Y_t)\,dt \leq \mathbb{E}\int_0^x \mathbb{E}\{e^{-r\tau(q)} P(0, Y_{\tau(q)})\}\,dq \leq K_y x < \infty,$$

and the conclusion follows. □

Example 3.4. Suppose as in Example 3.2 that Y is a geometric Brownian motion and P is of power type with constant prudence $\rho < 1$. Then we have that $P(0, Y_t) = Y_t/(1-\rho)$ and so, by the optional stopping theorem, condition (9) is equivalent to $\mu \leq r$.

Example 3.5. Suppose Y is an ergodic process with a unique invariant distribution Φ, and that

$$\int \Pi(0, y)\Phi(dy) < \infty.$$

Then by the ergodic theorem, $\int_0^\infty e^{-rt}\mathbb{E}\{\Pi(0, Y_t)\}\,dt < \infty$, and following the proof of Proposition 3.1, the value v is finite. When P is a power-type pricing function with constant prudence $\rho < 2$, $\Pi(0, Y)$ is proportional to Y^2, so the value is finite as long as $\int y^2\Phi(dy) < \infty$.

3.2 Markov Demand, HJB Equation & Demand Blockading

When demand is fluctuating randomly, there is the possibility, even in a monopoly, that the producer will, at times, halt production and wait for demand to rise, which we will refer to as *demand blockading*. We provide a condition on the demand model under which this scenario is eliminated. We denote explicitly the dependence of the producer's value on the initial conditions:

$$v(x,y) = \sup_{q\in\mathcal{A}(x,y)} \mathbb{E}\left\{\int_0^{\tau_x} e^{-rt} q_t P(q_t, Y_t)\, dt \mid X_0 = x, Y_0 = y\right\}. \tag{10}$$

Lemma 3.6. *For $x, y > 0$, we have*

$$\frac{\partial v}{\partial x} \leq \frac{v(x,y)}{x}.$$

Proof. For $\varphi \geq 1$, we have

$$\begin{aligned}
\varphi v(x,y) &= \sup_{q\in\mathcal{A}(x,y)} \mathbb{E}\left\{\int_0^{\infty} e^{-rt} \varphi q_t P(q_t, Y_t)\, dt \mid X_0 = x, Y_0 = y\right\} \\
&\geq \sup_{q\in\mathcal{A}(x,y)} \mathbb{E}\left\{\int_0^{\infty} e^{-rt} \varphi q_t P(\varphi q_t, Y_t)\, dt \mid X_0 = x, Y_0 = y\right\} \\
&= \sup_{q\in\mathcal{A}(\varphi x,y)} \mathbb{E}\left\{\int_0^{\infty} e^{-rt} q_t P(q_t, Y_t)\, dt \mid X_0 = \varphi x, Y_0 = y\right\} = v(\varphi x, y).
\end{aligned}$$

Then

$$\frac{\partial v}{\partial x} = \lim_{h\downarrow 0} \frac{v(x+h,y) - v(x,y)}{h} \leq \lim_{h\downarrow 0} \frac{\left(1+\frac{h}{x}\right) v(x,y) - v(x,y)}{h} = \frac{v(x,y)}{x}.$$

□

Assumption 3.7. *We suppose now that the demand factor Y is a continuous Markov process with infinitesimal generator $\mathcal{L}_y$. In particular, let us assume it is a positive time-homogeneous diffusion process driven by a Brownian motion W, defined starting from $Y_0 = y$ as the unique strong solution of the SDE:*

$$dY_t = \mu(Y_t)\, dt + \sigma(Y_t)\, dW_t, \quad \textit{and so} \quad \mathcal{L}_y = \frac{1}{2}\sigma^2(y)\frac{\partial^2}{\partial y^2} + \mu(y)\frac{\partial}{\partial y}.$$

We will work with the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) associated with the value function v in (10). Assuming sufficient regularity, a standard verification argument shows that $v(x,y)$ solves the PDE problem

$$rv = \sup_{q\geq 0} (qP(q,y) - v_x) + \mathcal{L}_y v, \quad x, y > 0, \qquad v(0,y) = 0, \tag{11}$$

where subscripts on v denote partial derivatives. The internal optimization problem is simply the static monopoly problem with v_x playing the role of a shadow cost, or scarcity. The optimizer is $q^*(v_x, y)$, with profit $\Pi(v_x, y)$, as given in Theorem 2.2, which gives the PDE

$$rv = \Pi(v_x, y) + \mathcal{L}_y v. \tag{12}$$

Proposition 3.8. *Suppose the price function P has finite choke price $P(0, Y)$ for all $Y > 0$. Then there is no demand blockading (that is $q_t^* > 0$) if and only if $(e^{-rt}P(0, Y_t))_{t\geq 0}$ is a nonnegative supermartingale.*

Proof. Suppose that $P(0, Y_t)\,\mathrm{e}^{-rt}$ is a nonnegative continuous supermartingale. We have that:

$$\begin{aligned} v(x, y) &= \sup_{q_t \in \mathcal{A}(x,y)} \mathbb{E}\left\{ \int_0^\infty q_t P(q_t, Y_t)\,\mathrm{e}^{-rt}\mathrm{d}t \ \mid X_0 = x, Y_0 = y \right\} \\ &\leq \sup_{q_t \in \mathcal{A}(x,y)} \mathbb{E}\left\{ \int_0^\infty q_t P(0, Y_t)\,\mathrm{e}^{-rt}\mathrm{d}t \ \mid X_0 = x, Y_0 = y \right\} \\ &=: u(x, y). \end{aligned} \tag{13}$$

Note that $u(x, y)$ is the monopolist's value function in a market without impact on price by the producer's quantity. Since $\int_0^\infty q_t\, dt > 0$ for $x > 0$, we know that the inequality is strict, *i.e.* that $v(x, y) < u(x, y)$ when $x > 0$. This is because any positive q in the function $P(X, q)$ results in a strictly lower price than $P(X, 0)$. For any strategy q_t that is admissible with $X_0 = x$, the strategy αq_t, for any $\alpha > 0$, is admissible with $X_0 = \alpha x$. As a result we have that:

$$\begin{aligned} u(\alpha x, y) &= \sup_{q_t \in \mathcal{A}(x,y)} \mathbb{E}\left\{ \int_0^\infty \alpha q_t P(0, Y_t)\,\mathrm{e}^{-rt}\, dt \ \mid X_0 = x, Y_0 = y \right\} \\ &= \alpha \sup_{q_t \in \mathcal{A}(x,y)} \mathbb{E}\left\{ \int_0^\infty q_t P(0, Y_t)\,\mathrm{e}^{-rt}\, dt \ \mid X_0 = x, Y_0 = y \right\} \\ &= \alpha u(x, y). \end{aligned}$$

Having thus shown that $u(x, y)$ is linear in x we can take $u(x, y) = U(y)x$, where $U(y) = u(1, y)$.

Now we turn to the problem of showing that $U(y) \leq P(0, y)$. First we remark that we consider only admissible strategies in which $\int_0^\infty q_t\, dt = 1$, since without loss of generality any other strategy can be scaled by a constant factor to satisfy this condition without reducing the expected payout. For every admissible strategy (q_t), we define the process $Q(t) = \int_0^t q_s\, ds$. For each continuous path of Y_t we have that $Q(t)$ is a continuous nondecreasing function with $\lim_{t\to\infty} Q(t) = 1$ and $\frac{dQ}{dt} = q_t$. We have:

$$\mathbb{E}\left\{ \int_0^\infty q_t P(0, Y_t)\,\mathrm{e}^{-rt}\, dt \ \mid X_0 = x, Y_0 = y \right\} = \mathbb{E}\left\{ \int_0^\infty P(0, Y_t)\,\mathrm{e}^{-rt}\, dQ(t) \ \mid X_0 = x, Y_0 = y \right\},$$

where the right-hand side is a Stieltjes integral in which the function $Q(t)$ is the cumulative density function of a measure with density q_t on $[0, \infty)$. Since $Q(t)$ is nondecreasing it has a generalized inverse $Q^{-1}(q)$. Since $Q^{-1}(q)$ is monotonic there are at most finitely many points of discontinuity, *i.e.* where the inverse is not explicitly defined. As a result we can write:

$$\mathbb{E}\left\{ \int_0^\infty P(0, Y_t)\,\mathrm{e}^{-rt}\, dQ(t) \ \mid X_0 = x, Y_0 = y \right\} = \mathbb{E}\left\{ \int_0^1 P\left(0, Y_{\tau(q)}\right)\mathrm{e}^{-r\tau(q)}\, dq \ \mid X_0 = x, Y_0 = y \right\},$$

where $\tau(q) = Q^{-1}(q) = \min\left\{t \geq 0 \middle| \int_0^t q_s\, ds \geq q\right\}$. In addition to being the generalized inverse each $\tau(q)$ is a stopping time which represents the point in time at which q units of inventory have been sold, with $1 - q$ remaining. We can write:

$$\mathbb{E}\left\{\int_0^1 P\left(0, Y_{\tau(q)}\right) \mathrm{e}^{-r\tau(q)}\, dq \;\mid X_0 = x, Y_0 = y\right\} = \int_0^1 \mathbb{E}\left\{P\left(0, Y_{\tau(q)}\right) \mathrm{e}^{-r\tau(q)} \;\mid X_0 = x, Y_0 = y\right\} dq.$$

The exchange in the order of integration is permitted by Tonelli's theorem since we have assumed that the integrand is nonnegative. Since $P(0, Y_t)\,\mathrm{e}^{-rt}$ is a nonnegative continuous supermartingale we can apply the optional stopping theorem, as given in [RY13, Theorem 3.3]. In particular, this variant of the optional stopping theorem does not require the stopping times $\tau(q)$ to have finite expectation. This yields:

$$\int_0^1 \mathbb{E}\left\{P\left(0, Y_{\tau(q)}\right) \mathrm{e}^{-r\tau(q)} \;\mid X_0 = x, Y_0 = y\right\} dq \leq \int_0^1 P(0, y)\; dq = P(0, y).$$

Therefore $U(y) \leq P(0, y)$ and we have $u(x, y) \leq P(0, y)\, x$. From Lemma 3.6 we have:

$$\frac{\partial v}{\partial x} \leq \frac{v(x, y)}{x} \leq \frac{u(x, y)}{x} \leq \frac{P(0, y)\, x}{x} = P(0, y).$$

We also know that this inequality is strict when $x > 0$, since $v(x, y) < u(x, y)$ in that case. From PDE (11) we know that the optimal quantity q_t^* is given by the solution to the maximization problem:

$$\sup_{q \geq 0} q\left(P(q, Y_t) - \frac{\partial v}{\partial x}(X_t, Y_t)\right), \tag{14}$$

which is only ever equal to 0 if the choke price $P(0, Y_t)$ is less than the marginal cost $\frac{\partial v}{\partial x}$. Since we showed that this is not true we know that $q_t^* > 0$ whenever $X_t > 0$ and $Y_t > 0$.

Now suppose that $P(0, Y_t)\,\mathrm{e}^{-rt}$ is a nonnegative continuous stochastic process which is not a supermartingale. Then it must be the case that for some times $t_2 > t_1$, demand level $y_1 > 0$ and $\epsilon > 0$, that we have:

$$\mathbb{E}\left\{P(0, Y_{t_2})\,\mathrm{e}^{-rt_2} \mid Y_{t_1} = y_1\right) = P(0, y_1)\,\mathrm{e}^{-rt_1} + \epsilon.$$

Since $P(0, Y_t)\,\mathrm{e}^{-rt}$ is a continuous stochastic process we can also find some $t_3 > t_2$ and $\delta > 0$ so that for every $t \in [t_2, t_3]$ we have:

$$\mathbb{E}\left\{P(\delta, Y_t)\,\mathrm{e}^{-rt} \mid Y_{t_1} = y_1\right) \geq P(0, y_1)\,\mathrm{e}^{-rt_1} + \frac{\epsilon}{2}.$$

Suppose that a monopolist has reserves at time t_1 of $X_{t_1} = x \leq \delta(t_3 - t_2)$. Then using a strategy of selling nothing from time t_1 until t_2, and then at rate $\delta' := x/(t_3 - t_2)$ during the interval $[t_2, t_3]$, which exhausts his remaining reserves x, gives the following lower bound:

$$v(x, y_1) \geq \mathbb{E}\left\{\int_{t_2}^{t_3} \delta' P(\delta, Y_t)\,\mathrm{e}^{-r(t-t_1)}\, dt \mid X_{t_1} = x, Y_{t_1} = y_1\right) \geq x\left(P(0, y_1) + \frac{\epsilon}{2}\right), \quad 0 \leq x \leq \delta(t_3 - t_2).$$

We can use this to bound the partial derivative:

$$\frac{\partial v}{\partial x}(0, y_1) \geq \lim_{x\to 0} \frac{x\left(P(0, y_1) + \frac{\epsilon}{2}\right) - 0}{x} = P(0, y_1) + \frac{\epsilon}{2}.$$

Then since $\frac{\partial v}{\partial x}$ is continuous in x we know that for some $x > 0$, $\frac{\partial v}{\partial x}(x, y_1) > P(0, y_1)$ and consequently the optimal strategy in (11) is $q^* = 0$, and so demand blockading can occur. □

Example 3.9. Suppose that $P(Q, Y)$ is a constant relative prudence price function given by (2.3). We know that any geometric Brownian motion (GBM) is a nonnegative supermartingale if and only if its growth rate is zero or negative. If Y is a GBM described by the SDE (8), then for any $\rho < 1$, the price process $P(0, Y_t)\,\mathrm{e}^{-rt} = e^{-rt}Y_t/(1-\rho)$ is a nonnegative supermartingale if and only if $\mu \leq r$. If this condition holds then the producer is never demand blockaded. Indeed, in this model, if we have the reverse, then the producer is always demand blockaded since it is always better to wait and sell later. This makes the problem degenerate because there is never any selling activity, but the monopolist's value function is infinite.

Common nonnegative mean-reverting processes do not satisfy the conditions of Proposition 3.8, because any sizeable upward drift when Y_t is low will violate supermartingality after discounting.

Example 3.10. Suppose that $P(Q, Y)$ is a constant relative prudence price function given by (2.3). The SDE for an exponential Ornstein-Uhlenbeck (expOU) processes is given by:

$$dY_t = \alpha\,(m - \log Y_t)\,Y_t\,dt + \sigma Y_t\,dW_t,$$

and so $P(0, Y_t)\,\mathrm{e}^{-rt} = e^{-rt}Y_t/(1-\rho)$ is a nonnegative supermartingale if and only

$$\alpha\,(m - \log Y_t) \leq r, \quad \forall Y_t > 0,$$

which is not true because $-\log Y_t$ is unbounded. Therefore with an OU process there is demand blockading for small enough demand Y_t.

Remark 3.11. *If the choke price $P(0^+, Y) = \infty$, then there will never be demand blockading of a monopolist, that is there will be no states in which $X_t > 0$ and $q_t^* = 0$. We can show this by observing that there is always some $q > 0$ so that $P(q, Y) > \frac{\partial v}{\partial x}$.*

3.3 Stochastic Hotelling's Rule

In this section, we derive the analog of Hotelling's rule for the monopolist under stochastic demand. We first note that in the static game of Section 2, when there is a finite choke price, the monopolist will be blockaded for c large enough or Y small enough, that is $q^*(c, Y) = 0$ for $c \geq P(0, Y)$. In our static monopoly problem, for $c < P(0^+, Y)$, $q^*(c, Y)$ is the unique solution to the first order condition

$$q^* P'(q^*, Y) + P(q^*, Y) - c = 0, \quad \text{and} \quad \Pi(c, Y) = q^*(P(q^*, Y) - c). \tag{15}$$

Clearly, by an implicit function theorem, q^* is continuous (and differentiable) in c, and in the cases of a finite choke price, $q^* = 0$ when $c = P(0, Y)$. Therefore q^* is a continuous function of c for all $c \geq 0$. Next, for $c < P(0^+, Y)$, we have

$$\Pi_c = q_c^*(q^* P'(q^*, Y) + P(q^*, Y) - c) - q^* = -q^*, \tag{16}$$

using the first order condition (15). Therefore Π is continuously differentiable in c for all $c \geq 0$, and the formula (16) for Π_c applies also when $q^* = 0$.

The optimal extraction path is the solution to the equation

$$dX_t = -q^*(v_x(X_t, Y_t), Y_t)\, dt, \quad X_0 = x.$$

Proposition 3.12. *The marginal value function $v_x(X_t, Y_t)$ along the optimal extraction path follows the dynamics*

$$dv_x(X_t, Y_t) = rv_x(X_t, Y_t)\, dt + \sigma(Y_t) v_{xy}(X_t, Y_t)\, dW_t. \tag{17}$$

Proof. Applying Itô's formula to $v_x(X_t, Y_t)$, we have

$$\begin{aligned} dv_x &= v_{xx}\, dX_t + v_{xy}\, dY_t + \frac{1}{2}\sigma(Y_t)^2 v_{xyy}\, dt \\ &= (-q^* v_{xx} + \mathcal{L}_y v_x)\, dt + \sigma(Y_t) v_{xy}\, dW_t. \end{aligned} \tag{18}$$

Next, differentiating the PDE (12) and using the expression (16) for the derivative of Π (which applies whether the player is blockaded or not), gives

$$rv_x = -q^*(v_x(x, y), y) v_{xx} + \mathcal{L}_y v_x. \tag{19}$$

Substituting (19) into (18) gives the conclusion. □

Remark 3.13. *When Y is a constant, and the value function $v = v(x)$, this is the classical Hotelling rule [Hot31]*

$$\frac{d}{dt} v'(x(t)) = rv'(x(t)).$$

Proposition 3.12 shows (under reasonable regularity) that the discounted marginal value function process $(e^{-rt} v_x(X_t, Y_t))_{t \geq 0}$ is a martingale. Similar stochastic Hotelling results, where the randomness comes from uncertain discoveries, rather than uncertain demand as here, have been found for instance in [DP83] and [LS11b]. In the deterministic case, v' satisfies an autonomous linear ODE and $v'(x(t)) = v'(x(0))e^{rt}$. In the stochastic case (17) *is not an autonomous equation since the volatility coefficient depends on v_{xy}.*

In the case of power-type pricing functions given in (4), with $\rho < 2$ for monopoly $N = 1$, the price, from Theorem 2.4, is given by

$$P(q^*(v_x(X_t, Y_t), Y_t) = \frac{Y_t + v_x(X_t, Y_t)}{2 - \rho},$$

and so is linear in v_x. If we define the demand-adjusted price process by

$$\tilde{P}_t = P(q^*(v_x(X_t, Y_t), Y_t) - \frac{Y_t}{2 - \rho},$$

then it follows from the stochastic Hotelling SDE (17) that

$$d\tilde{P}_t = r\tilde{P}_t\, dt + \frac{1}{(2 - \rho)} \sigma(Y_t) v_{xy}(X_t, Y_t)\, dW_t.$$

So, the adjusted price grows on average at the discount rate r, and its volatility is amplified by a factor that is large the closer ρ is to two in the infinite choke price models.

3.4 Variable Reduction in Monopoly with GBM Demand

In the case of linear demand (constant prudence $\rho = 0$ in (4)): $P(Q,Y) = Y - Q$ and GBM demand, there is a dimension reduction possible in the two-dimensional PDE (12).

Proposition 3.14. *Suppose that we have a stochastic demand model in which Y_t is a GBM* (8) *with $\mu < r$ and $2\mu + \sigma^2 < r$, and where $P(Q,Y) = Y - Q$. Then we can write $v(x,y) = y^2 W(x/y)$, where $W(\xi)$ is a function satisfying the ODE:*

$$\left(r - 2\mu - \sigma^2\right) W = \frac{1}{4}\left(1 - W'\right)^2 - \left(\mu + \sigma^2\right)\xi W' + \frac{1}{2}\sigma^2 \xi^2 W'', \tag{20}$$

with $W(0) = 0$ and $W'(\infty) = 0$.

Proof. When Y_t is a GBM with growth rate $\mu < r$ we know from Example 3.9 that the monopolist is never demand blockaded, so we can write HJB equation (40) as:

$$rv = \frac{1}{4}\left(y - \frac{\partial v}{\partial x}\right)^2 + \mu y \frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}. \tag{21}$$

Let $\xi = x/y$ and consider the substitution

$$v(x,y) = y^2 W\left(\frac{x}{y}\right).$$

We obtain the following partial derivatives:

$$\frac{\partial v}{\partial x} = yW', \quad \frac{\partial v}{\partial y} = 2yW - xW', \quad \frac{\partial^2 v}{\partial y^2} = 2W - \frac{2x}{y}W' + \frac{x^2}{y^2}W''.$$

Substituting into equation (21) yields:

$$rW = \frac{1}{4}\left(1 - W'\right)^2 + \mu\left(2W - \xi W'\right) + \frac{1}{2}\sigma^2\left(2W - 2\xi W' + \xi^2 W''\right), \tag{22}$$

which can be simplified to (20), which is a second order nonlinear ordinary differential equation in $W(\xi)$. For boundary conditions, we clearly have $W(0) = 0$ since $v(0,y) = 0$, and $\lim_{x\to\infty} v_x = 0$ implies $\lim_{\xi\to\infty} W'(\xi) = 0$. □

Clearly reduction to an ODE make computations simpler, and the optimal quantity q^* and market price $P(q_t^*, Y_t)$ are recovered from the formulas

$$q_t^* = \frac{1}{2}Y_t\left(1 - W\left(\frac{X_t}{Y_t}\right)\right), \qquad P(q_t^*, Y_t) = \frac{1}{2}Y_t\left(1 + W\left(\frac{X_t}{Y_t}\right)\right).$$

4 Deterministic Games with Renewables

We now introduce competition into the Cournot market, in the form of other energy producers with inexhaustible (or renewable) resources, but higher production costs. These could be solar or wind power compared to the effective lower cost of traditional fossil fuels, or production from a more plentiful and costlier technology such as shale oil from fracking. Such models have been studied in various (mostly deterministic) settings such as [HHS10, LS12, DS14] and [CS17], and a survey article is [LS15]. We first present some new results for deterministic games and then discuss games of this type in a stochastic demand environment.

In a deterministic setting, we will suppress dependence on the demand factor Y, which is just a constant here. Analogous to Section 2, we have Cournot market with price function $P(Q)$, which is smooth and decreasing, and we define the relative prudence

$$\rho(Q) = -Q\frac{P''(Q)}{P'(Q)}, \quad \text{and} \quad \bar{\rho} = \sup_{Q>0} \rho(Q).$$

4.1 Deterministic Monopoly

When there is a single player in a Cournot market, we can obtain explicit expressions for the value function and resource dynamics. The producer extracts his resource $x(t)$ and rate q_t:

$$\frac{dx}{dt} = -q_t \mathbb{1}_{\{x(t)>0\}}, \qquad x(0) = x,$$

and his value function is

$$v(x) = \sup_{q \in \mathcal{A}(x)} \int_0^{\tau_x} e^{-rt} q_t P(q_t)\, dt, \qquad \tau_x = \inf\{t > 0 \mid x(t) = 0\}. \tag{23}$$

We use the notations $q^*(c)$ and $\Pi(c)$ for the optimal quantity and profit in the static monopoly as a function of the player's marginal cost $c < P(0^+)$, where q^* is the unique solution of the first order condition

$$q^* P'(q^*) + P(q^*) - c = 0 \quad \text{and} \quad \Pi(c) = q^*(P(q^*) - c).$$

Then the Hamilton-Jacobi equation for v is

$$rv = \Pi(v'), \quad x > 0, \qquad v(0) = 0. \tag{24}$$

Lemma 4.1. *Let $u(x)$ solve the ODE problem*

$$ru = g(u') \quad x > 0, \qquad u(0) = 0,$$

for some constant $r > 0$ and non-negative, strictly decreasing continuously differentiable function g. Then u can be expressed as

$$u(x) = \frac{1}{r} g\left(h^{-1}(rx)\right), \quad \textit{where} \quad h(z) = \int_{g^{-1}(0)}^{z} \frac{g'(w)}{w}\, dw. \tag{25}$$

Proof. From the ODE $u' = g^{-1}(ru)$ and so

$$\int_0^{u(x)} \frac{d\tilde{u}}{g^{-1}(r\tilde{u})} = x.$$

Making the change of variable $w = g^{-1}(r\tilde{u})$ gives

$$\frac{1}{r} \int_{g^{-1}(0)}^{g^{-1}(ru(x))} \frac{g'(w)}{w}\, dw = x,$$

and the formula (25) follows. □

The profit function $\Pi(c)$ is strictly decreasing in the player's cost c, and is continuously differentiable following the argument leading to (16). Using Lemma 4.1 with $g = \Pi$, the solution of the ODE (24) for the value function is

$$v(x) = \frac{1}{r}\Pi\left(h^{-1}(rx)\right), \quad \text{where} \quad h(z) = \int_{\Pi^{-1}(0)}^{z} \frac{\Pi'(w)}{w}\, dw. \tag{26}$$

Proposition 4.2. *The optimal strategy is given by*

$$q_t^* = -\Pi'\left(h^{-1}(rx(t))\right), \tag{27}$$

and the optimal extraction path is

$$x(t) = \frac{1}{r}h\left(e^{rt}h^{-1}(rx(0))\right). \tag{28}$$

The time to exhaustion starting from $x(0) > 0$ *is given by*

$$\tau(x(0)) = \frac{1}{r}\log\left(\frac{P(0^+)}{h^{-1}(rx(0))}\right), \tag{29}$$

and so it is finite when the choke price is finite, and infinite when the choke price is infinite.

Proof. From (16), we have $q^*(c) = -\Pi'(c)$, and so $q_t^* = q^*(v'(x(t))) = -\Pi'(v'(x(t)))$. But

$$v'(x) = \Pi^{-1}(rv(x)) = \Pi^{-1}\left(\Pi\left(h^{-1}(rx)\right)\right) = h^{-1}(rx),$$

and so (27) follows. Then from $\frac{dx}{dt} = -q_t^*$, we have

$$\int_{x(0)}^{x(t)} \frac{d\tilde{x}}{\Pi'\left(h^{-1}(r\tilde{x})\right)} = t.$$

With $\xi = h^{-1}(r\tilde{x})$, we have

$$\int \frac{d\tilde{x}}{\Pi'\left(h^{-1}(r\tilde{x})\right)} = \frac{1}{r}\int \frac{h'(\xi)}{\Pi'(\xi)}\, d\xi = \frac{1}{r}\int \frac{d\xi}{\xi} = \frac{1}{r}\log\xi,$$

using, from (26), that $h'(\xi) = \Pi'(\xi)/\xi$. Thus (28) follows. Finally $x(t) = 0$ at time τ defined by $h^{-1}(0) = e^{r\tau}h^{-1}(rx(0))$. But $h^{-1}(0) = \Pi^{-1}(0) = P(0^+)$, which leads to (29). □

In the case of power-type pricing functions given in Definition 2.3, we have semi-explicit expressions for the value function.

Definition 4.3. The *exponential integral* $\mathrm{Ei}(z)$ is defined as the following definite integral:

$$\mathrm{Ei}\,(z) = -\int_{-z}^{\infty} \frac{\mathrm{e}^{-t}}{t}\mathrm{d}t.$$

Definition 4.4. The *hypergeometric function* ${}_2F_1(a,b;c;z)$ is defined as the following series:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where $(q)_n$ is a rising Pochhammer symbol, which is defined by:

$$ (q)_n = \begin{cases} 1 & n = 1 \\ q(q+1)\dots(q+n-1) & n > 0 \end{cases}. $$

Proposition 4.5. *Suppose that $P(Q)$ is a constant relative prudence price function given by equation* (4)*. Then the value function* (23) *is given by $v(x) = g\left(h^{-1}(rx)\right)$, and the monopolist's inventory is given by $x(t) = \frac{1}{r} h\left(e^{rt} h^{-1}(rx(0))\right)$, where when $\rho \neq 1$ the functions $g(z)$ and $h(z)$ are given by:*

$$ g(z) = \max\left\{0, Y^{\frac{\rho}{1-\rho}} \left(\frac{Y-(1-\rho)z}{2-\rho}\right)^{\frac{2-\rho}{1-\rho}}\right\}, $$

$$ h(z) = \max\left\{0, (\rho-1)\, Y^{\frac{\rho}{1-\rho}} \left(\frac{1-\rho}{2-\rho}\frac{Y-(1-\rho)z}{Yz^{-1}+\rho-1}\right)^{\frac{1}{1-\rho}} {}_2F_1\left(\frac{1}{\rho-1}, \frac{1}{\rho-1}, \frac{\rho}{\rho-1}; \frac{Yz}{1-\rho}\right)\right\}, $$

and when $\rho = 1$

$$ g(z) = Ye^{-1-z}, \qquad h(z) = Ye^{-1}\mathrm{Ei}(-z). $$

Proof. The result follows from (26) and Proposition 4.2 using the specific formulas in Theorem 2.4. □

In the case $\rho = 0$, the linear price function, these quantities can be expressed in terms of the Lambert-W function, as found in [HHS10, LS12].

Definition 4.6. The *Lambert-W* function $\mathbb{W}(x)$ is the inverse function of $f(z) = ze^z$, namely the real-valued solution to the relation $x = \mathbb{W}(x)e^{\mathbb{W}(x)}$. Since ze^z is not injective, the inverse is multi-valued for $z \in \left[e^{-1}, 0\right)$. The branch chosen here to represent the Lambert-W function is the unique branch that is continuous on $\left[e^{-1}, \infty\right)$, as opposed to the second branch which is defined only on $\left[e^{-1}, 0\right)$.

Then with $\rho = 0$, we have

$$ v(x) = \frac{Y^2}{4r}(1+\mathbb{W}(\theta(x))^2, \quad \theta(x) = -e^{-1-2rx/Y}, \qquad x(t) = x(0) - \frac{Y}{2}t + \frac{Y}{2r}(1-e^{rt})\mathbb{W}(\theta(x). $$

Remark 4.7. *If $P(Q)$ is a constant relative prudence price function with $\rho \geq 1$, then from Proposition 4.5 we can immediately determine that the monopolist's effective marginal cost $\frac{dv}{dx}$ is unbounded. That is:*

$$ \lim_{x\to 0} v'(x) = \infty. $$

This follows from the fact that $v(0) = 0$ requires having $g(v') = 0$.

Remark 4.7 shows an important distinction between finite choke price models and infinite choke price models, namely that since price is unbounded in the latter, the monopolist's marginal value will rise to infinity as the commodity becomes exhausted, and, as we saw in Proposition 4.2, that infinite choke price also means that inventory reserves never reach 0.

4.2 Deterministic Game

We now move to a game model with multiple players. Games with more than one exhaustible producers, with each player's reserves level being a state variable, lead to strongly coupled systems of nonlinear PDE which are difficult to handle, even numerically. See, for instance, [HHS10] in the context of nonzero sum stochastic differential Cournot games, or [LS11a] for dynamic Bertrand games. A workaround, at least in models with substitutable goods is approximation by continuum mean field games. See [CS14] and [CS17], again in the Cournot and Bertrand contexts. However games with many renewable players competing amongst themselves and with one player with finite reserves, can be handled.

There is a single exhaustible producer, player 0, whose reserves at time t are $x(t)$, and his production rate is $q_t^{(0)} \geq 0$: $\frac{dx}{dt} = -q_t^{(0)} \mathbb{1}_{\{x>0\}}$. His cost of production is assumed zero (for simplicity). In addition there are $N-1$ players with inexhaustible (or renewable) resources, but they have potentially higher costs c_i, where

$$0 \leq c_1 \leq c_2 \leq \cdots \leq c_{N-1} < \infty.$$

They produce at rates $q_t^{(i)}$, and the market price is $P(Q(t))$, where $Q(t) = \sum_{i=0}^{N-1} q_t^{(i)}$. In the dynamic Cournot game, where the exhaustible player has initial reserves $x(0) = x > 0$, the value functions of the exhaustible player v, and of the renewable players w_i are:

$$v(x) = \sup_{q^{(0)} \in \mathcal{A}(x)} \int_0^{\tau_x} e^{-rt} q_t^{(0)} P(Q(t))\, dt, \tag{30}$$

$$w_i(x) = \sup_{q^{(i)} \geq 0} \int_0^{\tau_x} e^{-rt} q_t^{(i)} \left(P(Q(t)) - c_i\right)\, dt + \frac{1}{r} e^{-r\tau_x} \Pi_i(c), \quad i = 1, \cdots, N-1. \tag{31}$$

The second term in (31) reflects that after the exhaustible player runs out of resource and exits the market at time τ_x (possibly infinite), the remaining $N-1$ firms accrue profits at rate $\Pi_i(c)$, discounted at rate r.

For the linear pricing function $P = 1 - Q$, this game can be solved explicitly as was done for one exhaustible competitor ($N = 2$) in [HHS10], and for general N in [LS12]. In the latter, a key issue is finding the blockading points, where the market price has increased sufficiently (due to oil running out) for each most costly player to enter, and a modified Hotelling rule in this non-monopoly game framework.

The HJ ODEs associated with the value functions $v(x)$ and $w_i(x)$ are

$$rv = \sup_{q_0 \geq 0} q_0 \left(P\left(q_0 + \sum_{j=1}^{N-1} q_j^* \right) - v' \right), \qquad v(0) = 0,$$

$$rw_i = \sup_{q_i \geq 0} q_i \left(P\left(q_i + \sum_{j=0, j\neq i}^{N-1} q_j^* \right) - c_i \right), \quad w_i(0) = \frac{1}{r}\Pi_i(c), \quad i = 1, \cdots, N-1,$$

and so the internal Nash equilibrium problem is the static N-player game of Section 2 with cost vector $(v', c_1, \cdots, c_{N-1})$. Therefore, we have

$$rv = \bar{\Pi}(v', c), \quad x > 0, \qquad v(0) = 0, \tag{32}$$

where $\bar{\Pi}(v', c)$ is the profit function Π_1 of the first player in Theorem 2.2, evaluated at cost vector $(v', c_1, \cdots, c_{N-1})$, and we have suppressed notational dependence on the demand factor Y in this deterministic case.

The equilibrium extraction rates are given by the static Nash equilibrium quantities (2) evaluated at cost vector $(v', c_1, \cdots, c_{N-1})$. In particular, these depend only on v', and not on the w_i, so it is sufficient just to solve the ODE (32). The aggregate production is given by $Q(t) = \sum_{j=0}^{N-1} q_t^{(j)*}$ and the market price by $P(Q(t))$. As time advances and $x(t)$ declines, the price goes up and higher cost producers who were hitherto blockaded may enter. In particular when, $c_{n-1} < P(Q(t)) < c_n$, renewable players $1, \cdots, n-1$ are producing in addition to player zero, and so $n \leq N$ players are active. The transition points can be found by solving (32) analytically or numerically.

We denote by $C_{n-1} = \sum_{j=1}^{n-1} c_j$.

Proposition 4.8. *When there are exactly $1 \leq n \leq N$ active players, the marginal value function for the exhaustible player along the equilibrium extraction path $v'(x(t))$ solves the modified Hotelling rule:*

$$\frac{d}{dt} v'(x(t)) = H(v'(x(t))) r v'(x(t)), \tag{33}$$

where the pre-factor H is given by

$$H(v') = \frac{n + 1 - \rho(Q_n)}{2n - 2\rho(Q_n) + \rho(Q_n)\Phi(v')}, \quad \Phi(v') = \frac{P(Q_n) - v'}{nP(Q_n) - (v' + C_{n-1})}, \tag{34}$$

and $Q_n = Q_n(v')$ is the solution of

$$Q_n P'(Q_n) + nP(Q_n) = v' + C_{n-1}, \tag{35}$$

whose existence and uniqueness is guaranteed by Theorem 2.2. The equilibrium production of the exhaustible player is $q_t^{(0)*} = \bar{q}_0(v'(x(t))$, where we define

$$\bar{q}_0(v') = \Phi(v') Q_n(v'),$$

so Φ is the fraction of total output that he produces. Moreover, when $n = 1$, $H \equiv 1$, and for $n \geq 2$, $0 \leq H \leq 1$.

Proof. From Theorem 2.2, when exactly n players are active, the ODE (32) applies, with

$$\bar{\Pi} = -\frac{1}{P'(Q_n)} (P(Q_n) - v')^2,$$

where $Q_n(v')$ solves (35). Then, differentiating the ODE (32) gives

$$rv' = \bar{q}_0 v'' \left(Q_n' \left[2P'(Q_n) - (P(Q_n) - v') \frac{P''(Q_n)}{P'(Q_n)} \right] - 2 \right). \tag{36}$$

Differentiating (35) with respect to v' gives

$$Q_n' P'(Q_n) + P''(Q_n) Q_n Q_n' + nP'(Q_n) Q_n' = 1,$$

from which

$$Q_n' = \frac{1}{P'(Q_n)(n + 1 - \rho(Q_n))}.$$

Then as $\frac{d}{dt}v'(x(t)) = -\bar{q}_0(v'(x(t)))v''(x(t))$, substituting for $\bar{q}_0 v''$ from (36) gives (33)-(34).

Note that when only players $0, 1, \cdots, n-1$ are active, the market price is $P(Q_n)$ satisfying $P(Q_n) \geq v'$ and $P(Q_n) \geq c_j$, $j = 1, \cdots, n-1$, from which it follows that $\Phi(v')$ given in (34) satisfies $0 \leq \Phi(v') \leq 1$. When $n = 1$, $C_{n-1} = 0$ and so $\varphi \equiv 1$, which leads to $H \equiv 1$, the classical monopoly Hotelling's rule. The bounds on H for $n \geq 2$ follow from $\rho \leq \bar{\rho} < 2$ (or, in the case of constant $\rho < N+1$, that, from Theorem 2.4, that there will always be $n \geq \lfloor \rho \rfloor$ players active). □

Remark 4.9. *In the case $\rho \equiv 0$, which corresponds to any linear pricing function such as $P(Q) = 1 - Q$, we have that H is constant (depending on n but not v'):*

$$H = \frac{1}{2} + \frac{1}{2n},$$

as was found in [LS12]. In general, as $H \leq 1$ for $n \geq 2$, competition slows the growth of the exhaustible player's marginal value function below the monopoly Hotelling rate e^{rt}.

Expressions for the optimal strategy, extraction path and exhaustion time, such as in Proposition 4.2 for the monopoly, can be given, but they are more complicated and not very informative, so we do not give them here. In practice, the ODEs are easily solved numerically, or analytically in the linear demand case.

5 Game with Stochastic Demand

In the stochastic demand version of the game introduced in Section 4.2, we replace the price function P by $P(Q, Y)$, which satisfies Assumption 2.1 and also the following.

Assumption 5.1. *The price function $P(Q, Y)$ has maximal relative prudence $\bar{\rho} < 2$ for all $Y > 0$; or P is of power type, as given in Definition 2.3, with constant relative prudence $\rho < N+1$ for all $Y > 0$.*

We shall assume the demand factor Y is a one-dimensional diffusion satisfying Assumption 3.7.

The single exhaustible producer, player 0, whose reserves at time t are X_t, chooses a production rate is $q_t^{(0)} \geq 0$ so we have

$$dX_t = -q_t^{(0)} \mathbb{1}_{\{X_t > 0\}}.$$

His cost of production is assumed zero (for simplicity). In addition there are $N-1$ players with inexhaustible (or renewable) resources, but they have potentially higher costs c_i, where

$$0 \leq c_1 \leq c_2 \leq \cdots \leq c_{N-1} < \infty.$$

They produce at rates $q_t^{(i)}$, and the market price is $P(Q(t), Y_t)$, where $Q(t) = \sum_{i=0}^{N-1} q_t^{(i)}$. Analogous to (31), the value functions for the stochastic game starting at resource level $X_0 = x > 0$ and demand factor $Y_0 = y > 0$ are

$$v(x, y) = \sup_{q^{(0)} \in \mathcal{A}(x,y)} \mathbb{E} \int_0^{\tau_x} e^{-rt} q_t^{(0)} P(Q(t), Y_t)\, dt, \tag{37}$$

$$w_i(x, y) = \sup_{q^{(i)} \geq 0} \mathbb{E} \left\{ \int_0^{\tau_x} e^{-rt} q_t^{(i)} \left(P(Q(t), Y_t) - c_i \right) dt + \int_{\tau_x}^{\infty} e^{-rt} \Pi_i(c, Y_t)\, dt \right\}, \quad i = 1, \cdots, N-1,$$

where $\tau_x = \inf\{t > 0 \mid X_t = 0\}$, and Π_i is given in Theorem 2.2, with $c = (c_1, \cdots, c_{N-1})$.

Assumption 5.2. *We will assume that the demand factor Y and discount rate r are such that, even if all the renewable players have zero cost, namely $c_i = 0$ for $i = 1, \cdots, N-1$, then*

$$\int_0^\infty e^{-rt}\mathbb{E}\Pi_i(0, Y_t)\,dt < \infty, \tag{38}$$

so even after the exhaustible player runs out, every remaining player's value function (which are equal under identical zero cost) remains finite.

Remark 5.3. *In the case of constant prudence pricing functions given in* (4)*, we have from Theorem 2.4 that $\Pi_i = kY^{2+\rho}$ when all the costs are zero, for some constant k. Therefore condition* (38) *becomes*

$$\int_0^\infty e^{-rt}\mathbb{E}Y_t^{2+\rho}\,dt < \infty.$$

When Y is a GBM (8)*, this condition becomes*

$$(2+\rho)\left(\mu + \frac{1}{2}\sigma^2(1+\rho)\right) < r.$$

5.1 HJB Equations & Stochastic Hotelling Rule for Games

The value functions in (37) have have associated HJB PDEs

$$rv = \sup_{q_0 \geq 0} q_0 \left\{ P\left(q_0 + \sum_{j=1}^{N-1} q_j^*, y\right) - v_x \right\} + \mathcal{L}_y v, \qquad v(0,y) = 0, \tag{39}$$

$$rw_i = \sup_{q_i \geq 0} q_i \left\{ P\left(q_i + \sum_{j=0, j\neq i}^{N-1} q_j^*, y\right) - c_i \right\} + \mathcal{L}_y w_i, \quad w_i(0,y) = \int_0^\infty e^{-rt}\mathbb{E}\Pi_i(c, Y_t)\,dt, \quad i = 1, \cdots, N-1,$$

and the boundary conditions for w_i are finite under our Assumption 5.2. The internal Nash equilibrium problem in the system (39) is the static N-player game of Section 2 with cost vector $(v_x, c_1, \cdots, c_{N-1})$. As is typical in these problems, the marginal value v_x plays the role of a shadow (or scarcity) cost for the exhaustible player. Therefore, we have

$$rv = \bar{\Pi}([v_x;\, c], y) + \mathcal{L}_y v, \quad x > 0, \qquad v(0,y) = 0, \tag{40}$$

where $\bar{\Pi}(v_x, c)$ is the profit function Π_1 of the first player in Theorem 2.2, evaluated at cost vector $[v_x;\, c] := (v_x, c_1, \cdots, c_{N-1})$. We note that the dynamic equilibrium strategies are given in terms of the static Nash equilibrium quantities q_i^*, defined in (2), by

$$q_t^{(i)*} = q_i^*([v_x(X_t, Y_t);\, c], Y_t), \qquad i = 0, 1, \cdots, N-1,$$

and only depend on v_x but not the w_i. Therefore, we focus on the v PDE henceforth.

First, we have the stochastic demand analogue of Proposition 4.8, Hotelling's rule for the game. We recall the relative prudence $\rho(Q, Y)$ defined in (1).

Proposition 5.4. *The marginal value function $v_x(X_t, Y_t)$ along the optimal extraction path follows the dynamics*

$$dv_x = \Big(H(v_x, Y_t) r v_x + (1 - H(v_x, Y_t))\mathcal{L}_y v_x\Big)\, dt + \sigma(Y_t) v_{xy}\, dW_t, \tag{41}$$

where the pre-factor H is

$$H(v_x, y) = \frac{n + 1 - \rho(Q_n, y)}{2n - 2\rho(Q_n, y) + \rho(Q_n, y)\Phi(v_x, y)}, \quad \Phi(v_x, y) = \frac{P(Q_n, y) - v_x}{nP(Q_n, y) - (v_x + C_{n-1})}, \tag{42}$$

and $Q_n = Q_n(v_x, y)$ is the solution of

$$Q_n P_Q(Q_n, y) + nP(Q_n, y) = v_x + C_{n-1}, \tag{43}$$

whose existence and uniqueness is guaranteed by Theorem 2.2.

Proof. Applying Itô's formula to $v_x(X_t, Y_t)$, we have

$$\begin{aligned} dv_x &= v_{xx}\, dX_t + v_{xy}\, dY_t + \frac{1}{2}\sigma(Y_t)^2 v_{xyy}\, dt \\ &= (-q_0^* v_{xx} + \mathcal{L}_y v_x)\, dt + \sigma(Y_t) v_{xy}\, dW_t. \end{aligned} \tag{44}$$

Next, differentiating the PDE (40) and using the computations in the proof of Proposition 4.8 gives

$$r v_x = \frac{1}{H(v_x, y)}(-q_0^* v_{xx}) + \mathcal{L}_y v_x.$$

Substituting for $-q_0^* v_{xx}$ into (44) gives (41). □

As in the monopoly case in Proposition 3.12, the stochastic Hotelling rule for the game is not an autonomous SDE as it involves v_{xy} and v_{xyy}. The drift is a linear combination of Hrv_x as in the deterministic game (Proposition 4.8) and $(1 - H)\mathcal{L}_y v_x$.

5.2 Demand Factor Models

We will consider two main diffusion models for the demand factor Y. If we think of the factor as being a proxy for global demand, or business income levels which drive demand for a commodity such as oil, a natural first choice is one that is mean-reverting, representing business cycles. A standard example is the exponential Ornstein-Uhlenbeck (expOU) process, which is positive and defined by the SDE

$$\frac{dY_t}{Y_t} = \alpha(m - \log Y_t)\, dt + \sigma\, dW_t, \qquad \alpha, \sigma > 0. \tag{45}$$

Here, $\log Y_t$ is an OU process.

However, while there is a clear business cycle pattern in the real world, it is not as regular or as 'predictable' as an expOU process. For example, Japan grew rapidly until the 1990s, becoming the second largest economy in the world, but has since stalled and only achieved minimal growth in the last 25 years. Further, [Chi13] found that oil prices exhibited mean-reverting behavior up until 1995, but since then have not been mean-reverting. Moreover, as we found in the monopoly with stochastic demand in Section 3.2, Example 3.10, the zero cost exhaustible producer may become demand blockaded in the expOU model, which seems rather unrealistic. As such, we conclude explicit mean-reversion induces too much "freeze production and wait for better", for even a zero cost monopolist, which is implausible in oil markets.

Therefore, for numerical and other examples, we will largely focus on the case where Y is a GBM (8), where as per Example 3.9, there is no monopoly demand blockading if $\mu \leq r$, as we shall assume. Indeed, if one thinks of Y as expected future demand (for instance from China), then it is reasonable to model Y as a conditional expectation $Y_t = \mathbb{E}\{D_T \mid \mathcal{F}_t\}$ for (uncertain) demand D at some future time $T > t$ and filtration $(\mathcal{F}_t)$, and therefore as a martingale ($\mu = 0$). Another possible model is a martingale bounded below by, for instance 1:

$$dY_t = \sigma(Y_t - 1)\, dW_t.$$

Remark 5.5. *It is also possible to obtain an explicit approximation to the value function in the fast mean-reverting limit. This is done when the demand factor Y is a two-state Markov chain in [LY14], and when Y is a continuous expOU process in [Fun17, Section 2.2.8] for the monopoly problem under linear demand. In this regime, demand blockading may occur infinitely often, while the exhaustion time goes to infinity. We do not pursue this approximation method here.*

5.3 Numerical Results

We present some numerical results when the demand factor is a GBM or an expOU process, and the pricing function is of power type. These are obtained from a finite difference approximation of the PDE (40). We do not give details of the numerical implementation here, but these can be found in [Bro16, Section 4.4.3] and [Fun17, Section 2.3.4]. In our examples, we have a duopoly between a zero cost exhaustible producer, and a renewable source with higher cost $c_1 > 0$.

Figure 2 shows the result of our numerical method for approximating $v(x, y)$ for a linear stochastic demand pricing function. The blue region represents the states where the renewable producer is blockaded, while the red region represents the states where both players are active. We observe that the exhaustible producer is never demand blockaded for these parameters, and that the value function is always concave in reserves.

For our second numerical example, in Figure 3, we use a constant relative prudence price function with $\rho = 1.5$, so the choke price is infinite. Once again the blue region is where the renewable producer is blockaded and the red region is where both players are active. Since we chose $\rho < 2$ there will still be states in which the renewable producer is blockaded, but in contrast to Figure 2, they cannot occur at $x = 0$, since at that point the renewable producer has a monopoly and the choke price is infinite. Indeed, we see that the blockading region pulls away from 0 much more sharply, though the resolution provided by the numerical PDE solution does not allow us to see that it never touches 0. While it does never reach 0, we see that the blockading region becomes arbitrarily close, as for very small values of Y, the exhaustible producer's production at almost any $x > 0$ will cause the renewable producer to become blockaded.

In our third example, demand is an expOU process as in (45). Figure 4 shows the numerical value function for linear pricing. The blue region represents the states where the renewable producer is blockaded, the red region represents the states where both players are active, and the yellow region represents the states in which the exhaustible producer is blockaded. The latter shows demand blockading, where the upward drift from mean reversion gives the exhaustible producer an incentive to wait out a period of low demand. Within this region there are some states

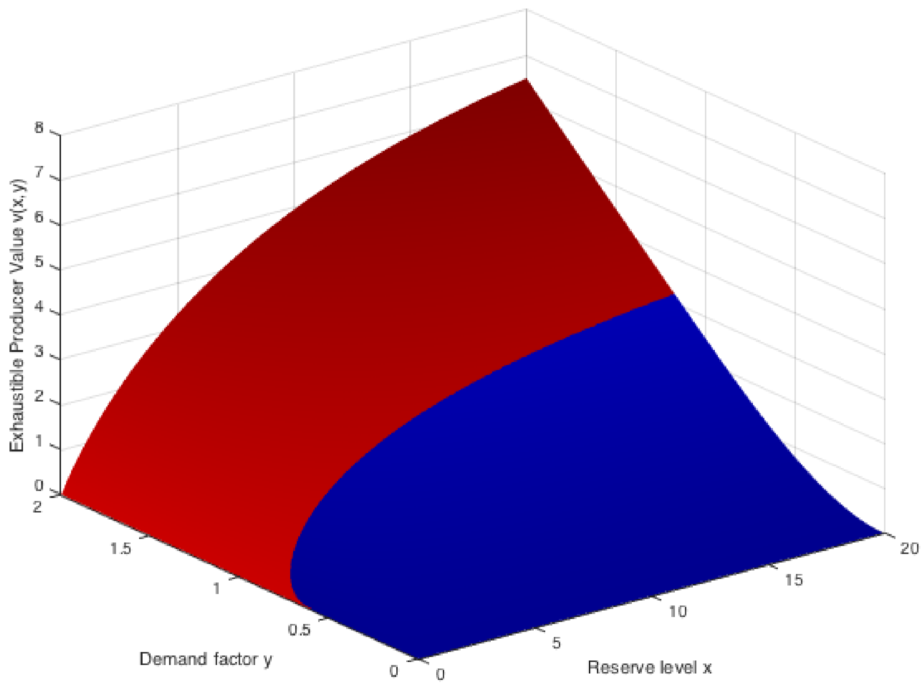

Figure 2: Value function $v(x, y)$ for exhaustible player with one renewable competitor with cost $c_1 = 0.6$. Price is $P(Q, Y) = Y - Q$ and Y is GBM with $\mu = 0, \sigma = 0.2$. Discount rate $r = 0.1$.

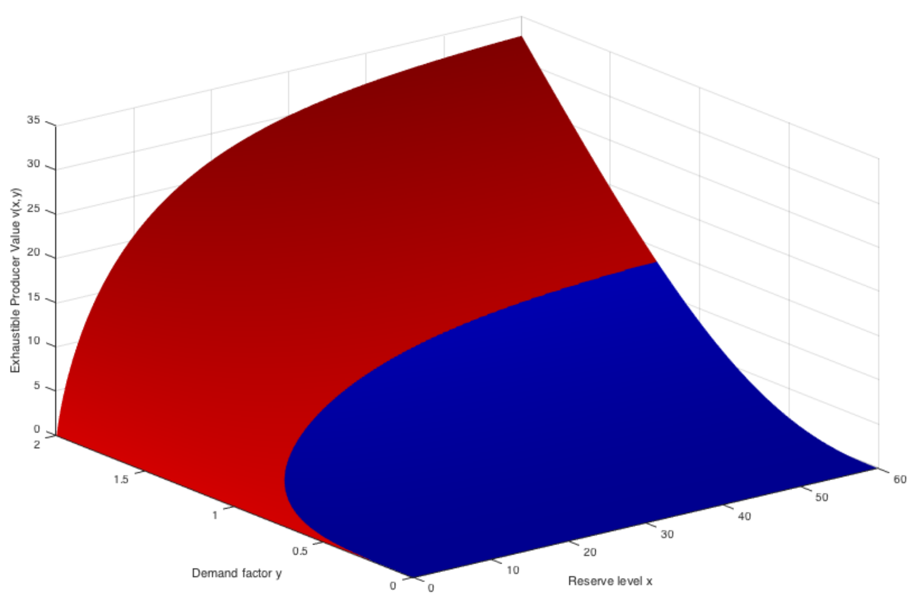

Figure 3: Value function $v(x, y)$ for exhaustible player with one renewable competitor with cost $c_1 = 2.5$. Price is as (4) with $\rho = 1.5$, and Y is GBM with $\mu = 0, \sigma = 0.2$. Discount rate $r = 0.1$.

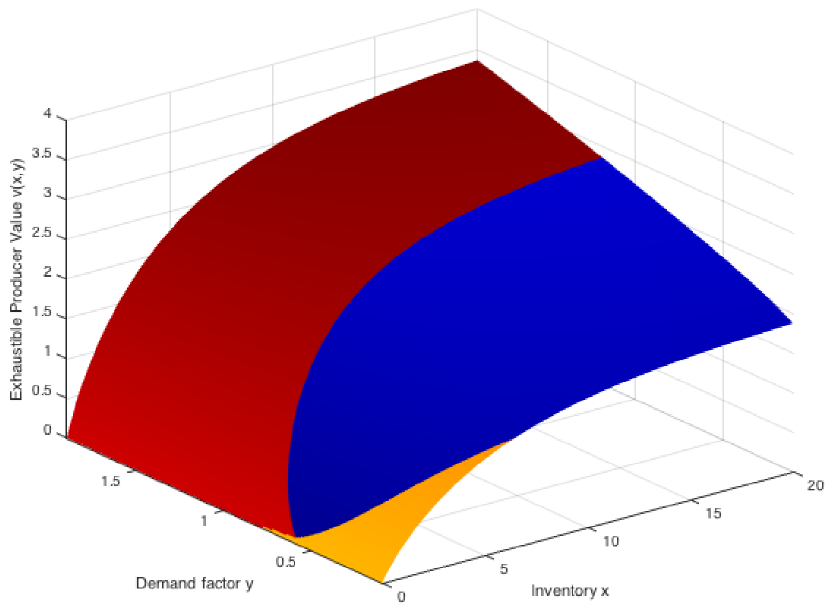

Figure 4: Value function $v(x, y)$ for exhaustible player with one renewable competitor with cost $c_1 = 0.6$. Price is $P(Q, Y) = Y - Q$ and Y is expOU as (45) with $\alpha = 0.5, m = 0, \sigma = 0.5$. Discount rate $r = 0.1$.

in which the renewable producer is blockaded and some in which they are active, depending on whether the choke price Y_t exceeds their marginal cost.

We compare now with a constant relative prudence price function with $\rho = 1.5$ and maintain expOU demand. We make the same modifications as we did in Figure 3 including an increase to the renewable competitor's cost. Figure 5 shows the resulting value function. As in the previous example, we see that the blockading region for the renewable producer does not extend to $x = 0$ as it does in Figure 3, because for any fixed value of the demand Y_t there will be some $X_t > 0$ for which the price is higher than the renewable producer's marginal cost c_1. The yellow region is no longer visible but it is still possible in theory for the exhaustible producer to be blockaded when saturation demand is very low, since the renewable producer's production will create a kind of finite choke price. Since there is a minimum level of total production equal to the renewable producer's strategy when $q^{(0)*} = 0$, we find that Remark 3.11, which states that infinite choke price models never result in demand blockading *in a monopoly*, does not translate to the Cournot competition with $N > 1$ players.

6 Dynamic Game with Inexhaustible Players

With the aim of better understanding short term oil price volatility, as shown in Figure 1, we consider dynamic games between many producers without the constraint of limited resources. This game again represents competition in oil markets in the short run, where the exhaustibility of reserves is not a constraint. We explore sample paths of the model, looking at the blockading of

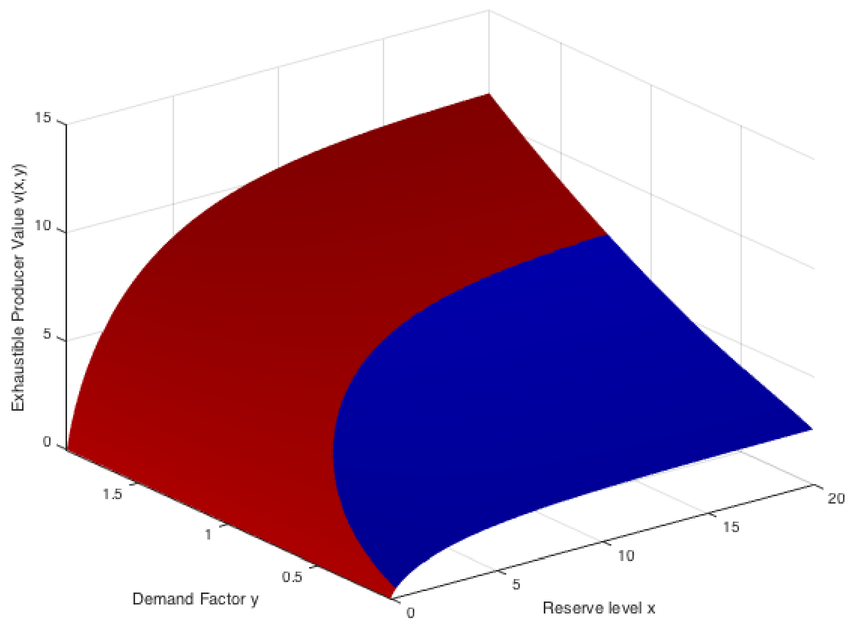

Figure 5: Value function $v(x, y)$ for exhaustible player with one renewable competitor with cost $c_1 = 0.6$. Price is as (4) with $\rho = 1.5$, and Y is expOU as (45) with $\alpha = 0.5, m = 0, \sigma = 0.5$. Discount rate $r = 0.1$.

firms, and examine price volatility and how blockading impacts it when there are many players.

6.1 Inexhaustible Game with Stochastic Demand

Without the concern of finite resources, the value functions of the N players ordered by marginal costs $c_1 \leq c_2 \leq \cdots \leq c_N$ are given by

$$v_i(y) = \sup_{q^{(i)}} \mathbb{E} \int_0^\infty e^{-rt} q_t^{(i)} \left(P(Q(t), Y_t) - c_i\right) dt, \quad i = 1, \cdots, N,$$

where $Y_0 = y$. They satisfy the system of ODEs

$$rv_i = \Pi_i(c, y) + \mathcal{L}_y v_i,$$

where Π_i are the static Nash equilibrium profits from Theorems 2.2 or 2.4, and c is the vector of costs. Consequently

$$v_i(y) = \int_0^\infty e^{-rt} \mathbb{E}\Pi_i(c, Y_t)\, dt, \quad \text{and} \quad q_t^{(i)*} = q_i^*(c, Y_t),$$

so each player's dynamic equilibrium strategy is simply to play the static strategy with the current level of the demand factor Y_t. The statistics of the Y process affect the value functions, but not the strategies, which in this sense can be described as myopic.

In this section, we will work with the stochastic linear inverse-demand curve

$$P(Q_t, Y_t) = Y_t - Q_t, \qquad Q_t = \sum_{i=1}^{N} q_t^{(i)},$$

and where Y_t is a GBM with the dynamics from equation (8). Then, from Theorem 2.4 with $\rho = 0$, the market price is given by

$$P_t = \frac{Y_t + C_{n_t}}{n_t + 1}, \qquad n_t = \arg\min_{1 \le n \le N} \frac{Y_t + C_n}{n+1}, \quad C_n = \sum_{i=1}^{n} c_i.$$

Here n_t is the number of active players at time t, with the remaining being blockaded. When Y reaches the threshold where, say, player n joins the first $n-1$ players in producing, we have

$$P_{t-} = \frac{Y_{t-} + C_{n-1}}{n} = c_n \quad \Rightarrow \quad P_t = \frac{Y_t + C_n}{n+1} = \frac{Y_t + C_{n-1} + c_n}{n+1} = c_n,$$

and so $P_{t-} = P_t$, using also the continuity of Y. So the change in the number of producers does not introduce discontinuity in the price process, but we will see how it affects volatility. The production rates of the players when n_t of them are active are given by

$$q_t^{(i)*} = \frac{1}{n_t + 1}\left(Y_t - n_t c_i + \sum_{j=1, j \neq i}^{n_t} c_j\right), \quad i = 1, \cdots, n_t,$$

and zero for $i > n_t$.

6.2 Illustration of Sample Paths

We start with a dynamic stochastic inexhaustible 1000-player model with the parameters

$$c_1 = 0, c_2 = 0.000004, c_3 = 0.000008, \cdots, c_{1000} = 0.004, \quad r = 0.5, \mu = 0, \sigma = 0.52, y = 1.$$

The left plot of Figure 6 illustrates the costs of the players. We use such small costs because we want to maintain an initial value of the demand factor of one. If the costs were larger with this initial value of the demand factor, most of the players would be blockaded at the start of the game. With these costs, 707 firms produce at the initial value of the demand factor and all firms produce at a demand factor of two. Figure 7 shows the number of producing firms at a given level of the demand factor for these costs. The higher the demand factor, the greater the number of producing firms. This is because as the demand factor rises, the price rises and higher cost players enter as they can produce profitably. The growth in the number of producing firms slows, as the demand factor gets large. The growth slows because the price rises at a slower rate as more firms enter the market and it becomes more competitive.

Figure 8 shows a sample path for this game over a two year period. In this simulation, the demand factor falls over the two year time period, which leads to a decrease in price, production levels and the number of firms producing. Rather than thinking of the players as industries, here we think of them as individual companies. The higher cost players produce less and the highest cost players are blockaded. As the bottom right plot of Figure 8 shows, there are originally just over 700 firms producing, but this number falls with the demand factor, until only about 450 firms are producing at the end of the second year.

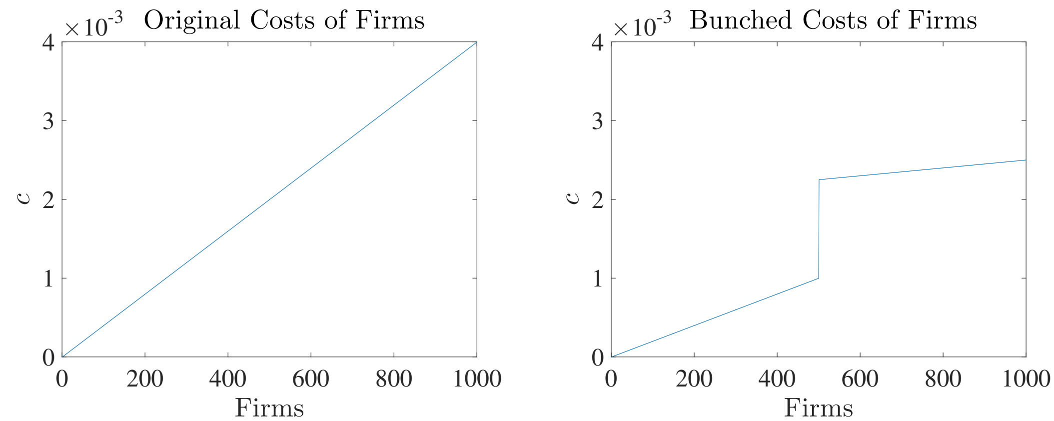

Figure 6: Costs of the players for the sample paths of the dynamic game with 1,000 players.

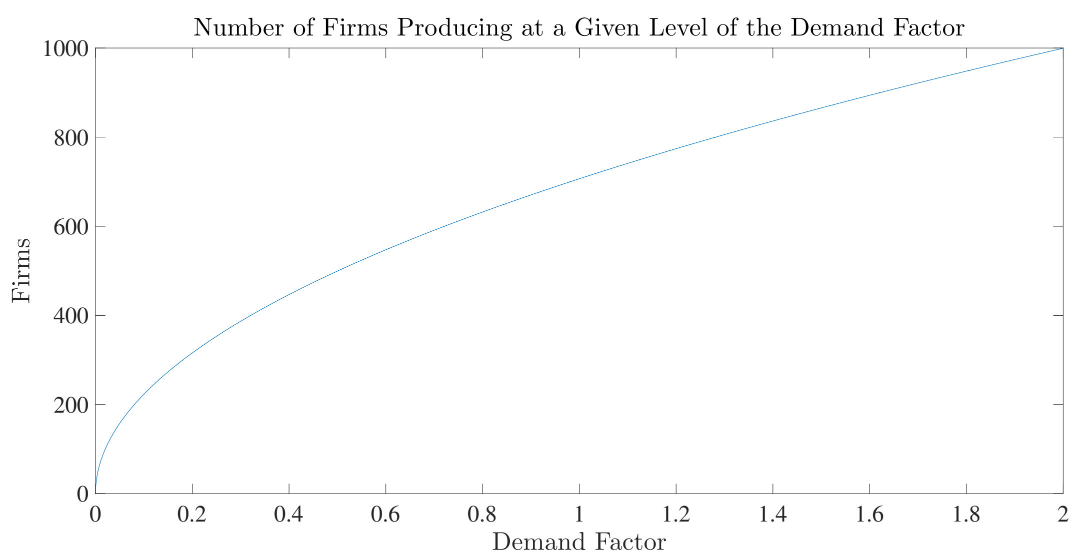

Figure 7: The number of firms producing at a given level of the demand factor in the dynamic game with 1,000 players. The costs illustrated here are $c_1 = 0$, $c_2 = 0.000004$, $c_3 = 0.000008, \ldots,$ $c_{1000} = 0.004$.

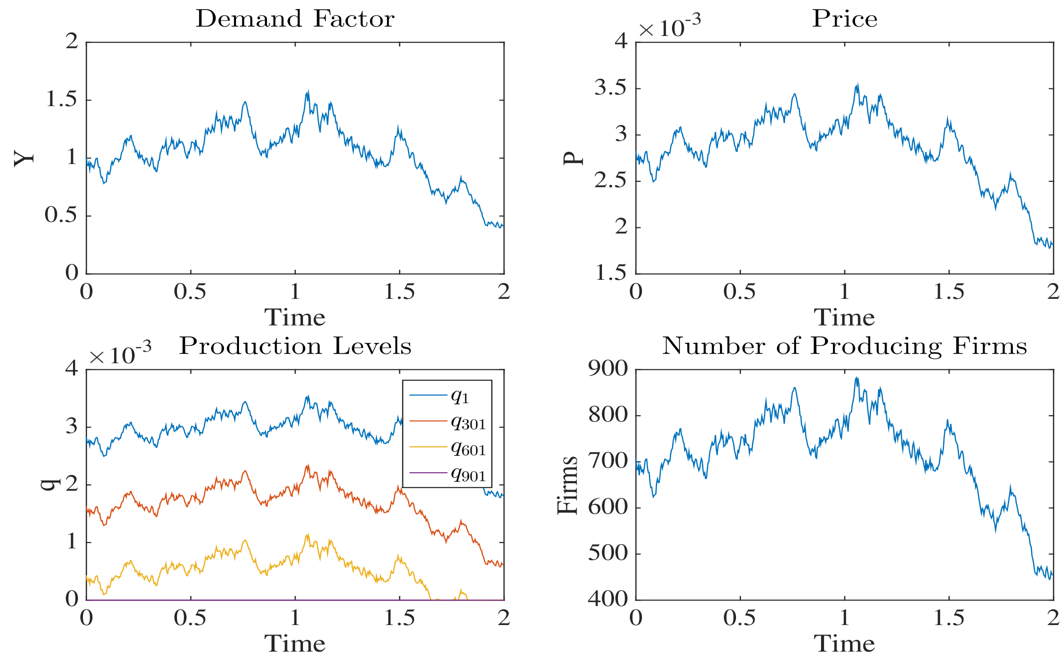

Figure 8: Sample path of the dynamic inexhaustible stochastic model with 1,000 players. The parameters illustrated here are $c_1 = 0$, $c_2 = 0.000004$, $c_3 = 0.000008$, $\ldots$, $c_{1000} = 0.004$, $r = 0.5$, $\mu = 0$, $\sigma = 0.52$ and $y = 1$.

6.3 Investigation of Volatility

Figure 9 shows a sample path of the number of firms producing and demand factor and price return volatility. In the figure, the number of firms producing varies from 450 to almost 900, but the price return volatility barely reaches 30% one time, even with demand factor volatility of 52%. The effect of the blockading here, however, is much harder to observe as the costs of the players are close together. Further, blockading's effect on price volatility is disguised because the number of firms is changing relatively slowly, and only hits the extremes for short periods of time.

In order to see more clearly the potential effect of blockading on price volatility, we adjust the costs of the players. Figure 10 shows another sample path of the number of firms producing and demand factor and price return volatility, using the same path of the demand factor as in Figure 8, but the costs of the players in this sample path are bunched in two groups (graphed in the right plot of Figure 6). Figure 11 shows the price and some production levels. In reality, costs of energy producers are bunched by technology. In this example, we think of the lower cost firms as the traditional oil producers and the higher cost firms as (say) fracking producers. In both sample paths, the discount rate and the parameters of the GBM are the same. In Figure 10, the number of producing firms varies from 500 to 1000, and the change in the price volatility due to blockading is more pronounced. The number of producing firms starts around 700 and rapidly falls to 500 and then rebounds to $1,000$. The rapid changes create volatility, especially when there is substantial blockading, such as at the end of the simulation when price volatility reaches over 35%.

By bunching the firms closer together and increasing the number of high cost producers relative to the low cost producers, we can create volatility that is closer to what is observed in real world

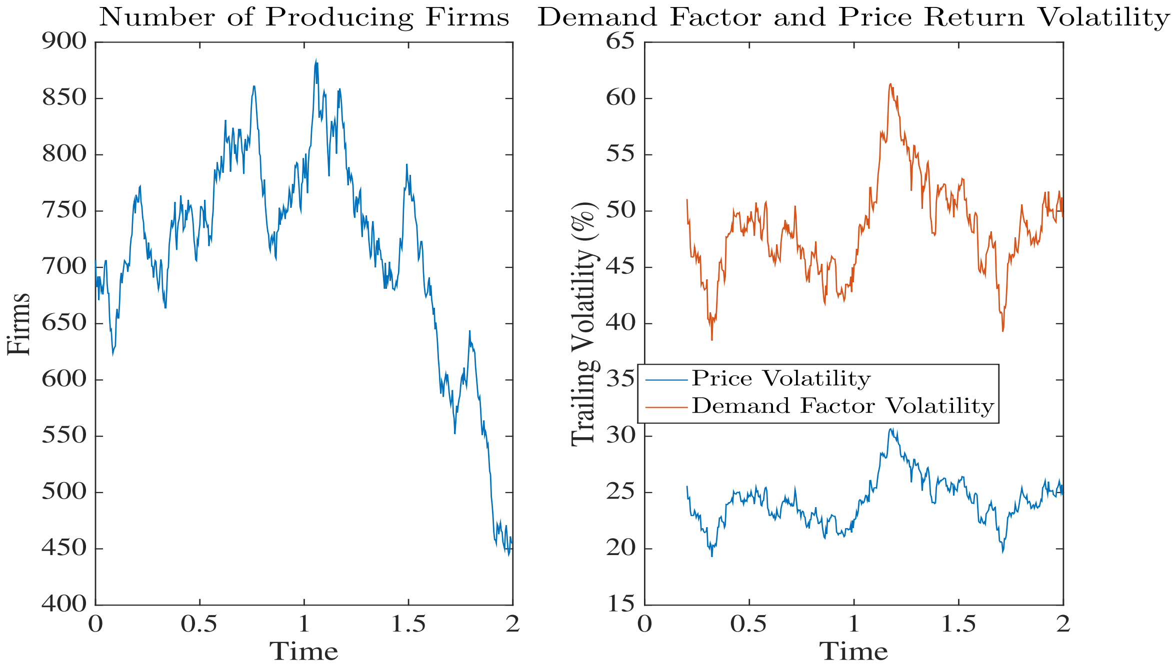

Figure 9: An illustration of the number of firms and demand factor and price return volatility in a sample path of the model with 1,000 players. The parameters illustrated here are $c_1 = 0$, $c_2 = 0.000004$, $c_3 = 0.000008, \dots, c_{1000} = 0.004$, $r = 0.5$, $\mu = 0$, $\sigma = 0.52$ and $y = 1$.

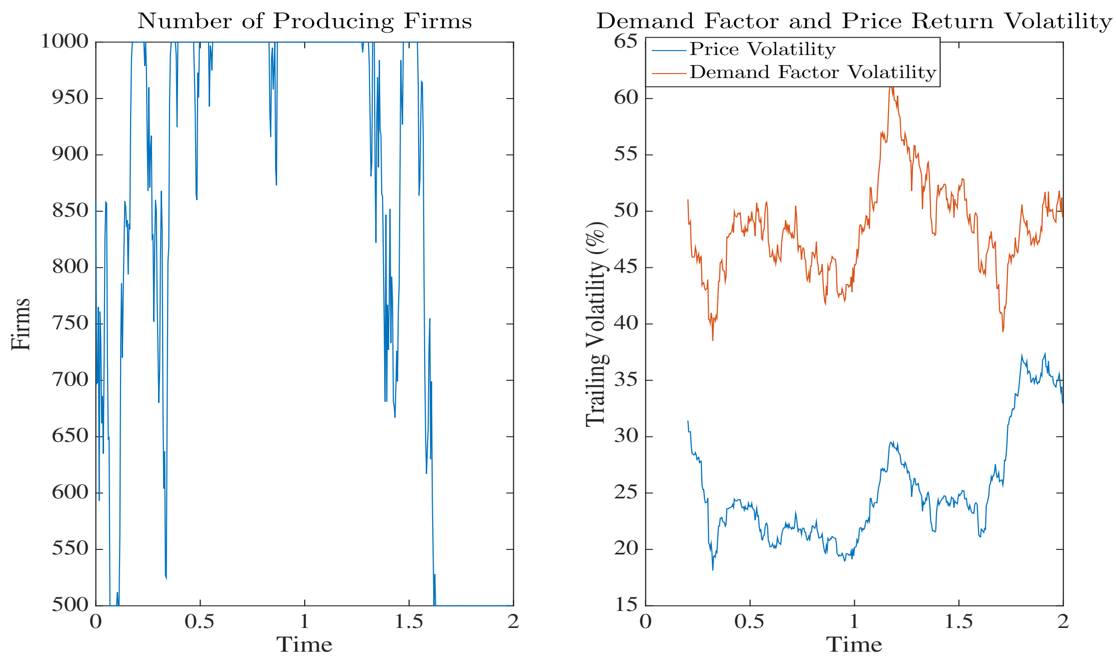

Figure 10: An illustration of the number of firms and demand factor and price return volatility in a sample path of the model with 1,000 players. The costs of the players are shown in the right plot of Figure 6 and the other parameters are $r = 0.5$, $\mu = 0$, $\sigma = 0.52$ and $y = 1$.

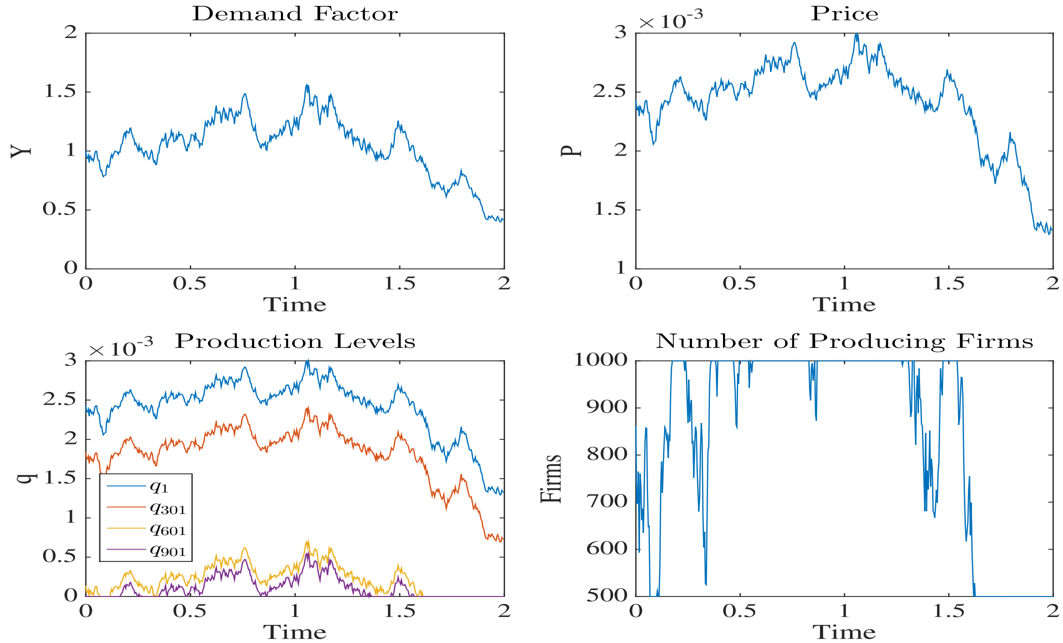

Figure 11: Game dynamics with 1,000 players. The costs of the players are shown in the right plot of Figure 6 and the other parameters are $r = 0.5$, $\mu = 0$, $\sigma = 0.52$ and $y = 1$.

prices. Figure 12 shows actual WTI crude oil price return volatility over the last eight years, for comparison with Figure 10. While the volatility is significantly lower in the sample path, the pattern of volatility is similar and blockading creates price spikes like those observed in the real world. Here, we confirm our finding above that lower demand leads to greater volatility through reduced competition and show that it applies even with many competitors. With bunching of producers' costs due to technology, there are more clear regimes in price volatility as prices change, which better fits with real world observations. This supports the argument that blockading can create volatility in oil prices.

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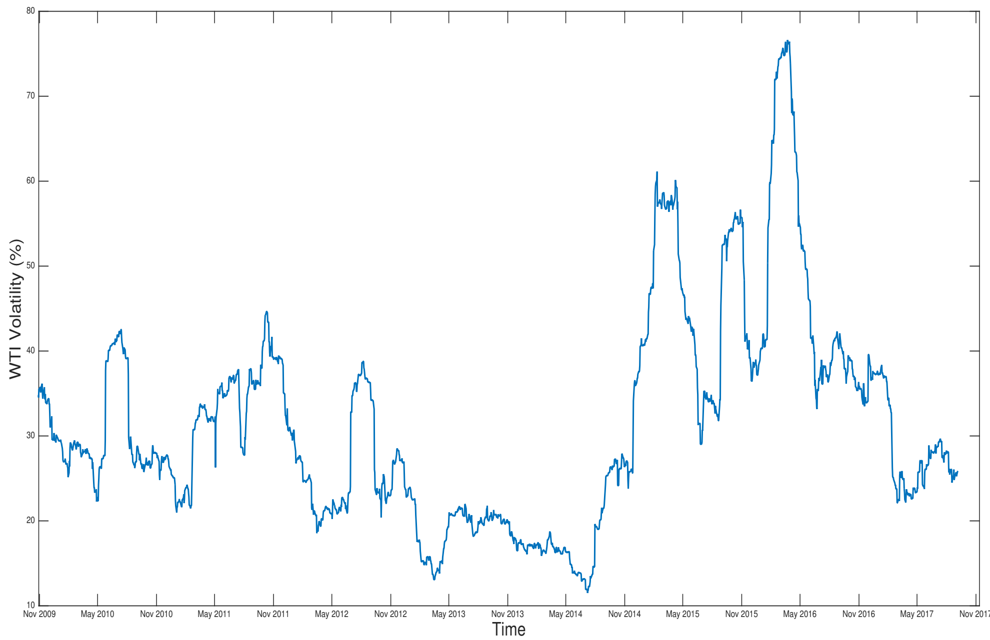

Figure 12: WTI price return volatility 2009-17.

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