

Queueing Theoretic Approaches to Financial Price Fluctuations*

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Abstract

One approach to the analysis of stochastic fluctuations in market prices is to model characteristics of investor behaviour and the complex interactions between market participants, with the aim of extracting consequences in the aggregate. This agent-based viewpoint in finance goes back at least to the work of Garman (1976) and shares the philosophy of statistical mechanics in the physical sciences. We discuss recent developments in market microstructure models. They are capable, often through numerical simulations, to explain many stylized facts like the emergence of herding behavior, volatility clustering and fat tailed returns distributions. They are typically queueing-type models, that is, models of order flows, in contrast to classical economic equilibrium theories of utility-maximizing, rational, “representative” investors. Mathematically, they are analyzed using tools of functional central limit theorems, strong approximations and weak convergence. Our main examples focus on investor inertia, a trait that is well-documented, among other behavioral qualities, and modelled using semi-Markov switching processes. In particular, we show how inertia may lead to the phenomenon of long-range dependence in stock prices.

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1 Introduction

Modeling market microstructure in order to understand the effects of many individual investors on aggregate demand and price formation is both a classical area of study in economics, and a rapidly growing activity among researchers from a variety of disciplines, partly due to modern-day computational power for large-scale simulations, and the increased availability of price and order-book data. Among the benefits of this type of analysis, whether mathematical or simulation-based, is the design of better models of macroscopic financial variables such as prices, informed by microscopic (investor-level) features, that can then be utilized for improved forecasts, investment and policy decisions.

The approach we discuss here is to identify characteristics common to large groups of investors, for example prolonged inactivity or inertia, and study the resulting price dynamics created by order flows. Typically, we are interested in understanding the microstructure effects on the aggregate quantity through approximations from stochastic process limit theorems when there is a large number of investors.

In this point of view, we model right away the behavior of individual traders rather than characterizing agents' investment decisions as solutions to individual utility maximization problems. Such an approach has also been taken in [45], [40], [65] and [39], for example. As pointed out by O'Hara in her influential book *Market Microstructure Theory* [72], it was Garman's 1976 paper

[45] that “inaugurated the explicit study of market microstructure”. There, he explains the philosophy of this approach as follows: “The usual development here would be to start with a theory of individual choice. Such a theory would probably include the assumption of a stochastic income stream [and] probabilistic budget constraints But here we are concerned rather with aggregate market behavior and shall adopt the attitude of the physicist who cares not whether his individual particles possess rationality, free will, blind ignorance or whatever, as long as his statistical mechanics will accurately describe the behavior of large ensembles of those particles”. This approach is also common in much of the econophysics literature (see the discussion in [36], for example), and is of course prevalent in queueing models of telephone calls or internet traffic [20], where interest is not so much on causes of phone calls or bandwidth demand, but on phenomenological models and their overall implications. As one econophysicist explained it in reaction to the usual battle-cry of the classical economist about rational behaviour, when AT&T uses queueing models, it doesn’t ask *why* you call your grandmother.

In this article, we provide an outline to recent surveys on agent-based computational models and analytical models based on dynamical systems, while our focus is on developing limit theorems for queueing models of investor behaviour, which apply modern methods from stochastic analysis to models based on economic intuition and empirical evidence. The goal is in obtaining insights into market dynamics by understanding price formation from typical behavioral qualities of individual investors.

The remainder of this paper is summarized as follows: in Section 2, we briefly survey some recent research on agent based models. These models relate the behavioral qualities of investors and quantitative features of the stock price process. We give a relevant literature review of Queueing Theory approaches to the modeling of stock price dynamics in Section 2.3. In Section 2.4, we discuss evidence of investor inertia in financial markets, and we study its effect on stock price dynamics in Section 3. Key tools are a functional central limit theorem for semi-Markov processes and approximation results for integrals with respect to fractional Brownian motion, that establish a link between investor inertia and long range dependence in stock price returns. These are extended in Section 3.2 to allow for the *feedback* of price of the stock into agents’ investment decisions, using methods and techniques from *state dependent queueing networks*. We establish approximation results for the stock price in a non-Walrasian framework in which the order rates of the agents depend on the stock price and exogenously specified investor sentiment. Section 4 concludes and discusses future directions.

2 Agent-Based Models of Financial Markets

In mathematical finance, the dynamics of asset prices are usually modelled by trajectories of some exogenously specified stochastic process defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Geometric Brownian motion has long become the canonical reference model of financial price dynamics. Since prices are generated by the demand of market participants, it is of interest to support such an approach by a microeconomic model of interacting agents. In recent years there has been increasing interest in agent-based models of financial markets where the demand for a risky asset comes from many agents with interacting preferences and expectations. These models are capable

of reproducing, often through simulations, many “stylized facts” like the emergence of herding behavior [39, 65]; volatility clustering [24, 66], or fat-tailed distributions of stock returns [25], that are observed in financial data.

In contrast to the traditional framework of an economy with a utility-maximizing representative agent, agent-based models typically comprise many *heterogeneous* traders who are so-called *boundedly rational*. Behavioral finance models assume that market participants do not necessarily share identical expectations about the future evolution of asset prices or assessments about a stock’s fundamental value. Instead, agents are allowed to use rule of thumb strategies when making their investment decisions and to switch randomly between them as time passes. Following the seminal article of Frankel and Froot [41], one typically distinguishes *fundamentalists*, *noise traders* and *chartists*¹. A fundamentalist bases his forecasts of future asset prices upon market fundamentals and economic factors such as dividends, quarterly earnings or GDP growth rates. He invests in assets he considers undervalued, that is, he invests in assets whose price is beneath his subjective assessment of the fundamental value. Chartists, on the other hand, base their trading strategy upon observed historical price patterns such as trends. Technical traders try to extrapolate future asset price movements from past observations. Fundamentalists and chartists typically coexist with fractions varying over time as agents are allowed to change their strategies in reaction to either the strategies’ performances or the choices of other market participants. Some of these changes can be self reinforcing when agents tend to follow the more successful strategies or agents. This may lead to temporary deviations of prices from the benchmark fundamental or rational expectations prices generating bubbles or crashes in periods when technical trading predominates. Fundamentalists typically have a stabilizing impact on stock prices.

In this section, we review some agent-based models of financial markets. Our focus will be on a class probabilistic models in which asset price dynamics are modelled as stochastic processes in a random environment of investor sentiment. These models are perhaps most amenable to rigorous mathematical results. Behavioral finance models based on deterministic dynamical systems are covered only briefly as they are discussed extensively in a recent survey by Hommes [49]. For results on evolutionary dynamics of financial markets we refer to [47], [32], or [75] and references therein.

2.1 Stock Prices as Temporary Equilibria in Random Media

Föllmer and Schweizer [40] argue that asset prices should be viewed as a sequence of temporary equilibrium prices in a random environment of investor sentiment; see also [37]. In reaction to a proposed price p in period t , agent $a \in \mathbb{A}$ forms a random excess demand $e_t^a(p, \omega)$, and the actual asset price $P_t(\omega)$ is determined by the market clearing condition of zero total excess demand. In [40], individual excess demand involves some *exogenous* liquidity demand and an *endogenous* amount obtained by comparing the proposed price p with some reference level \hat{S}_t^a . This dependence is linear on a logarithmic scale and individual excess demand takes the form

$$e_t^a(p, \omega) := c_t^a(\omega) \left(\log \hat{S}_t^a(\omega) - \log p \right) + \eta_t^a(\omega) \quad (2.1)$$

¹Survey data showing the importance of chartist trading rules among financial practitioners can be found in, e.g., [81] and [42]

with nonnegative random coefficients $c_t^a(\omega)$. Here $\eta_t^a(w)$ is the individual's liquidity demand. The logarithmic equilibrium price $S_t(\omega) := \log P_t(\omega)$ is then determined via the market clearing condition $\sum_{a \in \mathbb{A}} e^a(P_t(\omega), \omega) = 0$. It is thus formed from an aggregate of individual price assessments and liquidity demands. If the agents have no sense of the direction of the market and simply take the last logarithmic price S_{t-1} as their reference level, i.e., if $\log \hat{S}_t^a = S_{t-1}$, then the log-price dynamics reduces to an equation of the form

$$S_t = S_{t-1} + \eta_t$$

where η_t denotes the aggregate liquidity demand. In this case the dynamics of logarithmic prices reduces to a simple random walk model if the aggregate liquidity demand is independent and identically distributed over time. This is just the discretized version of the Black-Scholes-Samuelson geometric Brownian motion model.

A *fundamentalists* bases his investment decision on the idea that asset prices will move closer to his subjective benchmark fundamental value F^a . In a simple log-linear case

$$\log \hat{S}_t^a := S_{t-1} + \alpha_t^a (F^a - S_{t-1}) \quad (2.2)$$

for some random coefficient $0 < \alpha_t^a < 1$. If only such *information traders* are active on the market, the resulting logarithmic stock price process takes the form of a mean-reverting random walk in the random environment $\{\alpha_t\}_{t \in \mathbb{N}}$ ($\alpha_t = \{\alpha_t^a\}_{a \in \mathbb{A}}$). A combination of information trading and a simple form of *noise trading* where some agents take the proposed price seriously as a signal about the underlying fundamental value replacing F^a in (2.2) by p leads to a class of discrete time Ornstein-Uhlenbeck processes. Assuming for simplicity that subjective fundamentals equal zero the logarithmic price process takes the form

$$S_t - S_{t-1} = \tilde{\gamma}_t S_{t-1} + \gamma_t \quad (2.3)$$

with random coefficients $\tilde{\gamma}_t$ and γ_t . These coefficients describe the fluctuations in the proportion between fundamentalist and noise traders. When noise trading predominates, $\tilde{\gamma}_t$ becomes negative and the price process transient. Asset prices behave in a stable manner when the majority of the agents adopts a fundamentalist benchmark.

2.1.1 Random environment driven by interactive Markov processes

Let us now discuss a possible source of randomness driving the environment for the evolution of stock prices. Extending an earlier approach in [37], Horst [53] analyzes a situation with countably many agents located on some integer lattice \mathbb{A} where the environment is generated by an underlying Markov chain with an interactive dynamics. There is a set C of possible characteristics or trading strategies. An agent's state $x_t^a \in C$ specifies her reference level for the following period. The environment is then driven by a Markov chain

$$\Pi(x_t; \cdot) = \prod_{a \in \mathbb{A}} \pi_a(x_t; \cdot) \quad (2.4)$$

where $x_t = (x_t^a)_{a \in \mathbb{A}}$ denotes the current configuration of reference levels. The distribution $\pi_a(x_t; \cdot)$ of an agent's state in the following period may depend both on the current states of some "neighbors"

and signals about the aggregate behavior. Information about aggregate behavior is carried in the empirical distribution $\varrho(x_t)$ or, more generally, the empirical field $R(x_t)$ associated to the configuration x_t . The empirical field is defined as the weak limit

$$R(x_t) := \lim_{n \rightarrow \infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} \delta_{\theta^a x_t}(\cdot)$$

along an increasing sequence of finite sub-populations $\mathbb{A}_n \uparrow \mathbb{A}$ and $(\theta^a)_{a \in \mathbb{A}}$ denotes the canonical shift group on the space of all configurations. Due to the dependence of the transition probabilities $\pi_a(x; \cdot)$ on aggregate behavior, the kernel Π does not have the Feller property, and so standard convergence results for Feller processes on compact state spaces do not apply. As shown by Föllmer and Horst [38] and Horst [50] the evolution of aggregate behavior on the level of empirical fields can be described by a Markov chain. In [53], it is the $\{R(x_t)\}_{t \in \mathbb{N}}$ process that generates the environment:

$$(\tilde{\gamma}_t, \gamma_t) \sim Z(R(x_t); \cdot) \quad \text{for a suitable stochastic kernel } Z.$$

Under a weak interaction condition the process $\{R(x_t)\}_{t \in \mathbb{N}}$ settles down to a unique limiting distribution. Hence asset prices asymptotically evolve in a stationary and ergodic random environment. This allows us to approximate the discrete time process $\{S_t\}_{t \in \mathbb{N}}$ by the unique strong solution to the stochastic differential equation

$$dZ_t = Z_t dX_t + d\tilde{X}_t$$

where X and \tilde{X} are Brownian motions with drift and volatility; see [40] or [53] for details.

2.1.2 Feedback effects

The random environment in [53] is generated by a Markov process describing the evolution of individual behavior. While this approach is capable of capturing some interaction and imitation effects such as word-of-mouth advertising unrelated to market events, the dynamics of $\{x_t\}_{t \in \mathbb{N}}$ lacks a dependence on asset price dynamics. The model by Föllmer, Horst, and Kirman [39] captures feedback effects from stock prices into the environment. At the same time it allows for trend chasing. A trend chaser or chartist bases his expectation of future asset prices and hence his trading strategy upon observed historical price patterns such as trends. In [39], for instance, the chartist's benchmark level takes the form

$$\log \hat{S}_t^a := S_{t-1} + \beta_t^a (S_{t-1} - S_{t-2}). \quad (2.5)$$

A combination of the trading strategies (2.2) and (2.5) yields a class of asset price processes that can be described by a higher order stochastic difference equation. In [39], the agents use one of a number of predictors which they obtain from financial “gurus” to forecast future price movements. The agents evaluate the gurus' performance over time. Performances are measured by weighted sums of past profits the strategies generate. The probability of choosing a given guru is related to the guru's success. As a result, the configuration x_t of individual choices at time t is a random variable whose distribution depends on the current vector of performance levels U_{t-1} . This dependence

of the agents' choices on performances introduces a feedback from past prices into the random environment. Loosely speaking one obtains a difference equation of the form (2.3) where

$$(\tilde{\gamma}_t, \gamma_t) \sim Z(U_t; \cdot) \quad \text{for a suitable stochastic kernel } Z.$$

While prices can temporarily deviate from fundamental values, the main result in [39] shows that the price process has a unique stationary distribution, and time averages converge to their expected value under the stationary measure if the impact of trend chasing is weak enough.

2.1.3 Multiplicity of equilibria

As argued by Kirman [58], in a random economy with many heterogenous agents, a natural idea of an equilibrium is not a particular state, but rather a distribution of states reflecting the proportion of time the economy spends in each of the states. In the context of microstructure models where liquidity trading or interaction effects prevent asset prices from converging pathwise to some steady state, stationary distributions for asset prices are thus a natural notion of equilibrium. In this sense, the main result in [39] may be viewed as an existence and uniqueness result for equilibria in financial markets with heterogenous agents. Horst and Wenzelburger [54] study a related model with many small investors where performances are evaluated according historic returns or Sharpe ratios. In the limit of an infinite set of agents the dynamics of asset prices can be described by a path dependent linear stochastic difference equation of the form

$$Y_t = A(\varrho_{t-1})Y_{t-1} + B(\varrho_{t-1}, \epsilon_t).$$

Here $\{\epsilon_t\}_{t \in \mathbb{N}}$ is an exogenous i.i.d sequence of noise trader demand and ϱ_{t-1} denotes the empirical distribution of the random vector Y_0, Y_1, \dots, Y_{t-1} . While the model shares many of the qualitative features of [53] and [39], it allows for multiple limiting distributions of asset prices. If the interaction between different agents is strong enough, asset prices converge in distribution to a *random* limiting measure. Randomness in the limiting distribution may be viewed as a form of market incompleteness generated by contagious interaction effects.

2.1.4 Interacting agent models in an overlapping generations framework

The work in [54] is based on earlier work by Böhm *et al.* [12], Böhm and Wenzelburger [13], and Wenzelburger [82]. These authors developed a dynamic analysis of endogenous asset price formations in the context of overlapping generations economies where agents live for two periods and the demand for the risky asset comes from young households. They investigate the impact of different forecasting rules on both asset price and wealth dynamics under the assumption that agents are myopic and therefore boundedly rational, mean-variance maximizers. Böhm *et al.* [12] study asset prices and equity premia for a parameterized class of examples and investigate the role of risk aversion and of subjective as well as rational beliefs. It is argued that realistic parameter values explain Mehra and Prescott's equity premium puzzle ([70]). The model is generalized in [82] to a model with an arbitrary number of risky assets and heterogeneous beliefs, thus generalizing the classical CAPM. A major result is conditions under which a learning scheme converges to rational

expectations for one investor while other investors have non-rational beliefs. A second major result is the notion of a *modified market portfolio* along with a generalization of the security market line result stating that in a world of heterogeneous myopic investors, modified market portfolios are *mean-variance efficient in the classical sense of CAPM*, regardless of the diversity of beliefs of other agents. See [11] for a related approach.

2.1.5 Feedback Effects from Program Trading, Large Agents and Illiquidity

A different type of feedback effect, from the actions of a large group of *program traders* or large influential agents has been modelled in the financial mathematics literature. In the 1990s, following the Brady report that attributed part of the cause of the 1987 stock market crash to program trading by institutions following portfolio insurance strategies, researchers analyzed the feedback effect from option Delta-hedging by a significant fraction of market participants on the price dynamics of the underlying security. See, for example, Frey and Stremme [43], Sircar and Papanicolaou [78], Schönbucher and Wilmott [76] and Platen and Schweizer [73].

Related analyses can be found in models where there is a large investor whose actions move the price, for example Jonsson and Keppo [55], and where there is a market depth function describing the impact of order size on price, for example Cetin *et al.* [18]. A cautionary note on all such models is that, under sensible conditions, they do not explain the implied volatility smile/skew that is observed in modern options markets (in fact they predict a reverse smile). This would suggest that program trading, large agent or illiquidity effects are second order phenomena as far as derivatives markets are concerned, compared with the impacts of jumps or stochastic volatility.

There has also been some recent empirical work on estimating the market depth function, in particular the tail of the distribution governing how order size impacts trading price: see Farmer and Lillo [35] and Gabaix *et al.* [44].

2.2 Stock Prices and Random Dynamical Systems

An important branch of the literature on agent-based financial market models analyzes financial markets in which the dynamics of asset prices can be described by a deterministic dynamical system. The idea is to view agent-based models as highly nonlinear deterministic dynamical systems and markets as *complex adaptive systems*, with the evolution of expectations and trading strategies coupled to market dynamics. Many such models, when simulated, generate time paths of prices which switch from one expectations regime to another generating *rational routes to randomness*, i.e., chaotic price fluctuations. As these models are considerably more complex than the ones reviewed in the previous section, analytical characterizations of asset price processes are typically not available. However, when simulated, these model generate much more realistic time paths of prices explaining many of the stylized facts observed in real financial markets.

Particularly relevant contributions include the early work of Day and Huang [27], Frankel and Froot [41] and the work of Brock and Hommes [14]. The latter studies a model in which boundedly rational agents can use one of two forecasting rules or investment strategies. One of them is costly but when all agents use it, the emerging price process is stable. The other is cheaper but when used by many individuals induces unstable behavior of the price process. Their model has periods

of stability interspersed with bubble like behavior. In [15] the same authors introduced the notion of *Adaptive Belief Systems* (ABS), a “financial market application of the evolutionary selection of expectation rules” analyzed in [14]. An ABS may be viewed as asset pricing models derived from mean-variance optimization with heterogeneous beliefs. As pointed out in [49], “a convenient feature of an ABS is that it can be formulated in terms of (price) deviations from a benchmark fundamental and (...) can therefore be used in experimental and empirical testing of deviations from the (rational expectations) benchmark.” Recently, several modifications of ABSs have been studied. While in [15] the demand for a risky asset comes from agents with constant absolute risk aversion utility functions and the number of trader types is small, Chiarella and He [21] and Brock, Hommes, and Wagener [16] developed models of interaction of portfolio decisions and wealth dynamics with heterogeneous agents whose preferences are described by logarithmic CRRA utility functions and many types of traders, respectively. Gaunersdorfer [46] extends the work in [14] to the case of time-varying expectations about variances of conditional stock returns².

2.3 Queuing Models and Order Book Dynamics

The aforementioned models differ considerably in their degree of complexity and analytical tractability, but they are all based on the idea that asset price fluctuations can be described by a sequence of temporary price equilibria. All agents submit their demand schedule to a market maker who matches individual demands in such a way that markets clear. While such an approach is consistent with dynamic microeconomic theory, it should only be viewed as a first steps towards a more realistic modelling of asset price formation in large financial markets. In real markets, buying and selling orders arrive at different points in time, and so the economic paradigm that a Walrasian auctioneer can set prices such that the markets clear at the end of each trading period typically does not apply. In fact, almost all automated financial trading systems function as continuous double auctions. They are based on electronic *order books* in which all unexecuted limit orders are stored and displayed while awaiting execution. While analytically tractable models of order book dynamics would be of considerable value, their development has been hindered by the inherent complexity of limit order markets. So far, rigorous mathematical results have only been established under rather restrictive assumptions on aggregate order flows by, e.g., Mendelson [71], Luckock [62] and Kruk [60]. Statistical properties of continuous double auctions are often analyzed in the econophysics literature e.g., Smith *et al.* [79] and references therein.

Microstructure models with asynchronous order arrivals where orders are executed immediately rather than awaiting the arrival of a matching order and where asset prices move into the order to market imbalance are studied by, e.g. Garman [45]; Lux [63, 65, 64] or Bayraktar *et al.* ([5]). These models may be viewed as an intermediate step towards a more realistic modeling of electronic trading systems.

A convenient mathematical framework for such models, which we will develop in detail in Section 3.2, is based on the theory of state-dependent queuing networks (see [68] or [69] for detailed discussions of Markovian queuing networks). Underlying this approach is the idea that the dynamics

²There are many other papers utilizing dynamical system theory to analyze asset price dynamics in behavioral finance models. For a detailed survey, we refer the interested reader to [49].

of order arrivals follows a Poisson-type process with price dependent rates and that a buying (selling) order increases (decreases) the stock price by a fixed amount (on a possibly logarithmic scale to avoid negative prices).

More precisely, the arrival times of aggregate buying and selling orders are specified by independent Poisson processes Π_+ and Π_- with price and time dependent rates λ_+ and λ_- , respectively, that may also depend on investor characteristics or random economic fundamentals. In the simplest case the logarithmic price process $\{S_t\}_{t \geq 0}$ takes the form

$$S_t = S_0 + \Pi_+ \left(\int_0^t \lambda_+(S_u, u) du \right) - \Pi_- \left(\int_0^t \lambda_-(S_u, u) du \right).$$

The excess order rate $\lambda_+(S_u, u) - \lambda_-(S_u, u)$ may be viewed as a measure of aggregate excess demand while $\Pi_+ \left(\int_0^t \lambda_+(S_u, u) du \right) - \Pi_- \left(\int_0^t \lambda_-(S_u, u) du \right)$ denotes the accumulated net order flow up to time t . In a model with many agents and after suitable rescaling the asset price process may be approximated by a deterministic process while the fluctuations around this first order approximation can typically be described by an Ornstein-Uhlenbeck diffusion.

Recently, such queuing models have also been applied to modeling the credit risk of large portfolios by Davis and Esparragoza [26]. They approximate evolution of the loss distribution of a large portfolio of credit instruments over time. We further elaborate on queuing theoretic approaches to stock price dynamics in Section 3. Before that, we introduce a common investor trait, investor inertia, and show the effects of this common trait on stock prices.

2.4 Inertia in Financial Markets

The models mentioned previously assume that agents trade the asset in each period. At the end of each trading interval, agents update their expectations for the future evolution of the stock price and formulate their excess demand for the following period. However, small investors are not so efficient in their investment decisions: they are typically inactive and actually trade only occasionally. This may be because they are waiting to accumulate sufficient capital to make further stock purchases; or they tend to monitor their portfolios infrequently; or they are simply scared of choosing the wrong investments; or they feel that as long-term investors, they can defer action; or they put off the time-consuming research necessary to make informed portfolio choices. Long uninterrupted periods of inactivity may be viewed as a form of investor inertia.

2.4.1 Evidence of inertia

Investor inertia is a common experience and is well documented. The New York Stock Exchange (NYSE)'s survey of individual shareownership in the United States, "Shareownership2000" [33], demonstrates that many investors have very low levels of trading activity. For example they find that "23 percent of stockholders with brokerage accounts report no trading at all, while 35 percent report trading only once or twice in the last year". The NYSE survey also reports (Table 28) that the average holding period for stocks is long, for example 2.9 years in the early 90's. Empirical evidence of inertia also appears in the economic literature. For example, Madrian and Shea [67] looked at the reallocation of assets in employees' individual 401(k) (retirement) plans and found

“a status quo bias resulting from employee procrastination in making or implementing an optimal savings decision.” A related study by Hewitt Associates (a management consulting firm) found that in 2001, four out of five plan participants did not do any trading in their 401(k)s. Madrian and Shea explain that “if the cost of gathering and evaluating the information needed to make a 401(k) savings decision exceeds the short-run benefit from doing so, individuals will procrastinate.” The prediction of Prospect Theory (see [56]) that investors tend to hold onto losing stocks too long has also been observed in [77]. Another typical cause is that small investors seem to find it difficult to reverse investment decisions, as is discussed even in the popular press. A recent newspaper column (by Russ Wiles in the Arizona Republic, November 30, 2003) states: “Perhaps more than anything, investor inertia is a key force (in financial markets). When the news turns sour, people tend to hold off on buying rather than bail out. In 2002, the toughest market climate in a generation and a year with ample Wall Street scandals, equity funds suffered cash outflows of just one percent.”

2.4.2 Inertia and long range dependencies in financial time series

One of the outcomes of a limit analysis of an agent-based model of investor inertia is a stock price process based on fractional Brownian motion, which exhibits long-range dependence (that is correlation or memory in returns). This is discussed in Section 3.1. In particular, the limit fluctuation process is a *fractional Brownian motion*.

We recall that fractional Brownian motion B^H with Hurst parameter $H \in (0, 1]$ is an almost surely continuous and centered Gaussian process with auto-correlation

$$\mathbb{E} \{ B_t^H B_s^H \} = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}). \quad (2.6)$$

Remark 2.1 *Note that the case $H = \frac{1}{2}$ gives standard Brownian motion. Also note that the auto-correlation function is positive definite if and only if $H \in (0, 1]$.*

Bayraktar *et al.* [8] studied an asymptotically efficient wavelet-based estimator for the Hurst parameter, and analyzed high frequency S&P 500 index data over the span of 11.5 years (1989-2000). It was observed that, although the Hurst parameter was significantly higher than the efficient markets value of $H = \frac{1}{2}$ up through the mid-1990s, it started to fall to that level over the period 1997-2000 (see Figure 1). This might be explained by the increase in Internet trading in that period, which is documented, for example, in NYSE’s “Shareownership2000” [33], Barber and Odean [2], and Choi *et al.* [22], in which it is demonstrated that “after 18 months of access, the Web effect is very large: trading frequency doubles.” Indeed, as reported in [3], “after going online, investors trade more actively, more speculatively and less profitably than before”. Similar empirical findings to that of [8] were recently reached, using a completely different statistical technique by Bianchi [9].

Thus, the dramatic fall in the estimated Hurst parameter in the late 1990s can be thought of as a *posteriori* validation of the link the limit theorem in [5] provides between investor inertia and long-range dependence in stock prices. We review this model in Section 3.1. An extension based on state dependent queuing networks with semi-Markov switching is discussed in Section 3.2.

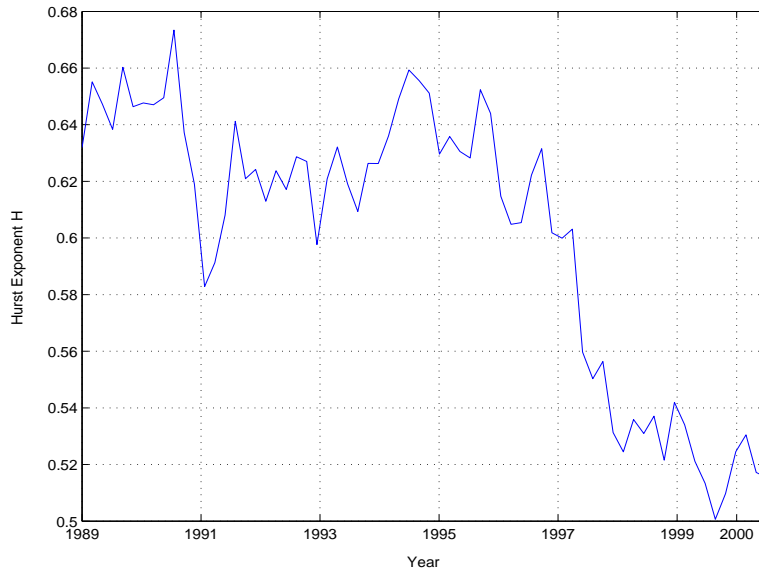


Figure 1: *Estimates of the Hurst exponent of the S&P 500 index over 1990s, taken from Bayraktar, Poor and Sircar [8].*

3 Microstructure Models with Inert Investors

We illustrate the use of microstructure, or agent-based models, combined with limit theorems by focusing on investor inertia as a very common characteristic among small and casual market participants. In Section 3.1 we summarize earlier work [5] that established a mathematical link between inertia, long-range dependence in stock returns and potential short-lived arbitrage opportunities for other ‘sophisticated’ parties. Section 3.2 contains an extension allowing for feedback effects from current prices into the agents’ order rates.

3.1 A Microstructure Model without Feedback

We now introduce the basic concepts and notation of the market microstructure model analyzed in [5] that will serve as basis for the more sophisticated model in Section 3.2. We start with a financial market with a set $\mathbb{A} := \{a_1, a_2, \dots, a_N\}$ of *agents* trading a single risky asset. Each agent $a \in \mathbb{A}$ is associated with a continuous-time stochastic process $x^a = \{x_t^a\}_{t \geq 0}$ on a finite state space E describing his *trading activity*.

We take a pragmatic approach to specify the demand. Instead of formulating an individual optimization problem under budget constraints for the agents, we start right away with the agent’s order rates. The agent $a \in \mathbb{A}$ accumulates the asset at a rate $\Psi_t x_t^a$ at time $t \geq 0$. Here x_t^a may be negative indicating that the agent is selling. The random process $\Psi = \{\Psi_t\}_{t \geq 0}$ describes the evolution of the size of a typical trade. It can also be interpreted as a stochastic elasticity coefficient (the reaction of the price to the market imbalance). We assume that Ψ is a continuous non-negative semi-martingale which is independent of the processes x^a and that $0 \in E$. The agents do not trade at times when $x_t^a = 0$. The holdings of the agent $a \in \mathbb{A}$ and the “market imbalance” at time $t \geq 0$

are thus given by, respectively,

$$\int_0^t \Psi_s x_s^a ds \quad \text{and} \quad \sum_{a \in \mathbb{A}} \int_0^t \Psi_s x_s^a ds. \quad (3.1)$$

Remark 3.1 *In our continuous time model, buyers and sellers arrive at different points in time. Hence the economic paradigm that a Walrasian auctioneer can set prices such that the markets clear at the end of each trading period does not apply. Rather, temporary imbalances between demand and supply will occur. Prices will reflect the extent of market imbalance.*

All the orders are received by a single market maker. The market maker clears all trades and prices in reaction to the evolution of market imbalances, the only component driving asset prices. Reflecting the idea that an individual agent has diminishing impact on market dynamics if the number of traders is large, we assume that the impact of an individual order is inversely proportional to the number of possible traders: a buying (selling) order increases (decreases) the logarithmic stock price by $1/N$. The pricing rule for the evolution of the logarithmic stock price process $S^N = \{S_t^N\}_{t \geq 0}$ is linear and taken to be:

$$dS_t^N = \frac{1}{N} \sum_{a \in \mathbb{A}} \Psi_t x_t^a dt. \quad (3.2)$$

In order to incorporate the idea of market inertia, the agents' trading activity is modelled by independent and identically distributed *semi-Markov* processes x^a . Semi-Markov processes are tailor-made to model individual traders' inertia as they generalize Markov processes by removing the requirement of exponentially distributed, and therefore thin-tailed, holding (or sojourn) times. Since the processes x^a are independent and identically distributed, it is enough to specify the dynamics of some "representative" process $x = \{x\}_{t \geq 0}$.

3.1.1 Semi-Markov Processes

A semi-Markov process x defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is specified in terms of random variables $\xi_n : \Omega \rightarrow E$ and $T_n : \Omega \rightarrow \mathbb{R}_+$, satisfying $0 = T_0 \leq T_1 \leq \dots$ almost surely and

$$\mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t | \xi_1, \dots, \xi_n; T_1, \dots, T_n\} = \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t | \xi_n\}$$

for each $n \in \mathbb{N}$, $j \in E$ and all $t \in \mathbb{R}_+$, through the relation

$$x_t = \sum_{n \geq 0} \xi_n \mathbf{1}_{[T_n, T_{n+1})}(t). \quad (3.3)$$

In economic terms, the representative agent's mood in the random time interval $[T_n, T_{n+1})$ is given by ξ_n . The distribution of the length of the interval $T_{n+1} - T_n$ may depend on the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ through the states ξ_n and ξ_{n+1} . This allows us to assume different distributions for the lengths of the agents' active and inactive periods, and in particular to model inertia as a heavy-tailed sojourn time in the zero state.

Remark 3.2 *In the present analysis of investor inertia, we do not allow for feedback effects of prices into agents' investment decisions. While such an assumption might be justified for small, non-professional investors, it is clearly desirable to allow active traders' investment decisions to be influenced by asset prices. We discuss such an extension in the next section.*

We assume that x is temporally homogeneous under the measure \mathbb{P} , that is,

$$Q(i, j, t) \triangleq \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t | \xi_n = i\} \quad (3.4)$$

is independent of $n \in \mathbb{N}$. By Proposition 1.6 in [17], this implies that $\{\xi_n\}_{n \in \mathbb{N}}$ is a homogeneous Markov chain on E whose transition probability matrix (p_{ij}) is given by

$$p_{ij} = \lim_{t \rightarrow \infty} Q(i, j, t).$$

Clearly, x is an ordinary temporally homogeneous Markov process if Q takes the form

$$Q(i, j, t) = p_{ij} (1 - e^{-\lambda_i t}). \quad (3.5)$$

We also assume that the *embedded Markov chain* $\{\xi_n\}_{n \in \mathbb{N}}$ satisfies $p_{ij} > 0$ so that $\{\xi_n\}_{n \in \mathbb{N}}$ has a unique stationary distribution. The conditional distribution function of the length of the n -th sojourn time, $T_{n+1} - T_n$, given ξ_{n+1} and ξ_n is specified in terms of the *semi-Markov kernel* $\{Q(i, j, t); i, j \in E, t \geq 0\}$ and the transition matrix P by

$$G(i, j, t) := \frac{Q(i, j, t)}{p_{ij}} = \mathbb{P}\{T_{n+1} - T_n \leq t | \xi_n = i, \xi_{n+1} = j\}. \quad (3.6)$$

The semi-Markov processes are assumed to satisfy the following conditions.

Assumption 3.3 (i) *The average sojourn time at state $i \in E$ is finite:*

$$m_i := \mathbb{E}[T_{n+1} - T_n | \xi_n = i] < \infty. \quad (3.7)$$

Here \mathbb{E} denotes the expectation operator with respect to \mathbb{P} .

(ii) *There exists a constant $1 < \alpha < 2$ and a locally bounded function $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is slowly varying at infinity (e.g. \log), i.e.,*

$$\lim_{t \rightarrow \infty} \frac{L(xt)}{L(t)} = 1 \quad \text{for all } x > 0,$$

such that

$$\mathbb{P}\{T_{n+1} - T_n \geq t | \xi_n = 0\} \sim t^{-\alpha} L(t). \quad (3.8)$$

Here we use the notation $f(t) \sim g(t)$ for two functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to mean that $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$.

(iii) *The distributions of the sojourn times at state $i \neq 0$ satisfy*

$$\lim_{t \rightarrow 0} \frac{\mathbb{P}\{T_{n+1} - T_n \geq t | \xi_n = i\}}{t^{-(\alpha+1)} L(t)} = 0. \quad (3.9)$$

(iv) *The distribution of the sojourn times in the various states have continuous and bounded densities with respect to Lebesgue measure on \mathbb{R}_+ .*

The key parameter is the tail index α of the sojourn time distribution of the inactive state zero. Condition (3.8) is satisfied if, for instance, the length of the sojourn time at state $0 \in E$ is distributed according to a Pareto distribution. The idea of inertia is then reflected by (3.9): the probability of long uninterrupted trading periods is small compared to the probability of an individual agent being inactive for a long time. In fact, it is natural to think of the sojourn times in the various active states as being thin tailed as in the exponential distribution since small investors typically do not trade persistently.

3.1.2 A Limit Theorem for Financial Markets with Inert Investors

We assume that the semi-Markov processes x^a are stationary. Stationarity can be achieved by a suitable specification of the common distribution of the initial states and initial sojourn times. We denote the resulting measure on the canonical path space by \mathbb{P}^* . Independence and stationarity of the semi-Markov processes guarantees that the logarithmic price process can be approximated pathwise by the process $\{s_t\}_{t \geq 0}$ defined by

$$s_t = \mu \int_0^t \Psi_s ds \quad \text{where} \quad \mu := \mathbb{E}^* x_0^a$$

when the number of agents grows to infinity. Our functional central limit theorem for stationary semi-Markov processes shows that after suitable scaling, the fluctuations around $(s_t)_{t \geq 0}$ can be approximated in law by a process with long range dependence. The convergence concept we shall use is weak convergence with respect to the measure \mathbb{P}^* of the Skorohod space \mathbb{D} of all right continuous processes. We write $\mathcal{L}\text{-}\lim_{n \rightarrow \infty} Y^n = Y$ if $\{Y^n\}_{n \in \mathbb{N}}$ is a sequence of \mathbb{D} -valued stochastic processes that converges weakly to the process Y .

The convergence result is formulated in terms of a scaling limit for the processes $\{x_{Tt}^a\}_{t \geq 0}$ ($T \in \mathbb{N}$). For T large, x_{Tt}^a is a “speeded-up” semi-Markov process. In other words, the investors’ individual trading dispensations are evolving on a faster scale than Ψ . Observe, however, that we are not altering the main qualitative feature of the model: agents still remain in the inactive state for relatively much longer times than in an active state. In the rescaled model the logarithmic asset price process $S^{N,T}$ is given by

$$S_t^{N,T} = \frac{1}{N} \int_0^t \sum_{a \in \mathbb{A}} \Psi_u x_{Tu}^a du. \quad (3.10)$$

The central limit theorem allows us to approximate the fluctuations around the first order approximation as $N \rightarrow \infty$. In terms of the Gaussian processes X^T and Y^T defined by

$$X_t^T \triangleq \mathcal{L}\text{-}\lim_{N \rightarrow \infty} T^{1-H} \frac{1}{\sqrt{N}} \sum_{a=1}^N (x_{Tt}^a - \mu t) \quad \text{and} \quad Y_t^T \triangleq \int_0^t X_s^T ds, \quad (3.11)$$

with $H = (3 - \alpha)/2$, the fluctuations around the first order approximation can be approximated by an integral of the elasticity coefficient with respect to Y^T :

$$\mathcal{L}\text{-}\lim_{N \rightarrow \infty} \sqrt{N} \left\{ S_t^{N,T} - \mu t \right\}_{0 \leq t \leq 1} = \left\{ \int_0^t \Psi_s dY_s^T \right\}_{0 \leq t \leq 1}.$$

In order to see more clearly the effects of investor inertia, we rescale the price process in space and time and T tend to infinity. In a benchmark model with many agents where $\Psi \equiv 1$ these, fluctuations when suitably normalized, can be described by a fractional Brownian motion B^H if $T \rightarrow \infty$. The Hurst coefficient is related to the degree of investor inertia.

Theorem 3.4 ([5]) *Let $H = \frac{3-\alpha}{2}$. Assume that $\Psi \equiv 1$, that Assumption 3.3 holds and that $\mu \neq 0$. Then there exists $\sigma > 0$ such that*

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} T^{1-H} \frac{\sqrt{N}}{\sqrt{L(T)}} \left\{ S_t^{N,T} - \mu t \right\}_{0 \leq t \leq 1} = \left\{ \sigma B_t^H \right\}_{0 \leq t \leq 1} \quad (3.12)$$

To generalize this result to a market in which the agents' order rates are coupled by a stochastic elasticity coefficient as in (3.2), we need the following approximation result for stochastic integrals of continuous semi-martingales with respect to fractional Brownian motion.

Theorem 3.5 ([5]) *Let $\{\Psi^n\}_{n \in \mathbb{N}}$ be a sequence of good semimartingales and $\{Z^n\}_{n \in \mathbb{N}}$ be a sequence of \mathbb{D} -valued stochastic processes that satisfy*

- (i) *The sample paths of the processes Z^n are almost surely of zero quadratic variation on compact sets, and $\mathbb{P}\{Z_0^n = 0\} = 1$.*
- (ii) *The stochastic integrals $\int \Psi^n dZ^n$ and $\int Z^n dZ^n$ exist as limits in probability of Stieltjes-sums, and the sample paths $t \mapsto \int_0^t Z_s^n dZ_s^n$ and $t \mapsto \int_0^t \Psi_s^n dZ_s^n$ are càdlàg.*

If Ψ is a continuous semimartingale and if B^H is a fractional Brownian motion process with Hurst parameter $H > \frac{1}{2}$, then the convergence $\mathcal{L}\text{-}\lim_{n \rightarrow \infty} (\Psi^n, Z^n) = (\Psi, B^H)$ implies the convergence

$$\mathcal{L}\text{-}\lim_{n \rightarrow \infty} \left(\Psi^n, Z^n, \int \Psi^n dZ^n \right) = \left(\Psi, B^H, \int \Psi dB^H \right).$$

As an immediate corollary to Theorem 3.5 we see that the fluctuations of the price process 3.10 around its first order approximation converge in distribution to a stochastic integral with respect to fractional Brownian motion.

Corollary 3.6 *Let Ψ be a continuous semi martingale with Doob-Meyer decomposition $\Psi = M + A$. If $\mathbb{E}\{[M, M]_T\} < \infty$, $\mathbb{E}\{|A|_T\} < \infty$ and $\mu \neq 0$, then there exists $\sigma > 0$ such that*

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} T^{1-H} \frac{\sqrt{N}}{\sqrt{L(T)}} \left\{ S_t^{N,T} - \mu \int_0^t \Psi_s ds \right\}_{0 \leq t \leq 1} = \left\{ \sigma \int_0^t \Psi_s dB_s^H \right\}_{0 \leq t \leq 1}. \quad (3.13)$$

The increments of a fractional Brownian motion with Hurst coefficient $H \in (\frac{1}{2}, 1]$ are positively correlated. The correlation increases in H . Thus, the limit theorem reveals that, in isolation, investor inertia may lead to long range dependence in asset returns. Indeed, a greater degree of inactivity, represented by a smaller tail index α , leads to a larger H , and so greater positive correlation between returns. Since fractional Brownian motion is not a semimartingale, it may also lead to arbitrage opportunities for other traders whose impact has not been considered in the model so far. Explicit arbitrage strategies for various models were constructed in, e.g. [6].

Remark 3.7 *In a model without inertia where all the sojourn time distributions are thin-tailed, the logarithmic stock price fluctuations can be approximated in law by a process of the form*

$$\left\{ \int_0^t \Psi_s dW_s \right\}_{0 \leq t \leq 1} \quad (3.14)$$

where W is a standard Brownian motion. Thus, when all traders' mood processes are standard Markov processes and Ψ is constant, we recover in the limit the standard Black-Scholes-Samuelson geometric Brownian motion model.

The approach of studying queuing systems through their limiting behaviour has a long history in many applications, see [83], for example. This analysis of investor inertia built upon the works of Taqqu *et al.* [80] on internet traffic. However, even the simple model we have discussed so far shows how economic applications lead to new mathematical challenges: in the teletraffic application, it is sufficient to consider a binary (on/off) state space, but when agents buy, sell or do nothing, there must be at least three states. This requires different techniques from the binary case. Our functional central limit theorems for stationary semi-Markov processes may also serve as a mathematical basis for proving heavy-traffic limits in the multilevel network models studied in, e.g. [30] and [29].

3.2 A Limit Theorem with Feedback Effects

The model in the previous section assumes that investors' actions affect the price, but prices did not affect the agents' demands. This assumption might be justified for Internet or new economy stocks where no accurate information about the actual underlying fundamental value is available. In such a situation, price is not always a good indicator of value and is often ignored by uninformed small investors. In general, however, it is certainly desirable to allow for feedback effects from current prices into the agents' order rates. In this section we extend our previous model to allow for feedback effects from prices into the agents' order rates. At the same time we provide a unified mathematical framework for analyzing microstructure models with asynchronous order arrivals. Our approach is based on methods and techniques from state dependent Markovian service networks. Mathematically, it extends earlier results in [1] beyond semi-Markov models with thin-tailed sojourn time distributions.

3.2.1 The dynamics of logarithmic asset prices

Let us now be more precise about the probabilistic structure our model. We assume that the agents' orders arrive with an order rate that depends on the price and the investor sentiment. Each order is good for one unit of the stock. Specifically, we associate to each agent $a \in \mathbb{A}$ two independent standard Poisson processes $\{\Pi_+^a(t)\}_{t \geq 0}$ and $\{\Pi_-^a(t)\}_{t \geq 0}$, a stationary semi-Markov process x^a on E satisfying Assumption 3.3, and bounded Lipschitz continuous *rate functions* $\lambda_{\pm} : E \times \mathbb{R} \rightarrow \mathbb{R}^+$. The rate functions along with the Poisson processes Π_{\pm}^a specify the arrivals times of buying and selling orders. The agent's holdings at time $t \geq 0$ are given by

$$\Pi_+^a \left(\int_0^t \lambda_+(x_u^a, S_u^N) du \right) - \Pi_-^a \left(\int_0^t \lambda_-(x_u^a, S_u^N) du \right) \quad (3.15)$$

where $\{S_t^N\}_{t \geq 0}$ denotes the logarithmic asset price process. As before, a buying (selling) order increases (decreases) the logarithmic price by $1/N$. Assuming for simplicity that $S_0^N = 0$, we thus obtain

$$S_t^N = \frac{1}{N} \sum_{a \in \mathbb{A}} \Pi_+^a \left(\int_0^t \lambda_+(x_u^a, S_u^N) du \right) - \frac{1}{N} \sum_{a \in \mathbb{A}} \Pi_-^a \left(\int_0^t \lambda_-(x_u^a, S_u^N) du \right). \quad (3.16)$$

Remark 3.8 (i) *In the model studied in the previous section, the agents continuously accumulated the stock at rates specified by semi-Markov processes. Our current models assume that stocks are purchased at random points in times. The arrival times of buying and selling times follow exponential distributions conditional on random arrival rates that depend on current prices and exogenous semi-Markov processes.*

(i) *As before, we think of x^a as being the investor's "mood" (for trading) process. Loosely speaking, $\lambda_+(x_t^a, s) - \lambda_-(x_t^a, s)$ may be viewed as the agent's excess demand at time t at a logarithmic price level s , given his trading mood x_t^a .*

(ii) *To develop a model of interaction, in which the participants are inert, out of (3.15), it is natural to assume that $\lambda_{\pm}(0, s) \equiv 0$ and that the buying and selling rates $\lambda_+(x, \cdot)$ and $\lambda_-(x, \cdot)$ are increasing, rep. decreasing, in the second variable meaning that meaning high (low) prices temper buying (selling) rates.*

The sum of independent Poisson processes is a Poisson process with intensity given by the sum of the intensities. As a result, the logarithmic price process satisfies the equality

$$S_t^N = \frac{1}{N} \Pi_+ \left(\sum_{a=1}^N \int_0^t \lambda_+(x_u^a, S_u^N) du \right) - \frac{1}{N} \Pi_- \left(\sum_{a=1}^N \int_0^t \lambda_-(x_u^a, S_u^N) du \right) \quad (3.17)$$

in *distribution* where Π_+ and Π_- are independent standard Poisson processes. Since our focus will be on a limit result for the *distribution* of the price process as the number of agents grows to infinity, we may with no loss of generality assume that the logarithmic price process is *defined* by (3.17) rather than (3.16).

Assumption 3.9 1. *The rate functions λ_{\pm} are uniformly bounded.*

2. *For each $x \in E$, the rate functions $\lambda_{\pm}(x, \cdot)$ are continuously differentiable with first derivative bounded in absolute value by some constant L .*

Our convergence results will be based on the following strong approximation result which allows for a pathwise approximation of a Poisson process by a standard Brownian motion living on the same probability space.

Lemma 3.10 ([61]) *A standard Poisson process $\{\Pi(t)\}_{t \geq 0}$ can be realized on the same probability space as a standard Brownian motion $\{B(t)\}_{t \geq 0}$ in such a way that the almost surely finite random variable*

$$\sup_{t \geq 0} \frac{|\Pi(t) - t - B(t)|}{\log(2 \vee t)}$$

has a finite moment generating function in the neighborhood of the origin and in particular finite mean.

In view of Assumption 3.9 (i), the strong approximation result yields the following alternative representation of the logarithmic asset price process:

$$S_t^N = \frac{1}{N} \left\{ \sum_{a=1}^N \int_0^t \lambda(x_u^a, S_u^N) du + B_+ \left(\sum_{a=1}^N \int_0^t \lambda_+(x_u^a, S_u^N) du \right) - B_- \left(\sum_{a=1}^N \int_0^t \lambda_-(x_u^a, S_u^N) du \right) \right\} + \mathcal{O}\left(\frac{\log N}{N}\right), \quad (3.18)$$

where $\lambda(x_u^a, \cdot)$ denotes the excess order rate of the agent $a \in \mathbb{A}$, given his mood for trading x_u^a and $\mathcal{O}(\log N/N)$ holds uniformly over compact time intervals. Using this representation of the logarithmic price process our goal is to prove approximation results for the process $\{S_t^N\}_{t \geq 0}$. In a first step we show that it can almost surely be approximated by the trajectory of an ordinary differential equation (“fluid limit”). In subsequent step, we apply a result from [5] to show that, after suitable scaling, the fluctuations around this first order approximation can be described in terms of a fractional process $\{Z_t\}_{t \geq 0}$ of the form

$$dZ_t = \mu_t Z_t dt + \sigma_t dB_t^H.$$

In a benchmark model without feedback, where the order rates do not depend on current prices, the process $\{Z_t\}_{t \geq 0}$ reduces to a fractional Brownian motion. That is, we recover the type of results of Section 3.1.2 with the alternative model presented in this section.

3.2.2 First order approximation

In order to prove our first convergence result, it is convenient to denote by

$$\lambda(x, s) \triangleq \lambda_+(x, s) - \lambda_-(x, s) \quad (3.19)$$

the accumulated net order rate at a given logarithmic price level $s \in \mathbb{R}$ and trading mood $x \in E$ and by

$$\bar{\lambda}(s) \triangleq \bar{\lambda}_+(s) - \bar{\lambda}_-(s)$$

the expected excess order flow where

$$\bar{\lambda}_\pm(s) \triangleq \int_E \lambda_\pm(x, s) \nu(dx),$$

and ν is the stationary distribution of the semi-Markov process x_t . We are first going to show that in a financial market with many agents the dynamics of the logarithmic price process can be approximated by the solution $\{s_t\}_{t \geq 0}$ to the ODE

$$\frac{d}{dt} s_t = \bar{\lambda}(s_t), \quad (3.20)$$

with initial condition $s_0 = 0$. To this end, we need to prove that the average excess order rate converges almost surely to the expected excess order flow uniformly on compact time intervals.

Lemma 3.11 *Uniformly on compact time intervals*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{a=1}^N \int_0^t \lambda_{\pm}(x_u^a, s_u) du = \int_0^t \bar{\lambda}_{\pm}(s_u) du \quad \mathbb{P}^* - a.s. \quad (3.21)$$

PROOF: The stationary semi Markov processes x^a are independent, and so the random variables $\int_0^t \lambda(x_u^a, s_u) du$ ($a = 1, 2, \dots$) are also independent. Thus, the law of large numbers for independent random variables along with Fubini's theorem (to exchange the sum and the integral) and bounded convergence theorem (to exchange the limit and the integral) yields convergence for each t . In order to prove that the convergence holds uniformly over compact time intervals we will use uniform law of large numbers of [74]. Denoting $D_E[0, t]$ the class of all càdlàg functions $y : [0, t] \rightarrow E$ we need to show that the maps $q_{\pm} : D_E[0, t] \times [0, t] \rightarrow \mathbb{R}$ defined by

$$q_{\pm}(y, t) \triangleq \int_0^t \lambda_{\pm}(y(u), s_u) du$$

are continuous. Since the rate functions are bounded, it is enough to show that the map $y \mapsto \int_0^t \lambda_{\pm}(y(u), s_u) du$ is continuous uniformly over compact time intervals.

To this end, we denote by d the metric defined in (3.5.2) in [34] which induces the Skorohod topology in $D_E[0, t]$ and recall that $\lim_{n \rightarrow \infty} d(y_n, y) = 0$ if and only if

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} |y_n \circ \tau_n(s) - y(s)| = 0 \quad (3.22)$$

for a suitable sequence of strictly increasing time-shifts τ_n ; see [34] page 117 for details. Let $\{y_n\}$ denote a sequence in $D_E[0, t]$ that converges to y and put

$$\lambda_{\pm}^n(u) \triangleq \lambda_{\pm}(y_n(u), s_u).$$

In view of the transformation formula for Lebesgue integrals and because $\tau(0) = 0$ and $\tau_n^{-1}(t) \leq t$ we obtain

$$\begin{aligned} \int_0^t [\lambda_{\pm}^n(u) - \lambda_{\pm}(u)] du &= \int_0^{\tau_n^{-1}(t)} [\lambda_{\pm}^n \circ \tau_n(u) \tau_n'(u) - \lambda_{\pm}(u)] du - \int_{\tau_n^{-1}(t)}^t \lambda_{\pm}(u) du \\ &= \int_0^{\tau_n^{-1}(t)} [\lambda_{\pm}^n \circ \tau_n(u) - \lambda_{\pm}(u)] du \\ &\quad + \int_0^{\tau_n^{-1}(t)} \lambda_{\pm}^n \circ \tau_n(u) [\tau_n'(u) - 1] du - \int_{\tau_n^{-1}(t)}^t \lambda_{\pm}(u) du. \end{aligned}$$

By (3.5.5)-(3.5.7) in [34]

$$\lim_{n \rightarrow \infty} \sup_{0 \leq u \leq t} |\tau_n'(u) - 1| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{0 \leq u \leq t} |\tau_n^{-1}(u) - u| = 0$$

so that the last two terms on the right hand side of the inequality above vanish uniformly on compact time intervals. As far as the first term is concerned, observe that boundedness of the rate function's derivative with respect to the second argument yields

$$|\lambda_{\pm}(y_n \circ \tau_n(u), s_{\tau_n(u)}) - \lambda_{\pm}(y(u), s_u)| \leq L |y_n \circ \tau_n(u) - y(u)| + L |s \circ \tau_n(u) - s(u)|.$$

As a continuous function s is uniformly continuous over compact time intervals. This, along with (3.22) yields

$$\lim_{n \rightarrow \infty} \sup_{0 \leq u \leq t} |\lambda_{\pm}(y_n \circ \tau_n(u), s_{\tau_n(u)}) - \lambda_{\pm}(y(u), s_u)| = 0$$

so that the maps q_{\pm} are indeed continuous. Thus, the uniform law of large numbers yields

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{a=1}^N q_{\pm}(x_u^a, u) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{a=1}^N \int_0^u \lambda_{\pm}(x_v^a, s_v) dv = \lambda_{\pm}(\mu, s_u)$$

almost surely uniformly on compact time intervals. \square

We are now ready to state and prove our functional law of large numbers.

Theorem 3.12 *As $N \rightarrow \infty$, the sequence of stochastic processes $\{S_t^N\}_{t \geq 0}$ ($N \in \mathbb{N}$) converges almost surely to the deterministic process $\{s_t\}_{t \geq 0}$:*

$$\lim_{N \rightarrow \infty} S_t^N = s_t \quad \mathbb{P}^* \text{-a.s.}$$

where the convergence is uniform over compact time intervals.

PROOF: In view of the strong approximation result formulated in Lemma 3.10 and because the rate functions are uniformly bounded,

$$\left| \Pi_{\pm} \left(\sum_{a=1}^N \int_0^t \lambda_{\pm}(x_u^a, S_u^N) du \right) - \sum_{a=1}^N \int_0^t \lambda_{\pm}(x_u^a, S_u^N) du - B_{\pm} \left(\sum_{a=1}^N \int_0^t \lambda_{\pm}(x_u^a, S_u^N) du \right) \right|$$

is of the order $O(\log N)$ almost surely where B_{\pm} are the Brownian motions used in (3.18). Since the rate functions are uniformly bounded, the law of iterated logarithm for Brownian motion yields

$$\lim_{N \rightarrow \infty} \sup_{u \leq t} \frac{1}{N} B_{\pm} \left(\sum_{a=1}^N \int_0^u \lambda_{\pm}(x_v^a, S_v^N) dv \right) = 0 \quad \mathbb{P}^* \text{-a.s.}$$

It follows from this and Lemma 3.11 above, that the quantities

$$B_t^N \triangleq \frac{1}{N} \left| B_+ \left(\sum_{a=1}^N \int_0^t \lambda_+(x_u^a, S_u^N) du \right) - B_- \left(\sum_{a=1}^N \int_0^t \lambda_-(x_u^a, S_u^N) du \right) \right|$$

and

$$\Lambda_t^N \triangleq \left| \frac{1}{N} \sum_{a=1}^N \int_0^t \{ \lambda(x_u^a, s_u) - \bar{\lambda}(s_u) \} du \right|$$

converge to zero uniformly over compact time intervals as $N \rightarrow \infty$.

Let us now fix $\epsilon > 0$. Due to Lemma 3.10 there exists $N^* \in \mathbb{N}$ such that for all $N \geq N^*$ and uniformly on compact time sets, for $l \leq t$ we can write

$$\begin{aligned} |S_l^N - s_l| &\leq \left| \frac{1}{N} \sum_{a=1}^N \int_0^l \lambda(x_u^a, S_u^N) du - \int_0^l \bar{\lambda}(s_u) du \right| + B_l^N + \epsilon \\ &\leq \left| \frac{1}{N} \sum_{a=1}^N \int_0^l \{ \lambda(x_u^a, S_u^N) - \lambda(x_u^a, s_u) \} du \right| + \Lambda_l^N + B_l^N + \epsilon \quad \mathbb{P}^* \text{-a.s.} \end{aligned}$$

Lipschitz continuity of the rate functions yields

$$\begin{aligned} |S_l^N - s_l| &\leq L \int_0^l \sup_{0 \leq r \leq u} |S_r^N - s_r| du + \Lambda_l^N + B_l^N + \epsilon \\ &\leq L \int_0^t \sup_{0 \leq r \leq u} |S_r^N - s_r| du + \sup_{0 \leq r \leq t} \Lambda_r^N + \sup_{0 \leq r \leq t} B_r^N + \epsilon \quad \mathbb{P}^*\text{-a.s.} \end{aligned}$$

for some $L > 0$ and so

$$\sup_{0 \leq r \leq t} |S_r^N - s_r| \leq L \int_0^t \sup_{0 \leq r \leq u} |S_r^N - s_r| du + \sup_{0 \leq r \leq t} \Lambda_r^N + \sup_{0 \leq r \leq t} B_r^N + \epsilon \quad \mathbb{P}^*\text{-a.s.} \quad (3.23)$$

Now, an application of Gronwall's lemma yields

$$\sup_{0 \leq r \leq t} |S_r^N - s_r| \leq \left(\sup_{0 \leq r \leq t} \Lambda_r^N + \sup_{0 \leq r \leq t} B_r^N + \epsilon \right) e^{Lt} \quad \mathbb{P}^*\text{-a.s.}$$

for all $N \geq N^*$. This proves our assertion. \square

3.2.3 Second order approximation

In this section we analyze the fluctuations of the logarithmic price process around its first order approximation. We are interested in the distribution of asset prices around their first order approximation as $N \rightarrow \infty$. In view of the representation (3.18) and by self-similarity of Brownian motion we may thus assume that $\{S_t^N\}_{t \geq 0}$ is defined by the integral equation:

$$\begin{aligned} S_t^N &= \frac{1}{N} \sum_{a=1}^N \int_0^t \lambda(x_u^a, S_u^N) du + \frac{1}{\sqrt{N}} B_+ \left(\frac{1}{N} \sum_{a=1}^N \int_0^t \lambda_+(x_u^a, S_u^N) du \right) \\ &\quad - \frac{1}{\sqrt{N}} B_- \left(\frac{1}{N} \sum_{a=1}^N \int_0^t \lambda_-(x_u^a, S_u^N) du \right) + O\left(\frac{\log N}{N}\right). \end{aligned} \quad (3.24)$$

As we shall see, the fluctuations around the first order approximation are driven by two Gaussian processes. The first,

$$X_t \triangleq B_+ \left(\int_0^t \bar{\lambda}_+(s_u) du \right) - B_- \left(\int_0^t \bar{\lambda}_-(s_u) du \right), \quad (3.25)$$

captures the randomness in the agents' trading times. The second, $\{Y_t\}_{t \geq 0}$, is defined in terms of the integral of a non-stationary Gaussian process whose covariance function depends on the first order approximation. It captures the second source randomness generated by the agents' trading activity. Specifically,

$$Y_t \triangleq \int_0^t y_s ds, \quad (3.26)$$

where $\{y_t\}_{t \geq 0}$ denotes the centered Gaussian process whose covariance function γ is given by the covariance function of the stochastic process $\{\lambda(x_t, s_t)\}_{t \geq 0}$, i.e.,

$$\gamma(t, u) \triangleq \mathbb{E}[\lambda(x_t, s_t) \lambda(x_u, s_u)] - \bar{\lambda}(s_t) \bar{\lambda}(s_u). \quad (3.27)$$

It turns out that the fluctuations can be approximated in distribution by the process $\{Z_t\}_{t \geq 0}$ which satisfies the integral equation

$$Z_t = \int_0^t \bar{\lambda}'(s_u) Z_u du + Y_t + X_t. \quad (3.28)$$

Our goal is to establish the following second order approximation for the asset price process in an economy with many market participants.

Theorem 3.13 *The fluctuations of the market imbalance $\{S_t^N\}_{0 \leq t \leq 1}$ around its first order approximation can be described by the process $\{Z_t\}_{0 \leq t \leq 1}$ defined in (3.28). More precisely,*

$$\mathcal{L}\text{-}\lim_{N \rightarrow \infty} \sqrt{N} \{S_t^N - s_t\}_{0 \leq t \leq 1} = \{Z_t\}_{0 \leq t \leq 1}.$$

The proof of Theorem 3.13 requires some preparation. For notational convenience we introduce stochastic processes $Q^N = \{Q_t^N\}_{0 \leq t \leq 1}$, $Y^N = \{Y_t^N\}_{0 \leq t \leq 1}$ and $X^N = \{X_t^N\}_{0 \leq t \leq 1}$ by, respectively,

$$Q_t^N \triangleq \sqrt{N}(S_t^N - s_t) \quad \text{and} \quad Y_t^N \triangleq \sum_{a=1}^N \int_0^t \frac{\lambda(x_u^a, s_u) - \bar{\lambda}(s_u)}{\sqrt{N}} du, \quad (3.29)$$

and

$$X_t^N \triangleq B_+ \left(\frac{1}{N} \sum_{a=1}^N \int_0^t \lambda_+(x_u^a, S_u^N) du \right) - B_- \left(\frac{1}{N} \sum_{a=1}^N \int_0^t \lambda_-(x_u^a, S_u^N) du \right). \quad (3.30)$$

We first prove convergence in distribution of the sequence $\{(X^N, Y^N)\}_{N \in \mathbb{N}}$ to (X, Y) .

Proposition 3.14 *The sequence $\{(X^N, Y^N)\}_{N \in \mathbb{N}}$ converges in distribution to the process (X, Y) defined by (3.25) and (3.26).*

PROOF: For any $\alpha \in (0, \frac{1}{2})$ and $T > 0$, there exist integrable and hence almost surely finite random variables M_{\pm} such that for all $t_1, t_2 \leq T$ we have

$$|B_{\pm}(t_1) - B_{\pm}(t_2)| \leq M_{\pm} |t_1 - t_2|^{\alpha} \quad \mathbb{P}^*\text{-a.s.},$$

see, for instance, Remark 2.12 in [57]. Thus, the first order approximation shows that the sequence of processes $\{X^N\}_{N \in \mathbb{N}}$ converges almost surely to X on any compact time interval. Since the processes

$$\int_0^t \frac{\lambda(x_u^a, s_u) - \bar{\lambda}(s_u)}{\sqrt{N}} du$$

have Lipschitz continuous sample paths and the semi-Markov processes are independent, the central limit theorem for Lipschitz processes ([83], Corollary 7.2.1) shows that $\{Y^N\}_{N \in \mathbb{N}}$ converges in distribution to the Gaussian process Y . As a result, both sequences $\{X^N\}_{N \in \mathbb{N}}$ and $\{Y^N\}_{N \in \mathbb{N}}$ are tight. Since $\{X^N\}_{n \in \mathbb{N}}$ is also C-tight, the sequence $\{(X^N, Y^N)\}_{N \in \mathbb{N}}$ is tight. It is therefore enough to prove weak convergence of the finite dimensional distributions of the process (X^N, Y^N) to the finite dimensional distributions of (X, Y) .

In order to establish weak convergence of the one-dimensional distributions we fix a Lipschitz continuous functions with compact support $F : \mathbb{R}^2 \rightarrow \mathbb{R}$. We may with no loss of generality assume that both the Lipschitz constant and the diameter of the support of F equal one. In this case

$$\left| \int F(X_t^N, Y_t^N) d\mathbb{P}^* - \int F(X_t, Y_t^N) d\mathbb{P}^* \right| \leq \int \min\{|X_t^N - X_t|, 1\} d\mathbb{P}^*.$$

In view of the convergence properties of the sequence $\{X^N\}_{N \in \mathbb{N}}$, there exists, for any $\epsilon > 0$, a constant $N^* \in \mathbb{N}$ such that

$$\sup_{0 \leq t \leq 1} \int \min\{|X_t^N - X_t|, 1\} d\mathbb{P}^* \leq \epsilon \quad \text{for all } N \geq N^*.$$

This yields

$$\lim_{N \rightarrow \infty} \left| \int F(X_t^N, Y_t^N) d\mathbb{P}^* - \int F(X_t, Y_t^N) d\mathbb{P}^* \right| = 0.$$

Since the random variables X_t and Y_t^N are independent, we also have that

$$\lim_{N \rightarrow \infty} \int F(X_t, Y_t^N) d\mathbb{P}^* = \int F(X_t, Y_t) d\mathbb{P}^*.$$

This proves vague convergence³ of the one-dimensional marginal distributions of (X^N, Y^N) to the one-dimensional distributions of (X, Y) and hence weak convergence. Weak Convergence of the finite dimensional distributions follows from similar considerations. \square

The following ‘‘compact containment condition’’ is key to the second order approximation.

Lemma 3.15 (i) *The sequence of stochastic processes $\{Q^N\}_{N \in \mathbb{N}}$ is bounded in probability. That is, for any $\epsilon > 0$, there exists $N^* \in \mathbb{N}$ and $K < \infty$ such that*

$$\mathbb{P}^* \left[\sup_{0 \leq t \leq 1} |Q_t^N| > K \right] < \epsilon \quad \text{for all } N \geq N^*. \quad (3.31)$$

(ii) *If $f^N = \{f_t^N\}_{t \geq 0}$ be a sequence of non-negative random processes such that*

$$\lim_{N \rightarrow \infty} \int_0^1 f_u^N du = 0 \quad \text{in probability,} \quad (3.32)$$

then, for all $\delta > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P}^* \left[\sup_{0 \leq t \leq 1} \left| \int_0^t Q_u^N f_u^N du \right| > \delta \right] = 0.$$

PROOF:

³A sequence of probability measure $\{\mu_n\}$ converges to a measure μ in the vague topology if $\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$ for all continuous functions f with bounded support. The vague limit μ is not necessarily a probability measure. However, if there is an *a priori* reason that μ is a probability measure, then weak convergence of $\{\mu_n\}$ to μ can be established by analyzing integrals of continuous and hence Lipschitz continuous functions with bounded support. See e.g. [4] and [10].

(i) The strong approximation for Brownian motion yields the representation

$$Q_t^N = \frac{\int_0^t \sum_{a=1}^N \{\lambda(x_u^a, S_u^N) - \lambda(x_u^a, s_u)\} du}{\sqrt{N}} + Y_t^N + X_t^N + O\left(\frac{\log N}{\sqrt{N}}\right). \quad (3.33)$$

By Proposition 3.14 the sequence $\{(X^N, Y^N)\}_{n \in \mathbb{N}}$ is tight, and hence it is bounded in probability (see e.g. [29]). As a result, Lipschitz continuity of the rate functions yields

$$\sup_{0 \leq t \leq 1} |Q_t^N| \leq L \int_0^T \sup_{0 \leq t \leq u} |Q_u^N| du + \sup_{0 \leq t \leq 1} |Y_t^N| + \sup_{0 \leq t \leq 1} |X_t^N| + \mathcal{O}\left(\frac{\log N}{\sqrt{N}}\right).$$

for some $L > 0$. Hence, by Gronwall's inequality,

$$\sup_{0 \leq t \leq 1} |Q_t^N| \leq e^{LT} \left[\sup_{0 \leq t \leq 1} |Y_t^N| + \sup_{0 \leq t \leq 1} |X_t^N| + \mathcal{O}\left(\frac{\log N}{\sqrt{N}}\right) \right] \quad \mathbb{P}^* \text{-a.s.}$$

This proves (i).

(ii) Let us fix $\epsilon > 0$. There exists a constant N^* such that when $N \geq N^*$ there exist sets Ω_N and A_N such that

$$\int_0^1 f_u^N du < \frac{\epsilon}{2} \quad \text{on } \Omega_N \quad \text{and such that} \quad \mathbb{P}^*[\Omega_N] \geq 1 - \frac{\epsilon}{2}.$$

and

$$\sup_{0 \leq t \leq 1} |Q_t^N| < K \quad \text{on } A_N \quad \text{and such that} \quad \mathbb{P}^*[A_N] \geq 1 - \frac{\epsilon}{2}.$$

Hence

$$\sup_{0 \leq t \leq 1} \left| \int_0^t Q_u^N f_u^N du \right| \leq \sup_{0 \leq t \leq 1} |Q_t^N| \int_0^1 f_u^N du < K\epsilon \quad \text{on } A_N \cap \Omega_N.$$

□

Proof of Theorem 3.13: Let us first define a sequence of stochastic processes $\tilde{Q}^N = \{\tilde{Q}_t^N\}_{0 \leq t \leq 1}$ by

$$\tilde{Q}_t^N \triangleq \int_0^t \bar{\lambda}'(s_u) \tilde{Q}_u^N du + Y_t^N + X_t^N.$$

By the continuous mapping theorem and Lemma 3.14 the sequence $\{\tilde{Q}^N\}_{N \in \mathbb{N}}$ converges in distribution to the process Z defined in (3.28). It is now enough to show that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq 1} |Q_t^N - \tilde{Q}_t^N| = 0 \quad \text{in probability.} \quad (3.34)$$

To this end, let $E_t^N \triangleq Q_t^N - \tilde{Q}_t^N$. From the definition of \tilde{Q}_t^N and the representation (3.33) of Q_t^N we obtain

$$\begin{aligned} E_t^N &= \int_0^t \bar{\lambda}'(s_u) E_u^N du + \frac{1}{\sqrt{N}} \int_0^t \sum_{a=1}^N \left\{ \lambda(x_u^a, S_u^N) - \lambda(x_u^a, s_u) \right\} du - \int_0^t \bar{\lambda}'(s_u) Q_u^N du \\ &= \int_0^t \bar{\lambda}'(s_u) E_u^N du + \int_0^t \left(\frac{1}{N} \sum_{a=1}^N \lambda'(x_u^a, s_u) - \bar{\lambda}'(s_u) \right) Q_u^N du \\ &\quad + \int_0^t \left(\frac{1}{N} \sum_{a=1}^N \lambda'(x_u^a, \xi_u^N) - \lambda'(x_u^a, s_u) \right) Q_u^N du. \end{aligned}$$

The second equality follows from the mean value theorem for $\lambda(x_u^a, \cdot)$,

$$\lambda(x_u^a, S_u^N) - \lambda(x_u^a, s_u) = \frac{1}{\sqrt{N}} \lambda'(x_u^a, \xi_u^N) Q_u^N,$$

where ξ_u^N lies between $\frac{1}{N} S_u^N$ and s_u . We put

$$f_u^{N,1} \triangleq \frac{1}{N} \sum_{a=1}^N \lambda'(x_u^a, s_u) - \bar{\lambda}'(s_u) \quad \text{and} \quad f_u^{N,2} \triangleq \frac{1}{N} \sum_{a=1}^N \lambda'(x_u^a, \xi_u^N) - \lambda'(x_u^a, s_u)$$

in order to obtain

$$\sup_{0 \leq s \leq t} |E_s^N| \leq L \int_0^t \sup_{0 \leq s \leq u} |E_s^N| du + \left| \sup_{0 \leq s \leq t} \int_0^s |f_u^{N,1}| Q_u^N du \right| + \left| \sup_{0 \leq s \leq t} \int_0^s |f_u^{N,2}| Q_u^N du \right|.$$

The processes $|f^{N,1}|$ and $|f^{N,2}|$ satisfy the condition (3.32) of Lemma 3.15 by the law of large numbers. Thus, an application of Gronwall's lemma yields (3.34). \square

3.2.4 Approximation by a fractional Ornstein-Uhlenbeck process

So far we have shown that the fluctuations of the logarithmic price process around its first order approximation can be described in terms of an Ornstein-Uhlenbeck process Z driven by two Gaussian processes X and Y . In order to see more clearly the effects of investor inertia on asset processes we need to better understand the dynamics of Y . As before, this will be achieved by a proper scaling of the semi-Markov processes x^a in time and the price process in space. Specifically, we introduce a family of processes $S^{N,T}$ ($T \in \mathbb{N}$) with initial value 0 by

$$S_t^{N,T} = \frac{1}{NT} \left\{ \Pi_+ \left(T \sum_{a=1}^N \int_0^t \lambda_+(x_{Tu}^a, S_u^{N,T}) du \right) - \Pi_- \left(T \sum_{a=1}^N \int_0^t \lambda_-(x_{Tu}^a, S_u^{N,T}) du \right) \right\}.$$

The strong approximation result for Poisson processes with respect to Brownian motion allow us to represent the process $\{S_t^{N,T}\}_{t \geq 0}$ as in (3.18) with the semi-Markov processes $\{x_t^a\}_{t \geq 0}$ replaced by the "speeded-up" processes $\{x_{Tt}^a\}_{t \geq 0}$. Moreover, by Lemma 3.11, the sequence of processes

$$\Lambda_t^{N,T} \triangleq \left| \frac{1}{N} \sum_{a=1}^N \int_0^t \{ \lambda(x_{Tu}^a, s_u) - \bar{\lambda}(s_u) \} du \right|$$

converges to zero uniformly over compact time intervals as $N \rightarrow \infty$. Following the same line of arguments as in the proof of Proposition 3.12 it can then be shown that for any $T > 0$

$$\lim_{N \rightarrow \infty} S_t^{N,T} = s_t \quad \mathbb{P}^* \text{-a.s.} \quad (3.35)$$

Here $\{s_t\}_{t \geq 0}$ denotes the deterministic process defined by the ordinary differential equation (3.20) with initial condition $s_0 = 0$ and the convergence holds uniformly over compact time intervals. Thus, the first order approximation is independent of T . By analogy to (3.24)-(3.28) introduce a Gaussian process Y^T by

$$Y_t^T \triangleq \int_0^t y_s^T ds, \quad (3.36)$$

where $\{y_t^T\}_{t \geq 0}$ denotes the centered Gaussian process with covariance function

$$\gamma^T(t, u) \triangleq \mathbb{E}[\lambda(x_{Tt}, s_t)\lambda(x_{Tu}, s_u)] - \bar{\lambda}(s_t)\bar{\lambda}(s_u).$$

Following the same arguments in the proof of Theorem 3.13, we see that as the number of agents tends to infinity the price fluctuations around the fluid limit can be approximated in distribution by a process $\{Z_t^T\}_{t \geq 0}$ of the form

$$Z_t^T = \int_0^t \bar{\lambda}'(s_u) Z_u^T du + Y_t^T + \frac{1}{\sqrt{T}} X_t.$$

Proposition 3.16 *For any T , the fluctuations of the logarithmic price process $\{S_t^{N,T}\}_{0 \leq t \leq 1}$ around its first order approximation can be described by the process $\{Z_t^T\}_{0 \leq t \leq 1}$. More precisely,*

$$\mathcal{L}\text{-}\lim_{N \rightarrow \infty} \sqrt{N} \left\{ S_t^{N,T} - s_t \right\}_{0 \leq t \leq 1} = \{Z_t^T\}_{0 \leq t \leq 1}.$$

To take the T -limit, we need the following assumption on the structure of the rate functions.

Assumption 3.17 *The rate function λ defined in (3.19) can be written as*

$$\lambda(x, s) = f(x)g(s) + h(s). \quad (3.37)$$

Moreover, the function f in (3.37) is one-to-one and $\hat{\mu} \triangleq f(0) \neq \mathbb{E}^* f(x_0)$.

Example 3.18 *The previous assumption is always satisfied if $(x_t^a)_{t \geq 0}$ is a stationary on/off process, i.e., if $E = \{0, 1\}$. In this case*

$$x_t^a = \frac{\lambda(x_t^a, s_t) - \lambda(0, s_t)}{\lambda(1, s_t) - \lambda(0, s_t)},$$

and the representation (3.37) holds with

$$f(x) = x, \quad g(s) = \lambda(1, s) - \lambda(0, s) \quad \text{and} \quad h(s) = \lambda(0, s).$$

We are now ready to show that the fluctuations of the logarithmic stock price around its first order approximation behaves like a fractional Ornstein-Uhlenbeck process.

Theorem 3.19 *Under the Assumptions 3.9 and 3.17 we have that*

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} T^{1-H} \frac{\sqrt{N}}{\sqrt{L(T)}} \left\{ S_t^{N,T} - s_t \right\}_{0 \leq t \leq 1} = \{\hat{Z}_t\}_{0 \leq t \leq 1}$$

Here \hat{Z} denotes unique solution to the stochastic differential equation

$$d\hat{Z}_t = \bar{\lambda}'(s_t)\hat{Z}_t dt + \sigma g(s_t) dB_t^H$$

where B^H is a fractional Brownian motion with Hurst coefficient $H = \frac{3-\alpha}{2}$. The integral with respect to B^H is understood as a limit in probability of Stieltjes sums.

PROOF: The proof uses modifications of arguments given in the proof of Theorem 3.13 and the approximation result for integrals with respect to fractional Brownian motion in [5].

(i) In a first step we study the dynamics of the process $\{Y_t^{N,T}\}_{t \geq 0}$ defined by

$$Y_t^{N,T} = \sum_{a=1}^N \int_0^t \frac{\lambda(x_{Tu}^a, s_u) - \bar{\lambda}(s_u)}{\sqrt{N}} du$$

Under Assumption 3.17 we can write

$$\begin{aligned} Y_t^{N,T} &= \sum_{a=1}^N \int_0^t \frac{1}{\sqrt{N}} [f(x_{Tu}^a)g(s_u) + h(s_u) - \bar{\lambda}(s_u)] du \\ &= \sum_{a=1}^N \int_0^t \frac{1}{\sqrt{N}} [f(x_{Tu}^a) - \hat{\mu}] g(s_u) du \end{aligned} \quad (3.38)$$

Since f is one-to-one, $(f(x_t^a))_{t \geq 0}$ is a semi-Markov process that has the same sojourn time structure as the underlying semi-Markov process $(x_t^a)_{t \geq 0}$. In particular, $f(0)$ is the state whose sojourn time distribution has heavy tails. Therefore it follows from Theorem 4.1 of [5] that

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} T^{1-H} \left\{ \frac{1}{\sqrt{L(T)}} Y_t^{N,T} \right\}_{0 \leq t \leq 1} = \left\{ \sigma \int_0^t g(s_u) dB_u^H \right\}_{0 \leq t \leq 1} \quad (3.39)$$

for some $\sigma > 0$ because $\hat{\mu} \neq f(0)$

(ii) Let us now define a family of stochastic processes $\tilde{Q}^{N,T} = \{\tilde{Q}_t^{N,T}\}_{0 \leq t \leq 1}$ by

$$\tilde{Q}_t^{N,T} \triangleq \int_0^t \bar{\lambda}'(s_u) \tilde{Q}_u^{N,T} du + \frac{T^{1-H}}{\sqrt{L(T)}} Y_t^{N,T} + \frac{T^{1/2-H}}{\sqrt{L(T)}} X_t^N.$$

Since the rate functions are bounded and $H > \frac{1}{2}$

$$\lim_{T \rightarrow \infty} \sup_N \sup_{0 \leq t \leq 1} \frac{T^{1/2-H}}{\sqrt{L(T)}} X_t^N = 0$$

almost surely, and the continuous mapping theorem along with (i) yields

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} \{\tilde{Q}_t^{N,T}\}_{0 \leq t \leq 1} = \{\hat{Z}_t\}_{0 \leq t \leq 1}$$

(iii) Let us put

$$Q_t^{N,T} \triangleq T^{1-H} \frac{\sqrt{N}}{\sqrt{L(T)}} (S_t^{N,T} - s_t).$$

Up to a term of the order $\frac{\log N}{\sqrt{N}}$ we obtain

$$Q_t^{N,T} = \frac{\int_0^t \sum_{a=1}^N \left\{ \lambda(x_{Tu}^a, S_u^{N,T}) - \lambda(x_{Tu}^a, s_u) \right\} du}{\sqrt{N}} + \frac{T^{1-H}}{\sqrt{L(T)}} Y_t^{N,T} + \frac{T^{1/2-H}}{\sqrt{L(T)}} X_t^N.$$

Using the same arguments as in the proof of Theorem 3.13 we thus see that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq 1} |Q_t^{N,T} - \tilde{Q}_t^{N,T}| = 0 \quad \text{in probability}$$

for all $T \in \mathbb{N}$. Hence the assertion follows from (ii). □

Remark 3.20 *In the case of Markov switching, i.e., when the process x_t is a Markov process, we obtain standard Ornstein-Uhlenbeck process, i.e., we have that*

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \mathcal{L}\text{-}\lim_{N \rightarrow \infty} \sqrt{T} \frac{\sqrt{N}}{\sqrt{L(T)}} \left\{ S_t^{N,T} - s_t \right\}_{0 \leq t \leq 1} = \{ \tilde{Z}_t \}_{0 \leq t \leq 1},$$

where \tilde{Z} denotes unique solution to the stochastic differential equation

$$d\tilde{Z}_t = \bar{\lambda}'(s_t) \tilde{Z}_t dt + \sigma g(s_t) dB_t,$$

with B a standard Brownian motion.

4 Outlook & Conclusion

We briefly outline two possible avenues of future research: microstructure models of fractional volatility and strategic interactions between “big players.”

4.1 Fractional Volatility

In this article, we suggested a microeconomic approach to financial price fluctuations that is capable of explaining the decay of the Hurst coefficient of the S&P 500 index in the late 1990s. We note that the evidence of long memory in stock price returns is mixed, there are several papers in the empirical finance literature providing evidence for the existence of long memory, yet there are several other papers that contradict these empirical findings; see e.g. [8] for an exposition of this debate and references. However, long memory is a well accepted feature in volatility (squared and absolute returns) and trading volume (see e.g. [23] and [28]). We are now going to illustrate how the mathematical results of this paper might also be seen as an intermediate step towards a microstructural foundation for this phenomenon. To ease notational complexity and to avoid unnecessary technicalities we restrict ourselves to the simplest case where the order rates do not depend on asset prices. Specifically, we assume that (after taking the N -limit) the dynamics of the asset price process can be described by a stochastic equation of the form

$$S_t^T = \frac{1}{T} \left\{ \Pi_+ \left(T \int_0^t \lambda_+(Y_u^T) du \right) - \Pi_- \left(T \int_0^t \lambda_-(Y_u^T) du \right) \right\}$$

where the Gaussian process Y^T defined in (3.11) converges in distribution to a fractional Brownian motion process. In view of the strong approximation of Poisson processes by Brownian motion,

and because the rate functions are bounded, the evolution of prices can be described in terms of an ordinary differential equation in a random environment generated by a fractional Brownian motion:

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \{S_t^T\}_{0 \leq t \leq 1} = \{\hat{s}_t\}_{t \leq 0 \leq 1} \quad \text{where} \quad d\hat{s}_t = \lambda(B_t^H) dt.$$

The fluctuations around this first order approximation satisfy

$$\sqrt{T} \left(S_t^T - \int_0^t \lambda(Y_u^T) du \right) = B_+ \left(\int_0^t \lambda_+(Y_u^T) du \right) - B_- \left(\int_0^t \lambda_-(Y_u^T) du \right),$$

up to a term of the order $\frac{\log T}{\sqrt{T}}$. Convergence of the Gaussian process Y^T to fractional Brownian motion along with continuity of the rate functions yields

$$\mathcal{L}\text{-}\lim_{T \rightarrow \infty} \left\{ B_{\pm} \left(\int_0^t \lambda_{\pm}(Y_u^T) du \right) \right\}_{0 \leq t \leq 1} = \left\{ \int_0^t \sqrt{\lambda_{\pm}(B_u^H)} dB_u^{\pm} \right\}_{0 \leq t \leq 1}.$$

Thus, for large T , logarithmic asset prices satisfy

$$\begin{aligned} S_t^T &\stackrel{\mathbb{D}}{\approx} \int_0^t \lambda(Y_u^T) du + \frac{1}{\sqrt{T}} \int_0^t \sqrt{\lambda_+(Y_u^T)} dB_u^+ - \frac{1}{\sqrt{T}} \int_0^t \sqrt{\lambda_-(Y_u^T)} dB_u^- \\ &\stackrel{\mathbb{D}}{\approx} \int_0^t \lambda(B_u^H) du + \frac{1}{\sqrt{T}} \int_0^t \sqrt{\lambda_+(B_u^H)} dB_u^+ - \frac{1}{\sqrt{T}} \int_0^t \sqrt{\lambda_-(B_u^H)} dB_u^-, \end{aligned}$$

i.e., the volatility is driven by a fractional Brownian motion process which is independent of the Wiener processes B^+ and B^- . We will further elaborate on the microstructure of fractional volatility in a separate paper.

4.2 Strategic Interactions

Together with the price taking small investors, it is also possible to incorporate the effects of large investors who influence the price. The existence of large agent price effects has been empirically described in several papers: [59], [48] and [19] describe the impacts of institutional trades on stock prices. In the presence of large agents there is limited liquidity in the market since the holdings of the stocks is concentrated in the hands of a few big traders. Trades of “big player’s” also affect stock prices due to large order sizes.

4.2.1 Stochastic Equations in Strategically Controlled Environments

Horst [51], [52] provides a mathematical framework for analyzing linear stochastic difference equation of the form (2.3) when the dynamics of the random environment is simultaneously controlled by the actions of strategically interacting agents playing a discounted stochastic game with complete information. In [51] we considered a simple microstructure models where small investors choose their current benchmarks in reaction to the actions taken by some “big players”. One may, for example, think of a central bank that tries to keep the “mood of the market” from becoming too optimistic and, if necessary, warns the market participants of emerging bubbles. One may also think of financial experts whose recommendations tempt the agents into buying or selling the stock. These market participants influence the stock price process through their impact on the behavior

of small investors, but without actively trading the stock themselves. It seems natural to assume that the big players anticipate the feedback effect their actions have on the evolution of stock prices and thus interact in a strategic manner. Under a weak interaction condition, the resulting stochastic game has a homogenous Nash equilibrium in Markovian strategies. It turns out that the main qualitative feature of the models studied in [40], [39] and [53], namely asymptotic stability of stock prices can be preserved even in a model of strategic interactions. However, the long run distribution of stock prices depends on the equilibrium strategy and is thus not necessarily uniquely determined. Hence, the presence of strategically interacting market participants can be an additional source of uncertainty.

4.2.2 Stochastic Games in a Non-Markovian Setting

Bayraktar and Poor [7] considered the strategic interaction of large investors and found an equilibrium stock price taking into account that the feedback effects of the large investors on the price. The large traders find themselves in a random environment due to the trades of small (i.e. price taking) investors. In [7], the institutional investors strategically interact through the controls they exert on the coefficients of a stochastic differential equation driven by a fractional Brownian motion. Here, the fractional Brownian motion models the effect of the price taking investors on the price. It can be argued that the observed stock price is the Nash-equilibrium price that arises as a result of the strategic interaction of the institutional investors this random environment. Bayraktar and Poor carries out an analysis of stochastic differential games in a non-Markov environment using the stochastic analysis for fractional Brownian motion developed in [31]. This analysis can be viewed as a first step toward incorporating the feedback effects of the large investors and the strategic interaction into the description of the stock price dynamics.

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