Games with Exhaustible Resources

Chris Harris* Sam Howison† Ronnie Sircar‡

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Abstract

We study N-player continuous-time Cournot games in an oligopoly where firms choose production quantities. These are nonzero-sum differential games, whose value functions may be characterized by systems of nonlinear Hamilton-Jacobi partial differential equations. When resources are in finite supply, such as oil, exhaustibility enters as boundary conditions for the PDEs. We analyze the problem when there is an alternative, but expensive, technology (for example solar power for energy production), and give an asymptotic approximation in the limit of small exhaustibility. We illustrate the two-player problem by numerical solutions, and discuss the impact of limited oil reserves on production and oil prices in the duopoly case.

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*Faculty of Economics, Cambridge University, Sidgwick Avenue, Cambridge CB3 9DD; *cjharris@econ.cam.ac.uk*

†OCIAM & Oxford-Man Institute of Quantitative Finance, Mathematical Institute, Oxford University, 24-29 St. Giles', Oxford OX1 3LB; *howison@maths.ox.ac.uk*

‡ORFE Department, Princeton University, Sherrerd Hall, Princeton NJ 08544; *sircar@princeton.edu*.
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1 Introduction

The problem of dwindling oil reserves and its impact on energy supply and prices is of long-standing importance. One way to analyze the issues is to view energy markets as being governed by a small number of competitive firms or countries, that is as oligopolies, and to model the formation of prices and supplies within this competitive framework. Game theory provides a natural way to frame the outcome of competition within different choices of market mechanism. Exhaustibility, meanwhile, requires analysis of how anticipation of changing resources impacts production and prices, and therefore leads to dynamic games. Here we study continuous-time (or differential) games arising from competition over a resource in limited supply. The games are nonzero-sum as all players act to maximize their own profits.

Typical models of industrial organization in the economics literature are restricted to the study of one or two-period games. For ordinary and stochastic *zero-sum* differential games, there is a fairly general theory [7, 12], including a viscosity theory for their associated (scalar) Hamilton-Jacobi-Bellman-Isaacs PDEs. In the nonzero-sum case where systems of equations for the value functions of all the players arise, there is, to our knowledge, no similar general theory. Some recent books that discuss nonzero-sum deterministic and stochastic differential games are [1, 6]. There is also a literature on exhaustibility (or capacity constraints), but not, primarily, in the context of continuous-time models. We mention [5] as a reference for the literature on exhaustible resources up through the 1970s.

When the quantity being produced is in finite supply, such as oil, exhaustibility is a "game-changer", and enters as boundary conditions for the PDEs. We analyze the problem when there is an alternative resource (for example solar technology for energy production), which is inexhaustible, but much more costly to produce than extracting oil. Of interest is the impact on oil extraction rates, and hence market prices, as reserves run out and energy production must switch to more expensive renewable sources.

We begin by analysing the static N-player Cournot game as a function of the costs of production of the firms. When we move to the dynamic problem in Section 3, exhaustibility acts like a varying cost, that depends on the dynamic game's value functions, in a static game at the infinitesimal level. Therefore, we devote some effort in Section 2 to establish existence and uniqueness of a Nash equilibrium for the static game with players who have different costs. The static game equilibrium production and profit functions are essential ingredients for the partial differential equations characterizing the dynamic game in Section 3. In Section 4, we derive a perturbation approximation for the case when the cost of the alternative technology is small. Section 5 studies in detail and with numerical PDE solutions a specific two-player example. In particular, we discuss the issue of one firm being blockaded out of competition by the dominance of the other. Section 6 concludes.

2 Static Cournot Game

In the bulk of the paper we will analyse a dynamic version of the Cournot model of competition: producers of the resource set quantities which they bring to the market, and the price is then determined from the total quantity produced. Before moving to the dynamic game with exhaustibility, we analyze the static (one-period, or stage) game and introduce notation from it that will be needed later.

We take as given a price (or inverse-demand) function $P : (0, \infty) \to \mathbb{R}$ that gives market price (per unit) as a function of quantity produced and put on the market. There are $N \geq 1$ players. Each player i chooses a quantity $q_i \in [0, \infty)$ to produce at unit cost of production $a_i \geq 0$, and the market price is determined from the total production. In the original example described by Cournot [4], the inexhaustible resource being produced was mineral water.

Once each player chooses his quantity, the market price is given by

$$P(Q), \quad \text{where} \quad Q = \sum_{j=1}^{N} q_j.$$

The profit of player i is the quantity he produces multiplied by price minus cost:

$$\pi(q_i, Q_{-i}, a_i) = \begin{cases} q_i \left(P(Q_{-i} + q_i) - a_i\right) & \text{if } q_i > 0, \\ 0 & \text{if } q_i = 0, \end{cases} \tag{1}$$

where $Q_{-i} = \sum_{j \neq i} q_j$ is total production by the players other than i. We allow for the possibility that $P(0^+) = +\infty$, but specify $\pi(q_i, Q_{-i}, a_i) = 0$ when $q_i = 0$: if a player does not produce anything, then he makes no profit.

Each player seeks to maximize his own profit, taking the quantities produced by the other players as given. More precisely:

Definition 2.1. A Nash equilibrium is a vector $\boldsymbol{q}^* = (q_1^*, q_2^*, ..., q_N^*) \in [0, \infty)^N$ such that, for all i,

$$\pi(q_i^*, Q_{-i}^*, a_i) = \max_{q_i \in [0,\infty)} \pi(q_i, Q_{-i}^*, a_i), \tag{2}$$

where $Q_{-i}^* = \sum_{j \neq i} q_j^*$. That is, each player's equilibrium production q_i^* maximizes his own profit $\pi(\cdot, Q_{-i}^*, a_i)$ when the other $N-1$ players produce their equilibrium quantities. If, in addition, $q_i^* > 0$ for all i, then we call $\boldsymbol{q}^*$ an interior Nash equilibrium.

The main aim of this section is to show that, under suitable conditions on the price function P and the cost vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_N)$, a Nash equilibrium exists and is unique. Much of the literature on static Cournot games assumes identical costs among the players (so that all or none will participate), or assumes all costs are small enough that there is an interior Nash equilibrium. In the models we study here, costs are linear in quantity (constant marginal cost), but we shall need to consider the situation of hetergeneous costs among the players, which will be related to different proximities to exhaustion in the dynamic game of Section 3. An important issue arising from this is that it may be too costly for some players to participate. To our knowledge, this aspect has not been fully addressed before, even in the static game. We refer to [17, Chapter 4] for a discussion and references on general existence results for static Cournot games.

Our primary assumptions on P are:

Assumption 2.2. The price function P is twice continuously differentiable, with $P' < 0$ everywhere on $(0, \infty)$; and there exists $\eta \in (0, \infty)$ such that $P(\eta) = 0$.

The first part of this assumption is natural: the greater total production, the less the market will be willing to pay per unit. The second part implies that $P(Q) < 0$ for $Q > \eta$: if there is over-production, players have to pay to have the surplus removed. Nonetheless, it should be noted that negative prices will play no role in our analysis, because profit-maximizing players with positive costs will never produce at a level at which prices are negative. We shall refer to η as the *saturation point*.

We order the firms by their costs and assume they are strictly less than the *choke price* $P(0^+)$:

$$0 \leq a_1 \leq a_2 \leq ... \leq a_N < P(0^+). \tag{3}$$

When some firms have equal costs, the ordering is arbitrary and does not affect the analysis that follows. The assumption that $a_i < P(0^+)$ for all i ensures that, in any Nash equilibrium $\boldsymbol{q}^*$, total production $Q^* = \sum_{i=1}^N q_i^*$ will be strictly positive, for if all players other than i produce nothing, so that $q_j = 0$ for all $j \neq i$, then player i can make a strictly positive profit $q_i(P(q_i) - a_i)$ by producing a small positive amount q_i (and may be able to do much better). Having all players produce nothing is therefore not a Nash equilibrium. The assumption that $a_i \geq 0$ for all i ensures that $P(Q^*) > 0$. In particular, $Q^* < \eta$.

The behaviour of P is best characterized in terms of the *relative prudence* of P, namely

$$\rho(Q) = -\frac{Q\,P''(Q)}{P'(Q)}. \tag{4}$$

Our terminology here is adapted from [13]. [1] We also define

$$\overline{\rho} = \sup_{Q \in (0,\infty)} \rho(Q). \tag{5}$$

We turn now to the detailed analysis of Nash equilibrium, first for general price functions P, and then, in Section 2.2, for a convenient family of examples which are amenable to explicit calculations.

2.1 General Price Functions

Suppose that $\boldsymbol{q}^*$ is an interior Nash equilibrium. Then, for all $i \in \{1, 2, ..., N\}$, q_i^* must satisfy the first-order condition

$$0 = \frac{\partial \pi}{\partial q_i}(q_i^*, Q_{-i}^*, a_i) = q_i^* P'(Q_{-i}^* + q_i^*) + P(Q_{-i}^* + q_i^*) - a_i. \tag{6}$$

[1] The relationship between the definition (4) of relative prudence and the usual definition in economics involving the third derivative of a utility function can be understood as follows. Suppose that a consumer has quasilinear utility function $u(Q) + m$, where Q is the quantity consumed and m is money. Then her inverse demand function is $u'(Q)$ and her coefficient of relative prudence is $\rho(Q) = -\frac{Q\,u'''(Q)}{u''(Q)}$. Putting $u'(Q) = P(Q)$ in this latter formula, we recover (4). That is, our ρ is in fact the relative prudence of u in the usual sense. Another example in which relative prudence plays a role in establishing the uniqueness of equilibrium in a game-theoretic setting is [11].

Summing over i, we obtain

$$0 = Q^* P'(Q^*) + N\, P(Q^*) - A_N, \tag{7}$$

where $Q^* = Q^*_{-i} + q^*_i = \sum_{i=1}^N q^*_i$ is total production and $A_N = \sum_{i=1}^N a_i$ is the sum of the unit costs. In other words, Q^* must satisfy the scalar equation $f_N(Q) = A_N$, where

$$f_N(Q) = QP'(Q) + N\, P(Q), \quad Q > 0.$$

On the other hand, given a solution of this equation, making q^*_i the subject of equation (6), a candidate Nash equilibrium is

$$q^*_i = \frac{P(Q^*) - a_i}{-P'(Q^*)}. \tag{8}$$

Using (7), we can also express the candidate equilibrium quantities in (8) as

$$q^*_i = \left(\frac{P(Q^*) - a_i}{\sum_{j=1}^N (P(Q^*) - a_j)} \right) Q^*.$$

This has the interpretation that, once the equilibrium total quantity Q^* is determined, the fraction produced by player i is the deviation of his cost a_i from the market price $P(Q^*)$ relative to the total deviation of all players' costs from the price.

However, some q^*_i defined by (8) may be negative, and so we must also consider Nash equilibria in which some players do not produce. For $1 \le n \le N$, we define

$$f_n(Q) = QP'(Q) + n\, P(Q), \quad Q > 0, \tag{9}$$

and $A_n = \sum_{i=1}^n a_i$. We then have the following lemma.

Lemma 2.3. *Fix $n \in \{1, \cdots, N\}$, and suppose that $\overline{\rho} < n + 1$. Then there is a unique $Q^*_n \in (0, \eta)$ such that $f_n(Q^*_n) = A_n$.*

Proof. It is sufficient to show that f_n is decreasing with $f_n(0^+) > A_n$ and $f_n(\eta) < 0$, so there is a unique root of $f_n(Q) = A_n$ in $(0, \eta)$. We compute

$$\begin{aligned} f'_n(Q) &= QP''(Q) + (n+1)\, P'(Q) \\ &= (n + 1 - \rho(Q))\, P'(Q) \\ &\le (n + 1 - \overline{\rho})\, P'(Q). \end{aligned} \tag{10}$$

Hence $f'_n < 0$ on $(0, \infty)$. We also have $f_n(\eta) = \eta P'(\eta) < 0$ (since $P(\eta) = 0$). We therefore need only show that $f_n(0^+) > A_n$. In the case where $P(0^+)$ is finite and $P'(0^+)$ exists and is finite, $f_n(0^+) = nP(0^+) > A_n$, from (3). The case where the choke price is infinite ($P(0^+) = +\infty$) is a little more involved, and is handled in the appendix. □

Then, for each $n > \max(0, \overline{\rho} - 1)$, we have the following n-player candidate Nash equilibrium:

$$q^*_i = \begin{cases} \frac{P(Q^*_n) - a_i}{-P'(Q^*_n)} & \text{for } 1 \le i \le n, \\ 0 & \text{for } n + 1 \le i \le N, \end{cases} \tag{11}$$

where Q^*_n is the unique solution of $f_n(Q) = A_n$, and we recall that the players are ordered by their production costs a_i. This candidate equilibrium can fail to be a Nash equilibrium of the game as a whole in one of three ways:

1. it may happen that $q_i^* < 0$ for some $1 \le i \le n$;
2. it may happen that $a_i < P(Q_n^*)$ for some $n+1 \le i \le N$; or
3. it may happen that, for some $1 \le i \le n$, q_i^* is not a global maximum of $\pi(\cdot, Q_{-i}, a_i)$.

The first case occurs if and only if $a_i > P(Q_n^*)$, that is, the unit cost of player i is greater than or equal to the market price and player i would be better off not producing anything. In other words, we should look for a Nash equilibrium with a smaller number $n' < n$ of active players. This is only possible if $\overline{\rho}$ satisfies the stricter inequality $\overline{\rho} < n'$. In the second case, the unit cost of player i is less than the market price and player i would therefore be better off participating by producing some $q_i^* > 0$. In other words, we should look for a Nash equilibrium with a larger number $n'' > n$ of active players. In the third case, player i will want to deviate from the candidate equilibrium, and it is not clear where we should look for an alternative equilibrium. This third possibility can be eliminated by a hypothesis on P, namely that $\overline{\rho} < 2$.

Lemma 2.4. *Suppose that $\overline{\rho} < 2$ and $Q_{-i} \ge 0$. Then $g(q_i) := \pi(q_i, Q_{-i}, a_i)$ has a unique global maximum, which is attained in $[0, \eta)$.*

Proof. As $P(0^+)$ may not be finite, the details of the proof depend on whether $Q_{-i} > 0$ or $Q_{-i} = 0$. Suppose first that $Q_{-i} > 0$. Then $g(q_i) = q_i(P(Q_{-i} + q_i) - a_i)$ $(q_i \ge 0)$ is twice continuously differentiable everywhere, and in particular at $q_i = 0$. Moreover, for $q_i \ge 0$,

$$\begin{aligned} g''(q_i) &= P''(Q_{-i}+q_i)\, q_i + 2\, P'(Q_{-i}+q_i) \\ &= \left(2 - \frac{q_i}{Q_{-i}+q_i}\, \rho(Q_{-i}+q_i)\right) P'(Q_{-i}+q_i) \qquad (12) \\ &\le \left(2 - \frac{q_i}{Q_{-i}+q_i}\, \overline{\rho}\right) P'(Q_{-i}+q_i) \\ &\le (2-\overline{\rho})\, P'(Q_{-i}+q_i) \\ &< 0. \end{aligned}$$

Hence g has a unique global maximum, which is attained in $[0, \eta)$ since $g(0) = 0$ and $g(q_i) < 0$ for $q_i \ge \eta$.

In the case $Q_{-i} = 0$,

$$g(q_i) = \begin{cases} q_i(P(q_i) - a_i) & \text{if } q_i > 0, \\ 0 & \text{if } q_i = 0. \end{cases}$$

In particular, $g(q_i)$ may be discontinuous at $q_i = 0$. As $a_i < P(0^+)$ by assumption, $P^{-1}(a_i) \in (0, \eta]$ exists and is unique, and $g > 0$ on $(0, P^{-1}(a_i))$. Consequently, $g(0^+) \ge 0$. We also have $g'(q_i) = q_i P'(q_i) + P(q_i) - a_i$, so that

$$g'(0^+) = f_1(0^+) - a_i.$$

It follows from the calculations in the proof of Lemma 2.3 that $g'(0^+) > 0$. Finally,

$$g''(q_i) = (2 - \rho(q_i))\, P'(q_i) \le (2 - \overline{\rho})\, P'(q_i) < 0$$

for all $q_i > 0$. Thus since $g < 0$ for $q > P^{-1}(a_i)$, g has a unique global maximum, which is attained in $(0, P^{-1}(a_i)) \subset [0, \eta)$. ∎

The assumption that $\overline{\rho} < 2$ does more than simply eliminate the possibility of competing local maxima: it allows us to implement the approach to characterizing equilibria sketched above. Starting from the one-player equilibrium with player one, who has the lowest cost, we look at whether player two, who has the second lowest cost, wants to participate, in other words if his cost is less than the one-player market price: $a_2 < P(Q_1^*)$. If so, we ask if both the first two players want to participate in the two-player equilibrium, and so on. The following lemma establishes the crucial step.

Lemma 2.5. *Suppose for some $n < N$, we have n- and $(n+1)$-player candidate equilibria with aggregate production quantities Q_n^* and Q_{n+1}^* respectively, and the individual production levels given by (11) with the appropriate Q^*. Then player $n+1$ will want to participate in the n-player equilibrium if and only if he wants to participate in the $(n+1)$-player equilibrium.*

Proof. From (11), player i participates in an n-player candidate equilibrium if and only if $a_i < P(Q_n^*)$. Recall from Lemma 2.3 that each $Q_n^* \in (0, \eta)$ satisfies $f_n(Q_n^*) = A_n$, where the functions $f_n(Q)$ were defined in (9), and are decreasing on $(0, \eta)$. For $1 \leq n < N$, it is straightforward to see that $f_{n+1}(Q_n^*) = A_n - P(Q_n^*)$, and therefore

$$f_{n+1}(Q_{n+1}^*) - f_{n+1}(Q_n^*) = a_{n+1} - P(Q_n^*). \tag{13}$$

Similarly, $f_n(Q_{n+1}^*) = A_{n+1} - P(Q_{n+1}^*)$, so that

$$f_n(Q_{n+1}^*) - f_n(Q_n^*) = a_{n+1} - P(Q_{n+1}^*). \tag{14}$$

Then

$$\begin{array}{rcllll} a_{n+1} < P(Q_n^*) & \iff & f_{n+1}(Q_{n+1}^*) & < & f_{n+1}(Q_n^*) & \text{from (13)}, \\ & \iff & Q_{n+1}^* & > & Q_n^* & \text{as } f_{n+1} \text{ is decreasing}, \\ & \iff & f_n(Q_{n+1}^*) & < & f_n(Q_n^*) & \text{as } f_n \text{ is decreasing}, \\ & \iff & a_{n+1} & < & P(Q_{n+1}^*) & \text{from (14)}, \end{array}$$

and the conclusion follows. □

Proposition 2.6. Suppose that $\overline{\rho} < 2$. Then there is a unique Nash equilibrium.

Proof. If $\overline{\rho} < 2$, then Q_n^* is well-defined for all $1 \leq n \leq N$. Now, it is obvious that the single player in the one-player candidate equilibrium will not wish to leave this equilibrium. Suppose for $n < N$, we have a Nash equilibrium in which the first n players participate with positive production. If $a_{n+1} < P(Q_n^*)$, then, from Lemma 2.5, player $n+1$ wishes to enter, and he will participate in the $(n+1)$-player equilibrium, as will the other n players since their costs are lower than or equal to his. Therefore every candidate equilibrium with n or fewer players is overturned by entry. We can proceed adding players until either (i) no further players wish to enter; or (ii) there are no further players, and therefore we have uniqueness. □

We note that even if we have the case that $a_{n+1} = a_{n+2} = \cdots = a_{n+k}$ for some $k > 1$, it is sufficient just to keep adding the players with equal costs one-by-one in any order. It is clear that if player $n+1$ wishes to enter, then so will players $n+2$ through $n+k$, but it suffices to carry out the argument in unit increments. Proposition 2.6 also shows that as players wish to join candidate equilibria, the sequence $\{Q_n^* \mid 1 \leq n \leq N\}$ is first strictly increasing: $a_{n+1} < P(Q_n^*) \iff Q_{n+1}^* > Q_n^*$. It may then become constant: $Q_{n_1}^* = Q_{n_1+1}^* = ... = Q_{n_2}^*$ for some $1 \leq n_1 < n_2 \leq N$ if and only if (i) player $n_1 + 1$ is exactly indifferent between entering and remaining out of the n_1-player candidate equilibrium; and (ii) $a_{n_1+1} = ... = a_{n_2}$. Finally, if there is an $n' < N$ where $a_{n'+1} \geq P(Q_{n'}^*)$ so that the remaining players after n' are costed out of participating in any Nash equilibrium, then $Q_{n'+1}^* \geq Q_{n'}^*$, and the sequence (Q_n^*) is non-increasing thereafter, and strictly decreasing once some $a_{n+1} > P(Q_n^*)$.

When a Nash equilibrium exists and is unique, we denote the equilibrium production of player i as a function of the vector of costs $\boldsymbol{a} = (a_1, a_2, ..., a_N)$ by $q_i^*(\boldsymbol{a})$, and the equilibrium profit of player i by

$$G_i(\boldsymbol{a}) = \pi(q_i^*(\boldsymbol{a}), Q_{-i}^*(\boldsymbol{a}), a_i), \tag{15}$$

where $Q_{-i}^*(\boldsymbol{a}) = \sum_{j \neq i} q_j^*(\boldsymbol{a})$. The functions q_i^* and G_i are essential building blocks in the system of PDEs in Section 3. We have the following corollary.

Corollary 2.7. *Suppose that $\overline{\rho} < 2$. Then the unique Nash equilibrium can be constructed as follows. Let $\bar{Q}^* = \max\{Q_n^* \mid 1 \leq n \leq N\}$. Then the unique Nash equilibrium quantities are given by*

$$q_i^*(\boldsymbol{a}) = \max\left\{\frac{P\left(\bar{Q}^*\right) - a_i}{-P'\left(\bar{Q}^*\right)}, 0\right\}, \quad 1 \leq i \leq N,$$

and the corresponding profits are

$$G_i(\boldsymbol{a}) = q_i^*(\boldsymbol{a})(P(\bar{Q}^*) - a_i), \quad 1 \leq i \leq N.$$

In particular, q_i^* and G_i are Lipschitz continuous, and the number of active players in the unique equilibrium is $m = \min\left\{n \mid Q_n^* = \bar{Q}^*\right\}$.

Lipschitz continuity follows from the fact that q_i^* and G_i are constructed from compositions of $\mathcal{C}^1$ functions and max operations. Notice that kinks occur in the q_i^* and G_i only when $a_j = P\left(\bar{Q}^*\right)$ for some j, that is, when player j is exactly indifferent between participating or not. Or, to put the same point another way, the q_i^* and G_i are as smooth as P' on any region of unit-cost space $[0, P(0^+))^N$ on which the set $\{i \mid q_i^*(\boldsymbol{a}) > 0\}$ of active players is constant.

2.2 Example: Constant Prudence Price Curves

In this section we present formulae for Nash equilibria under a tractable family of price functions for which the relative prudence ρ, defined in (4), is constant. In this case, P satisfies the second-order ordinary differential equation $Q\,P'' + \rho P' = 0$. The most natural choices for the two constants of integration are the saturation point $\eta > 0$, and the slope at the saturation point, $-\zeta < 0$. With these choices, we have $P(Q) = -\zeta\eta C\left(\frac{Q}{\eta}\right)$, where

$$C(z) = \begin{cases} \dfrac{z^{1-\rho} - 1}{1 - \rho} & \text{if } \rho \neq 1, \\ \log z & \text{if } \rho = 1, \end{cases}$$

is the canonical solution with $\eta = \zeta = 1$. However, the only role for the constant ζ is to translate the units in which P is measured into the units in which a is measured, so we set $\zeta = 1$. This leaves us with

$$P(Q) = \begin{cases} \frac{\eta}{1-\rho}\left(1 - \left(\frac{Q}{\eta}\right)^{1-\rho}\right) & \rho \neq 1, \\ \eta(\log \eta - \log Q) & \rho = 1. \end{cases} \tag{16}$$

For $\rho < 1$, the choke price $P(0^+) = \eta/(1-\rho)$ is finite. For $\rho \geq 1$, the choke price is infinite. On $(0, \eta]$, the pricing curve is convex for $\rho > 0$, affine for $\rho = 0$, and concave for $\rho < 0$.

As in the general case, we work with the functions $f_n(Q) = QP'(Q) + nP(Q)$, but now we have much more precise information about these functions. We define

$$n_\rho = \max(1, \lfloor \rho \rfloor),$$

where $\lfloor \rho \rfloor$ denotes the largest integer less than or equal to ρ.

Lemma 2.8. *There is no Nash equilibrium for $n < \lfloor \rho \rfloor$. For $n \geq n_\rho$, there is a unique solution $Q_n^* \in (0, \eta)$ to $f_n(Q) = A_n$, for all $0 \leq A_n < nP(0^+)$.*

Proof. For the specific functional forms arising when ρ is constant, we have

$$f_n(Q) = (n + 1 - \rho)\, P(Q) - \eta.$$

Hence, when $\rho > 1$ and $n \leq \rho - 1 < \lfloor \rho \rfloor$, then $f_n < 0$ on $(0, \eta)$, and there is no solution Q_n^* to the equation $f_n(Q) = A_n$ for $A_n \geq 0$. When $n \geq n_\rho$, f_n is decreasing with $f_n(\eta) = -\eta$. If $P(0^+) < \infty$, then $f_n(0^+) \geq nP(0^+)$ and otherwise $f_n(0^+) = \infty$. Therefore, there is a unique solution, lying in $(0, \eta)$, to $f_n(Q) = A_n$, for $0 \leq A_n < nP(0^+)$. □

We can then prove existence by starting from the n_ρ-player candidate equilibrium and adding players until either (i) no further players wish to enter; or (ii) there are no further players. Of course, we must assume that $\rho < N + 1$, for otherwise there is no $n \leq N$ for which an n-player candidate equilibrium exists.

The only real obstacle to this program is the possibility that, in an n -player candidate equilibrium, the production level q_i of player i is not the global maximum of $\pi(\cdot, Q_{-i}, a_i)$. Since the case $\rho < 2$ is already covered by Lemma 2.4, we can restrict attention to the case in which $\rho \geq 2$. Moreover, if Q_{-i} were to vanish for some i in an n-player candidate equilibrium that was feasible in all other respects, then that candidate equilibrium would also be a one-player candidate equilibrium; and if $\rho \geq 2$ then there are no one-player candidate equilibria. We can therefore further restrict attention to the case $Q_{-i} > 0$.

Lemma 2.9. *Suppose that $2 \leq \rho < N + 1$, and $Q_{-i} > 0$. Then $g(q_i) := \pi(q_i, Q_{-i}, a_i)$ has a unique global maximum, which is attained in $[0, \eta)$.*

Proof. Now $g(0) = 0$ and $g(q_i) < 0$ for all $q_i \in [\eta, \infty)$, so g has a global maximum attained in $[0, \eta]$. From (12), we have

$$g''(q_i) = \left(2 - \frac{q_i}{Q_{-i} + q_i}\,\rho\right) P'(Q_{-i} + q_i)$$

for all $q_i \in [0,\infty)$. There are then two cases to consider. First, if $\rho = 2$, then $g'' < 0$ everywhere on $[0,\infty)$, and so g has a global maximum attained in $[0,\eta)$. Second, if $\rho > 2$, then $g'' < 0$ on $\left[0, \frac{2Q_{-i}}{\rho-2}\right)$ and $g'' > 0$ on $\left(\frac{2Q_{-i}}{\rho-2}, \infty\right)$. Since $g'' > 0$ on $\left(\frac{2Q_{-i}}{\rho-2}, \infty\right)$ and g is bounded above, we must have $g' < 0$ on $\left[\frac{2Q_{-i}}{\rho-2}, \infty\right)$, and uniqueness follows. □

Proposition 2.10. *Suppose that $\rho < N+1$. Then there is a unique Nash equilibrium given as follows:*

$$q_i^*(\boldsymbol{a}) = \left(\frac{\bar{Q}}{\eta}\right)^{\rho} \max\left\{\bar{P} - a_i, 0\right\}, \quad 1 \le i \le N,$$

where

$$\bar{P} = \min\left\{P_n \mid n_\rho \le n \le N\right\}, \qquad P_n = \frac{A_n + \eta}{n+1-\rho}, \quad n_\rho \le n \le N, \tag{17}$$

and

$$\bar{Q} = \begin{cases} \eta\left(1 - (1-\rho)\frac{\bar{P}}{\eta}\right)^{\frac{1}{1-\rho}} & \text{if } \rho \neq 1, \\ \\ \eta \exp\left(-\frac{\bar{P}}{\eta}\right) & \text{if } \rho = 1. \end{cases}$$

The corresponding profits are $G_i(\boldsymbol{a}) = q_i^*(\boldsymbol{a})(\bar{P} - a_i)$. In particular, q_i^* and G_i are Lipschitz continuous, and the number of active players in the unique equilibrium is

$$m = \min\left\{n \mid n_\rho \le n \le N, \quad P_n = \bar{P}\right\}.$$

Proof. For the constant prudence price curves, the formulas are best expressed in terms of the equilibrium prices. By direct calculation, for each $n_\rho \le n \le N$, the unique solution to $f_n(Q) = A_n$ is given by

$$P(Q_n^*) = \frac{A_n + \eta}{n+1-\rho} =: P_n, \tag{18}$$

since $f_n(Q) = (n+1-\rho)P(Q) - \eta$. Computing P^{-1} gives

$$Q_n^* = \begin{cases} \eta\left(1 - (1-\rho)\frac{P_n}{\eta}\right)^{\frac{1}{1-\rho}} & \text{if } \rho \neq 1, \\ \eta \exp\left(-\frac{P_n}{\eta}\right) & \text{if } \rho = 1. \end{cases}$$

The candidate n-player equilibria are

$$q_i^* = \frac{P_n - a_i}{-P'(Q_n^*)} = \left(\frac{Q_n^*}{\eta}\right)^{\rho} (P_n - a_i).$$

We prove existence and uniqueness as in Proposition 2.6. The only difference is that we now confine attention to $n_\rho \le n \le N$. We start with the n_ρ-player candidate equilibrium. If $n_\rho = 1$, then it is obvious player one will wish to make profit and not leave. If $n_\rho > 1$, then $0 < n_\rho + 1 - \rho \le 1$ and the candidate price P_{n_ρ} is guaranteed to exceed the cost of player n_ρ:

$$P_{n_\rho} = \frac{A_{n_\rho} + \eta}{n_\rho + 1 - \rho} \ge A_{n_\rho} + \eta > a_{n_\rho}.$$

That is, player n_ρ (and therefore all players $i < n_\rho$) will not want to leave the n_ρ-player candidate equilibrium.

Then Lemma 2.5, Proposition 2.6 and Corollary 2.7 hold as in the general case, except we no longer need the restriction $\bar{\rho} < 2$ to guarantee unique global maxima. It remains to characterize the transition point where players stop entering the game. As in Corollary 2.7, this occurs when the equilibrium aggregate production level Q_n^* first becomes non-increasing, or we reach the maximum number of players N. This is equivalent to when the candidate prices P_n first become non-decreasing or we reach N, and hence the Nash equilibrium price is given by $\bar{P}$. $\square$

As in the general case, kinks occur in the q_i^* and G_i only when $a_j = P\left(\bar{Q}\right)$ for some j, that is when player j is exactly indifferent between producing or not. Moreover the q_i^* and G_i are as smooth as P' on any region of unit-cost space $[0, P(0^+))^N$ on which the set $\{i \mid q_i^*(\boldsymbol{a}) > 0\}$ of active players is constant.

Remark 2.11. We note that the general formula for P_n (obtained without setting $\zeta = 1$) is $(A_n + \zeta\,\eta)/(n+1-\rho)$, which resolves the seemingly inconsistent dimensions in the numerator of (17).

Remark 2.12. There is an even more explicit characterization of m. Consider the function h given by the formula $h(n) = \eta + A_{n-1} - (n-\rho)\,a_n$. The requirement that $a_1 < P(0^+)$ ensures that $h(1) = \eta - (1-\rho)\,a_1 > 0$. Moreover $h(n+1) - h(n) = (\rho - n)\,(a_{n+1} - a_n)$. Hence $h(n+1) - h(n) \geq 0$ if and only if $n \leq \rho$. Then m is the largest $n \in \{1, 2, ..., N\}$ such that $h(n) > 0$.

In the numerical solutions in Section 5.3, we shall use the linear price function corresponding to $\rho = 0$, that is, $P(Q) = \eta - Q$. In this case, the market price is

$$\bar{P} = \min\left\{\frac{A_n + \eta}{n+1} \mid 1 \leq n \leq N\right\}; \tag{19}$$

the quantity produced by player i is $q_i^*(\boldsymbol{a}) = \max\left\{\bar{P} - a_i, 0\right\}$ for all $1 \leq i \leq N$; and the profit of player i is $G_i(\boldsymbol{a}) = q_i^*(\boldsymbol{a})(\bar{P} - a_i) = \left(\max\left\{\bar{P} - a_i, 0\right\}\right)^2$. Alternatively, in terms of the number of active players $m = \min\{1 \leq n \leq N \mid P_n = \bar{P}\}$, we have

$$q_i^*(\boldsymbol{a}) = \frac{1}{m+1}\left(\eta - m a_i + \sum_{j=1, j\neq i}^{m} a_j\right), \quad G_i(\boldsymbol{a}) = \left(q_i^*(\boldsymbol{a})\right)^2,$$

for $1 \leq i \leq m$, and $q_i^* = G_i = 0$ for $i > m$.

Finally, we mention that ordering of the players by costs is, of course, not crucial to defining the functions q_i^* and G_i in this section. Given a general costs vector, the constructions above are simply modified to first temporarily relabel the firms according to their costs, compute the Nash equilibrium as above, and then return the equilibrium quantities and profits in the original labelling order.

3 Differential Game & Exhaustibility

We now introduce the dynamic Cournot game under exhaustibility constraints. Each player i has reserves of a traditional and cheap-to-produce resource (for example oil, by extraction), denoted by $x_i(t)$ at time $t \geq 0$. We take for simplicity the cost of production from this source to be zero, but reserves are finite (exhaustible). There is also an alternative source that is inexhaustible, but expensive-to-produce (solar power in the energy example), with constant unit cost of production $c \in [0, P(0^+))$.

Player i chooses a dynamic production rate $\bar{q}_i$ that is a *Markov strategy*:[2] $\bar{q}_i = \bar{q}_i(\boldsymbol{x}(t))$, where $\boldsymbol{x}(t) = (x_1(t), \cdots, x_N(t))$. As long as $x_i > 0$, player i has the choice between producing from the cheap or expensive sources. After x_i hits zero, he can only produce from the costlier alternative, which never runs out. We shall suppose, at first, that no player produces from the more expensive source as long as the cheaper one is available,[3] and we will discuss how this could be validated *a posteriori*. Therefore, reserves of his traditional resource deplete according to

$$\frac{dx_i}{dt} = -\bar{q}_i(\boldsymbol{x}(t)), \quad x_i > 0.$$

(To lighten the notation, we do not denote the dependence of $\boldsymbol{x}$ on the $\bar{q}_i$.)

The market price is governed by a Cournot competition with the price function P as before, satisfying Assumption 2.2. We assume that, given a cost vector $\boldsymbol{a}$ satisfying (3), there is a unique Nash equilibrium $\boldsymbol{q}^*(\boldsymbol{a})$ of the *static* Cournot game. Some general conditions for this were given in Proposition 2.6, and, for a specific family of price functions, in Proposition 2.10.

3.1 Dynamic Cournot Competition

Given initial reserves $x_i(0) \geq 0$, player i wants to maximize his discounted lifetime profit

$$\int_0^\infty e^{-rt} \pi\left(\bar{q}_i(\boldsymbol{x}(t)), \bar{Q}_{-i}(\boldsymbol{x}(t)), c\mathbb{I}_{\{x_i(t)=0\}}\right) dt,$$

where $\mathbb{I}$ denotes the indicator function, $r > 0$ is a discount rate, the profit function π was defined in (1), and $\bar{Q}_{-i} = \sum_{j \neq i} \bar{q}_j$. Note that the cost of production rises from zero to c when reserves x_i run out, as denoted in the third argument of π in the integral.

We look for a *Markov Perfect* Nash equilibrium $\bar{\boldsymbol{q}}^*(\boldsymbol{x}(t)) = (\bar{q}_1^*(\boldsymbol{x}(t)), \cdots, \bar{q}_N^*(\boldsymbol{x}(t)))$ such that, for each player i, and each initial state $\boldsymbol{x}(0)$, $\bar{q}_i^*$ is the best response when all the other players play their equilibrium strategies. Therefore, with the notation $\bar{Q}_{-i}^* = \sum_{j \neq i} \bar{q}_j^*$,

$$\int_0^\infty e^{-rt} \pi\left(\bar{q}_i^*(\boldsymbol{x}(t)), \bar{Q}_{-i}^*(\boldsymbol{x}(t)), c\mathbb{I}_{\{x_i(t)=0\}}\right) dt \geq \int_0^\infty e^{-rt} \pi\left(\bar{q}_i(\boldsymbol{x}(t)), \bar{Q}_{-i}^*(\boldsymbol{x}(t)), c\mathbb{I}_{\{x_i(t)=0\}}\right) dt,$$

for any Markov strategy $\bar{q}_i$ of player i, and for all $\boldsymbol{x}(0) \in \mathbb{R}_+^N = [0, \infty)^N$. The requirement that the equilibrium strategies are independent of the initial resource levels $\boldsymbol{x}(0)$ is equivalent,

[2]Markov strategies are also sometimes called *feedback* or *closed-loop* strategies.

[3]According to the lead editorial in The Times of London on 13 July, 2009: "No sane energy company would, while fossil fuels are still plentiful, voluntarily opt for a more expensive, less reliable energy source."

in our setting, to the requirement that the equilibrium is *perfect*, sometimes called subgame perfect. This excludes equilibria with so-called "incredible threats" whereby players may make extreme, but unrealistic, threats of increased production if another player deviates from a certain path. We refer to [15] and the textbooks [9, 10] for further discussion and references on this issue.

We give an informal motivation for the dynamic programming PDEs we shall use to construct Nash equilibria for these problems. First, consider any continuous Markov strategy $\{\bar{q}_j(\boldsymbol{x(t)}) \mid t \geq 0, 1 \leq j \leq N\}$, and the profits starting at time $s \geq 0$:

$$v_i^{\bar{q}}(\boldsymbol{x}(s)) = \int_s^\infty e^{-r(t-s)}\pi\left(\bar{q}_i(\boldsymbol{x}(t)), \bar{Q}_{-i}(\boldsymbol{x}(t)), c\mathbb{I}_{\{x_i(t)=0\}}\right) dt, \quad 1 \leq i \leq N. \tag{20}$$

Let $\boldsymbol{x} = \boldsymbol{x}(0)$ denote any interior point ($x_j > 0$, for all $1 \leq j \leq n$), so all players start with some initial reserves.[4] Then, differentiating (20) with respect to s and setting $s = 0$ gives the partial differential equation

$$\pi(\bar{q}_i(\boldsymbol{x}), \bar{Q}_{-i}(\boldsymbol{x}), 0) - \sum_{j=1}^N \bar{q}_j(\boldsymbol{x})\frac{\partial v_i^{\bar{q}}}{\partial x_j} - rv_i^{\bar{q}} = 0, \tag{21}$$

which can be re-written as

$$\pi\left(\bar{q}_i(\boldsymbol{x}), \bar{Q}_{-i}(\boldsymbol{x}), \frac{\partial v_i^{\bar{q}}}{\partial x_j}\right) - \sum_{j\neq i} \bar{q}_j(\boldsymbol{x})\frac{\partial v_i^{\bar{q}}}{\partial x_j} - rv_i^{\bar{q}} = 0. \tag{22}$$

Then Bellman-principle arguments[5] reduce each player's optimization problem to a local optimization, and the search for a Markov perfect equilibrium to a search for a local static Nash equilibrium, which, in this case, amounts to optimizing π in (22) with respect to its first argument, with the second argument fixed at the other players' equilibrium strategies. Writing $v_i = v_i^{\bar{q}^*}$ for the value functions using the equilibrium policies:

$$v_i(\boldsymbol{x}) = \int_0^\infty e^{-rt}\pi\left(\bar{q}_i^*(\boldsymbol{x}(t)), \bar{Q}_{-i}^*(\boldsymbol{x}(t)), c\mathbb{I}_{\{x_i(t)=0\}}\right) dt,$$

we have

$$\max_{q_i\geq 0}\left[\pi\left(q_i, \bar{Q}_{-i}^*(\boldsymbol{x}), \frac{\partial v_i}{\partial x_i}\right)\right] - \sum_{j\neq i} \bar{q}_j^*(\boldsymbol{x})\frac{\partial v_i}{\partial x_j} - rv_i = 0, \quad i = 1, \cdots, N. \tag{23}$$

where $\boldsymbol{x} \in \mathbb{R}_+^N$ with $x_j > 0$ for $1 \leq j \leq N$, that is when all players have some reserve of the traditional resource.

We assume throughout that each v_i is continuously differentiable up to the axes: $v_i \in \mathcal{C}^1(\mathbb{R}_+^N)$; and that $\bar{q}_i^*(\boldsymbol{x})$ is continuous at all $\boldsymbol{x} \in \mathbb{R}_+^N$. We observe from (23) that $\frac{\partial v_i}{\partial x_i}$ enters as a "shadow cost" for player i at the differential level. The interpretation as a cost is legitimate as we naturally expect $\frac{\partial v_i}{\partial x_i} \geq 0$: adding more reserves increases the value function.

[4]The boundary cases when some $x_j = 0$ are dealt with in Section 3.2.

[5]See, for instance, [16], [8, Section 8.2], [1, Section 6.5.2], or [6, Section 4.2].

Comparison with (2) reveals the differential Nash equilibrium problem in the PDEs is just the one-period game with the role of the costs a_i played by the partial derivatives $\partial v_i/\partial x_i$. For a fixed $\boldsymbol{x} \in \mathbb{R}_+^N$, if there is a unique Nash equilibrium $\boldsymbol{q}^*(\boldsymbol{a})$ for the static game with

$$a_i = \frac{\partial v_i}{\partial x_i}(\boldsymbol{x}), \quad i = 1, \cdots, N,$$

then we re-write equations (23) as

$$G_i(\mathcal{D}\boldsymbol{v}) - \sum_{j \neq i} q_j^*(\mathcal{D}\boldsymbol{v}) \frac{\partial v_i}{\partial x_j} - r v_i = 0, \quad i = 1, \cdots, N, \tag{24}$$

where we define

$$\mathcal{D}\boldsymbol{v} = \mathrm{diag}(\nabla \boldsymbol{v}) = \left(\frac{\partial v_1}{\partial x_1}, \cdots, \frac{\partial v_N}{\partial x_N} \right),$$

and recall that $G_i(\boldsymbol{a}) = q_i^*(\boldsymbol{a})(P(Q^*) - a_i)$ is equilibrium profit function of the static game. The equilibrium production rates of the exhaustible resource at time t are given by $\bar{q}_i^*(\boldsymbol{x}(t)) = q_i^*(\mathcal{D}\boldsymbol{v}(\boldsymbol{x}(t)))$.

Note that the definition of Nash equilibrium and the constructions of $q_i^*(\boldsymbol{a})$ and $G_i(\boldsymbol{a})$ in Propositions 2.6 and 2.10 take care of the fact that not all players may participate at all resource levels $\boldsymbol{x}$, depending on the vector of shadow costs $\boldsymbol{a} = \mathcal{D}\boldsymbol{v}(\boldsymbol{x})$ at that point. However they encompass the fact that there are always potentially N active players. For the majority of the paper, we shall treat the cases where all players participate, but we shall discuss the situation where one player may be blockaded in the two-player dynamic game in Section 5.

At this level of generality, we are not able to provide reasonable conditions for existence and uniqueness of a solution to the system (24), equipped with appropriate boundary conditions discussed in the following section, let alone solutions with sufficient regularity to generate a unique Nash equilibrium with well-behaved strategies. We will proceed by staying relatively close to a case which is well-understood. In the next section, we address the issue of exhaustibility and boundary conditions.

3.2 Exhaustibility

When a player has exhausted his reserves of the cheap resource, he can turn to the alternative means of production which, while more costly than the original one, allows the exhausted player to remain in the game, but in a disadvantaged position. In the energy example, there are alternative "backstop" technologies, such as solar power or steam-extracted oil shales, that an energy supplier may resort to when his reserves of oil run out, both of which are more expensive than delivering energy by extracting oil.

We consider the case $x_i = 0$, when player i has exhausted his supply. Then we have

$$\frac{dx_i}{dt} = 0.$$

The HJ equation for $v_i(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_N)$ becomes

$$\max_{q_i \geq 0} \pi\left(q_i, \bar{Q}_{-i}^*(\boldsymbol{x}), c\right) - \sum_{j \neq i} q_j^*(\mathcal{D}\boldsymbol{v}) \frac{\partial v_i}{\partial x_j} - r v_i = 0,$$

or

$$G_i(\mathcal{D}_{-i}\boldsymbol{v}) - \sum_{j \neq i} q_j^* \left(\mathcal{D}_{-i}\boldsymbol{v}\right) \frac{\partial v_i}{\partial x_j} - r v_i = 0, \tag{25}$$

where we define

$$\mathcal{D}_{-i}\boldsymbol{v} = \left(\frac{\partial v_1}{\partial x_1}, \cdots, \frac{\partial v_{i-1}}{\partial x_{i-1}}, c, \frac{\partial v_{i+1}}{\partial x_{i+1}}, \cdots, \frac{\partial v_N}{\partial x_N}\right).$$

Similarly, the HJ equation for $v_j(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_N), j \neq i$ becomes

$$G_j(\mathcal{D}_{-i}\boldsymbol{v}) - \sum_{k \notin \{i,j\}} q_k^*(\mathcal{D}_{-i}\boldsymbol{v}) \frac{\partial v_j}{\partial x_k} - r v_j = 0. \tag{26}$$

These are then a system of N equations on the hyperplane $x_i = 0$ involving only partial derivatives along the plane. We can proceed similarly to the cases in which more and more players have exhausted their reserves until we reach the fully exhausted case. Here, all the players are using the inexhaustible alternative resource, and so produce at the constant rate $q_i^*(c\mathbf{1})$, where $\mathbf{1}$ denotes the N-vector of ones. It follows that the value functions are just the constant

$$v_i(\mathbf{0}) = \frac{G_i(c\mathbf{1})}{r}. \tag{27}$$

This serves as the initial condition for the one-player ODEs on the lines. Once solved, these are axis Dirichlet boundary conditions for the two-player PDEs on the planes, and so on.

Now if the value functions are found to satisfy

$$\frac{\partial v_i}{\partial x_i} < c \quad \text{in } \{x_i > 0\},$$

then our initial hypothesis, that no producer will use the alternative technology while the cheaper one is available, will be validated. This is clear because if we introduced additional control variables in the interior so that each player could choose both a quantity from reserves and a quantity from the alternative source, then in each game at the differential level, none would be produced from the alternative as long as the shadow cost $\frac{\partial v_i}{\partial x_i}$ is smaller than c. This is indeed the case for the approximate solutions in Section 4.2, and the numerical solutions of Section 5.3, and we shall assume it to be the case for the remainder of the paper.

3.3 Neumann Boundary Conditions

When all players participate at all resource levels, it is possible to replace the Dirichlet boundary conditions coming from (26) by simpler Neumann boundary conditions. We reiterate that the Dirichlet conditions for the value functions are found by solving the chain of games starting with the case of all players exhausted, then the game where one has reserves and $N-1$ players are using the alternative technology, up to $N-1$ with reserves and one using the alternative.

Recall that we have assumed that we have continuity of the first derivatives of the value functions up to the boundaries and continuity of the equilibrium policies $\bar{q}_i^*(\boldsymbol{x})$ at all $\boldsymbol{x} \in \mathbb{R}_+^N$.

Fix $1 \leq i \leq N$ and a point $\boldsymbol{x} \in \mathbb{R}_+^N$ with all $x_j > 0$, and let $\hat{\boldsymbol{x}}$ denote the projection of $\boldsymbol{x}$ onto $\{x_i = 0\}$. For player i, his equilibrium production on $x_i = 0$ is given by

$$\bar{q}_i^*(\hat{\boldsymbol{x}}) = q_i^*(\mathcal{D}_{-i}\boldsymbol{v}(\hat{\boldsymbol{x}})) = q_i^*\left(\frac{\partial v_1}{\partial x_1}(\hat{\boldsymbol{x}}), \cdots, \frac{\partial v_{i-1}}{\partial x_{i-1}}(\hat{\boldsymbol{x}}), c, \frac{\partial v_{i+1}}{\partial x_{i+1}}(\hat{\boldsymbol{x}}), \cdots, \frac{\partial v_N}{\partial x_N}(\hat{\boldsymbol{x}})\right),$$

whereas in the interior it is

$$\bar{q}_i^*(\boldsymbol{x}) = q_i^*(\mathcal{D}\boldsymbol{v}(\boldsymbol{x})) = q_i^*\left(\frac{\partial v_1}{\partial x_1}(\boldsymbol{x}), \cdots, \frac{\partial v_N}{\partial x_N}(\boldsymbol{x})\right).$$

From continuity of $\bar{q}_i^*(\boldsymbol{x})$, $q_i^*(\mathcal{D}\boldsymbol{v}(\boldsymbol{x})) \to q_i^*(\mathcal{D}_{-i}\boldsymbol{v}(\hat{\boldsymbol{x}}))$ as $\boldsymbol{x} \to \hat{\boldsymbol{x}}$. In general, we expect that $q_i^*(\boldsymbol{a})$ is a continuous function, which was true in the cases of Corollary 2.7 and Proposition 2.10. Further, we expect the static equilibrium quantity $q_i^*(\boldsymbol{a})$ to be decreasing in its i-th argument, as long as $q_i^* > 0$. Specifically, if it is the case that

$$\frac{\partial q_i^*}{\partial a_i}(\mathcal{D}_{-i}\boldsymbol{v}(\hat{\boldsymbol{x}})) < 0 \quad \text{and} \quad q_i^*(\mathcal{D}_{-i}\boldsymbol{v}(\hat{\boldsymbol{x}})) > 0,$$

then we can conclude that

$$\frac{\partial v_i}{\partial x_i} = c \quad \text{on } x_i = 0. \tag{28}$$

The interpretation of this expression is that, on hitting the boundary $x_i = 0$, the shadow cost of player i turns into the real cost c. As long as player i participates in the game on $x_i = 0$, then we have the Neumann boundary condition (28).

Comparison of (26) with (24), which we re-write as

$$G_j(\mathcal{D}\boldsymbol{v}) - \sum_{k \neq j} q_k^*(\mathcal{D}\boldsymbol{v})\frac{\partial v_j}{\partial x_k} - rv_j = 0,$$

yields

$$\frac{\partial v_j}{\partial x_i} = 0 \quad \text{on } x_i = 0, \quad j \neq i \tag{29}$$

provided $q_i^*(\mathcal{D}_{-i}\boldsymbol{v}) \neq 0$. Therefore, as long as player i still participates in the game on $x_i = 0$, when he is forced to use the alternative technology, the shadow cost of the other players is zero.

Indeed, when the cost c is small enough, all players will participate at all resource levels. For example, in the case of the constant prudence price curves of Section 2.2, we need only consider the extreme case where there are $N-1$ producers with the minimum unit cost of zero, and one with the maximum possible unit cost c. If $\rho \in [N, N+1)$, then only a full N-player Nash equilibrium is possible. As the candidate market price from (17) is

$$P_N = \frac{\eta + c}{N + 1 - \rho} > c,$$

since $0 < N + 1 - \rho \leq 1$, all players participate in this case for any $c < \infty$. In the case $\rho < N$, the candidate market price $P_{N-1} = \eta/(N - \rho)$ exceeds c if

$$c < \frac{\eta}{N - \rho}, \tag{30}$$

so all players participate if c is smaller than this bound.

In the case $c = 0$, the system (24) with Neumann boundary conditions (28)-(29) has the constant solution

$$v_i^{(0)}(\boldsymbol{x}) = \frac{G_i(\mathbf{0})}{r}.$$

This corresponds to both the alternative and traditional technologies having zero cost of production, so supplies are inexhaustible, and players produce at the same constant rate $q_i^*(\mathbf{0})$, which corresponds to the static game being played repeatedly. This, of course, is the inexhautible limiting case, since the alternative technology is also costless and so the resource is effectively in infinite supply. The static zero-cost game is played repeatedly and

$$v_i^{(0)}(\boldsymbol{x}) = \int_0^\infty e^{-rt} G_i(\mathbf{0})\, dt = \frac{G_i(\mathbf{0})}{r}. \tag{31}$$

In Section 4.2, we develop an approximation for small $c > 0$.

We remark that the inexhaustible limit (31) is also the behaviour for large discounting rate r or large resources. Indeed if we write the value functions as $v_i(\boldsymbol{x}; r)$ to stress the discount rate, it is easy to check from the PDEs (24) and boundary conditions (28) and (29), that

$$v_i(\boldsymbol{x}; r) = \frac{1}{r} v_i(r\boldsymbol{x}; 1),$$

so formal asymptotics in the limits of large (or small) r or $||\boldsymbol{x}||$ are analogous calculations.

4 Small Exhaustibility Approximation

When c is small, we are close to the inexhaustible game played repeatedly, and we may expect that all players participate at all resource levels. In preparation for an approximation in this case for the dynamic game, we first analyze the effect of small-costs on the static game.

4.1 Static Game Small Cost Perturbation

We return to the static game, whose Nash optimal strategies are given by $q_i^*(\boldsymbol{a})$, and equilibrium profits are

$$G_i(\boldsymbol{a}) = q_i^*(\boldsymbol{a})(P(Q^*) - a_i), \tag{32}$$

where Q^* satisfies (7). We assume costs $\boldsymbol{a}$ are such that all players participate in the equilibrium and the $q_i^*(\boldsymbol{a})$ (and hence the $G_i(\boldsymbol{a})$) are differentiable. A small costs Taylor expansion gives

$$G_i(\boldsymbol{a}) \approx G_i(\mathbf{0}) + A a_i + B \sum_{j \neq i} a_j,$$

where we define the constants

$$A = \frac{\partial G_i}{\partial a_i}(\mathbf{0}), \qquad B = \frac{\partial G_i}{\partial a_j}(\mathbf{0}), \quad j \neq i, \tag{33}$$

which are independent of i and j. Similarly, the strategies are given approximately by

$$q_i^*(\boldsymbol{a}) \approx \gamma + \lambda a_i + \mu \sum_{j \neq i} a_j, \tag{34}$$

where we define

$$\gamma = q_i^*(\mathbf{0}), \qquad \lambda = \frac{\partial q_i^*}{\partial a_i}(\mathbf{0}), \qquad \mu = \frac{\partial q_i^*}{\partial a_j}(\mathbf{0}), \quad j \neq i, \tag{35}$$

which are again independent of i and j. Further, we define the constant of relative prudence at the zero cost equilibrium solution by

$$\rho_0 = -N\gamma \, \frac{P''(N\gamma)}{P'(N\gamma)}, \tag{36}$$

which is just $\rho(N\gamma)$, where $\rho(q)$ was defined in (4).

Then we have the following expressions for (A, B, λ, μ) in terms of $(\gamma, \rho_0, P'(N\gamma))$.

Proposition 4.1. We assume $\rho_0 < (N+1)$. The perturbation coefficients (A, B, λ, μ) can be expressed as

$$A = -\gamma \left[\frac{2N - (2 - N^{-1})\rho_0}{(N+1) - \rho_0} \right], \tag{37}$$

$$B = \gamma \left[\frac{2 - N^{-1}\rho_0}{(N+1) - \rho_0} \right], \tag{38}$$

$$\lambda = \frac{1}{P'(N\gamma)} \left[\frac{N - (1 - N^{-1})\rho_0}{(N+1) - \rho_0} \right], \tag{39}$$

$$\mu = -\frac{1}{P'(N\gamma)} \left[\frac{1 - N^{-1}\rho_0}{(N+1) - \rho_0} \right]. \tag{40}$$

Proof. Let $\theta = \lambda + (N-1)\mu$. Differentiating the summed first-order conditions (7) with respect to a_i and setting $\boldsymbol{a} = \mathbf{0}$ gives

$$(N+1)\theta P'(N\gamma) - 1 + N\gamma\theta P''(N\gamma) = 0,$$

which gives

$$\theta = \frac{1}{(N+1)P'(N\gamma) + N\gamma P''(N\gamma)} = \frac{1}{P'(N\gamma)((N+1) - \rho_0)}.$$

Differentiating (6) with respect to a_i and setting $\boldsymbol{a} = \mathbf{0}$ gives

$$\theta P'(N\gamma) - 1 + \lambda P'(N\gamma) + \gamma\theta P''(N\gamma) = 0,$$

which yields

$$\lambda = \frac{1}{P'(N\gamma)}(1 - \theta(P'(N\gamma) + \gamma P''(N\gamma))).$$

Re-arranging leads to (39) and (40).

Next, differentiating (32) with respect to a_i and setting $\boldsymbol{a} = \mathbf{0}$ gives

$$A = \lambda P(N\gamma) + \gamma(\theta P'(N\gamma) - 1).$$

But, from (7) with $A_N = \sum_j a_j = 0$, we have $P(N\gamma) = -\gamma P'(N\gamma)$. Therefore,

$$A = \gamma(\theta - \lambda)P'(N\gamma) - \gamma = \gamma(N-1)\mu P'(N\gamma) - \gamma,$$

and (37) follows after substituting from (40). Similarly, differentiating (32) with respect to a_j $(j \neq i)$ and setting $\boldsymbol{a} = \mathbf{0}$ gives

$$B = \gamma(\theta - \mu)P'(N\gamma),$$

and (38) follows after substitution for θ and μ. □

We comment that since $\rho_0 < (N+1)$ (and $N \geq 2$), the formulas (38-40) imply that

$$B > 0, \quad \text{and} \quad \lambda < 0, \tag{41}$$

while (37) implies that

$$A \leq 0 \quad \text{for} \quad \rho_0 \leq \frac{2N^2}{2N-1}, \tag{42}$$

and $A > 0$ otherwise. Formula (40) yields

$$\mu \geq 0 \quad \text{for} \quad \rho_0 \leq N,$$

and $\mu < 0$ otherwise. In other words, player i's profits increase when any other player's cost a_j is increased from zero, and he also increases his production for $\rho_0 \leq N$. When his own cost a_i is increased from zero, he decreases his production, but his profit may increase or decrease depending on ρ_0.

Finally, if costs for all the players are increased by the same amount, $a_i = c, \forall i$, then

$$G_i(c\mathbf{1}) \approx G_i(\mathbf{0}) + (A + (N-1)B)c,$$

so each player's equilibrium profit increases with cost according to the sign of $A + (N-1)B$. From (37) and (38), we have

$$A + (N-1)B = \gamma\frac{(\rho_0 - 2)}{(N+1) - \rho_0},$$

so profits actually increase with costs for $\rho_0 > 2$, suggesting that sufficient curvature induces a degree of mutually beneficial production. Similarly, the optimal production quantities under a symmetric increase in costs are approximated as

$$q_i^*(c\mathbf{1}) \approx q_i^*(\mathbf{0}) + (\lambda + (N-1)\mu)c.$$

From (39) and (40),

$$\lambda + (N-1)\mu = \frac{1}{P'(N\gamma)((N+1) - \rho_0)} < 0, \tag{43}$$

so each player's equilibrium production decreases with an across-the-board cost increase.

4.2 Differential Game Small Cost Perturbation

We look for an approximate solution of (24), with boundary conditions (28)-(29), of the form

$$v_i(\boldsymbol{x}) = \frac{G_i(\mathbf{0})}{r} + cv_i^{(1)}(\boldsymbol{x}) + o(c),$$

for some functions $v_i^{(1)}$ to be found. Inserting the expansion into the PDEs and boundary conditions leads to the linearized system

$$\begin{aligned} A\frac{\partial v_i^{(1)}}{\partial x_i} + B\sum_{j\neq i}\frac{\partial v_j^{(1)}}{\partial x_j} - \gamma\sum_{j\neq i}\frac{\partial v_i^{(1)}}{\partial x_j} - rv_i^{(1)} &= 0, & (44)\\ \frac{\partial v_i^{(1)}}{\partial x_i}\,|_{x_i=0} &= 1, & (45)\\ \frac{\partial v_i^{(1)}}{\partial x_j}\,|_{x_i=0} &= 0\ (j\neq i), & (46)\end{aligned}$$

where (A, B, γ) were defined in (33) and (35). We have the following explicit solution.

Proposition 4.2. Assume that

$$\rho_0 < \frac{2N^2}{(2N-1)}. \tag{47}$$

Then the small exhaustibility corrections to the value functions are given by

$$v_i^{(1)}(\boldsymbol{x}) = \begin{cases} \frac{A}{r}e^{\frac{r}{A}x_i} + \sum_{j\neq i}\frac{B}{r(\gamma+A)}\left(Ae^{\frac{r}{A}x_j} + \gamma e^{-\frac{r}{\gamma}x_j}\right), & A\neq-\gamma \quad (\iff \rho_0\neq N),\\ \frac{A}{r}e^{\frac{r}{A}x_i} + \sum_{j\neq i}\frac{B}{r}e^{\frac{r}{A}x_j}, & A=-\gamma \quad (\iff \rho_0 = N).\end{cases} \tag{48}$$

Proof. We make the additively separable *ansatz*:

$$v_i^{(1)}(\boldsymbol{x}) = g(x_i) + \sum_{j\neq i} f(x_j),$$

for some functions f and g, and the solution follows immediately from (44)-(46). The assumption $\rho_0 < 2N^2/(2N-1)$ guarantees that $A < 0$ from (42). Therefore, the terms in (48) go to zero at large resource levels, corresponding to the inexhaustible limit. □

It follows from (34) with the replacements

$$a_i \mapsto c\frac{\partial v_i^{(1)}}{\partial x_i}, \qquad a_j \mapsto c\frac{\partial v_j^{(1)}}{\partial x_j},$$

that the optimal strategies under small exhaustibility are approximated by

$$q_i^*(\boldsymbol{x}) = \gamma + c\left(\lambda e^{\frac{r}{A}x_i} + \mu\sum_{j\neq i} e^{\frac{r}{A}x_j}\right) + o(c). \tag{49}$$

4.3 Time to Exhaustion in the Symmetric Game

When there is a small cost of alternative technology c, the resource levels $x_i(t)$ diminish according to the approximate dynamics

$$\frac{dx_i}{dt} = -\left[\gamma + c\left(\lambda e^{\frac{r}{A}x_i} + \mu\sum_{j\neq i} e^{\frac{r}{A}x_j}\right)\right], \quad i = 1, \cdots, N. \tag{50}$$

In the symmetric game, when the initial reserves are equal, $x_i \equiv x_0$, for $i = 1, \cdots, N$, it is clear that $x_i(t) \equiv x(t)$, where $x(t)$ solves

$$\frac{dx}{dt} = -\gamma - c(\lambda + (N-1)\mu)e^{rx/A}, \qquad x(0) = x_0.$$

The solution is given by

$$x(t) = -\gamma t - \frac{A}{r}\log\left(e^{-\frac{r}{A}x_0} + \frac{\tilde{c}}{\gamma}(e^{-\gamma\frac{r}{A}t} - 1)\right), \tag{51}$$

where $\tilde{c} := -c(\lambda + (N-1)\mu) > 0$.

We compute the time t_f when resources run out, namely $x(t_f) = 0$. When there is no alternative technology ($c = 0$), the inexhaustible game is played repeatedly with constant production rate γ. In this case, the time to use x_0 units is simply x_0/γ. When $c > 0$ but is small, it follows from (51) that

$$t_f = \frac{1}{\gamma}x_0 - \frac{A}{\gamma r}\log\left(\frac{\gamma - \tilde{c}e^{\frac{r}{A}x_0}}{\gamma - \tilde{c}}\right). \tag{52}$$

This formula is illustrated in Figure 1 using parameter values for $(\gamma, A, \lambda, \mu)$ corresponding to the pricing curves (16) of constant ρ. Note that, for relatively small c, the effect of exhaustibility is to slow down extraction, with diminishing effect as ρ_0 increases (left panel). The effect becomes smaller as N increases (right).

4.4 Illustration: Duopoly

We illustrate the effects of exhaustibility using the two-player duopoly problem. The use of asymptotic expansions used to analyze equilibria for differential games is not common, but we mention [3] which studied a very different kind of duopoly model with small parameter approximations. When $N = 2$, we shall use the notation

$$(x_1, x_2) = (x, y), \qquad (v_1, v_2) = (v, w).$$

The system of equations (24) are then

$$G_1(v_x, w_y) - q_2^*(v_x, w_y)v_y - rv = 0, \tag{53}$$
$$G_2(v_x, w_y) - q_1^*(v_x, w_y)w_x - rw = 0, \tag{54}$$

with exhaustibility boundary conditions

$$v_x = c, w_x = 0 \quad \text{on } x = 0; \qquad v_y = 0, w_y = c \quad \text{on } y = 0, \tag{55}$$

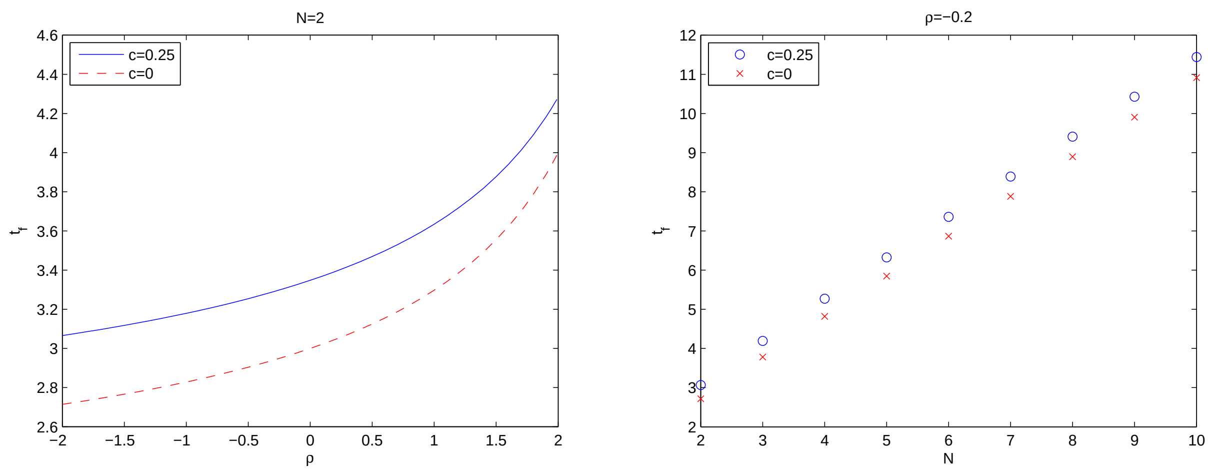

Figure 1: *Time t_f to exhaust $x_0 = 1$ units given by the small c approximation formula (52) as a function of ρ for the two-player game and pricing curves (16) of constant ρ, with and without alternative technology (left); and as a function of N for pricing function with constant $\rho = -0.2$.*

and we use subscripts for partial derivatives.

Our expansion in small c is denoted

$$v = \frac{G_1(0,0)}{r} + cv^{(1)} + \cdots, \qquad w = \frac{G_2(0,0)}{r} + cw^{(1)} + \cdots,$$

where

$$v^{(1)} = \frac{A}{r}e^{\frac{r}{A}x} + \frac{B}{r(A+\gamma)}\left(Ae^{\frac{r}{A}y} + \gamma e^{-\frac{r}{\gamma}y}\right), \tag{56}$$

and $w^{(1)}(x,y) = v^{(1)}(y,x)$. Recall that $\gamma = q_1^*(0,0) = q_2^*(0,0)$, the optimal extraction rate of the zero-cost stage game, and the correction term depends on

$$\rho_0 = -2\gamma\,\frac{P''(2\gamma)}{P'(2\gamma)},$$

the constant of relative prudence, via the relations (37)-(38), which here are

$$\frac{A}{\gamma} = \frac{3\rho_0 - 8}{2(3-\rho_0)}, \qquad \frac{B}{\gamma} = \frac{4-\rho_0}{2(3-\rho_0)}. \tag{57}$$

We assume the restriction (47) in Proposition 4.2, namely $\rho_0 < \frac{8}{3}$.

Our first observation is about the effect of exhaustibility on the firms' extraction rates. The first-order correction to the inexhaustible extraction rate γ (for player 1) is, from (49),

$$c\left(\lambda e^{\frac{r}{A}x} + \mu e^{\frac{r}{A}y}\right),$$

where

$$\lambda = \frac{1}{2P'(2\gamma)}\left(\frac{4-2\rho_0}{3-\rho_0}\right), \qquad \mu = -\frac{1}{2P'(2\gamma)}\left(\frac{2-\rho_0}{3-\rho_0}\right). \tag{58}$$

From (41), we know $\lambda < 0$ for all admissible ρ_0, and $\mu \leq 0$ for $\rho_0 \geq N = 2$. In this case, production rates decrease at all resource levels (x, y) under the introduction of exhaustibility.

In the case $\rho_0 < N = 2$, player 1's production rate increases when

$$x > y + I, \qquad I := -\frac{A}{r} \log\left(\frac{-\lambda}{\mu}\right) = \frac{\gamma}{2r}\left(\frac{8 - 3\rho_0}{3 - \rho_0}\right) \log\left(\frac{4 - \rho_0}{2 - \rho_0}\right).$$

Similarly, player 2 increases his production over the inexhaustible rate when $y > x + I$. We have $I > 0$ if $-\lambda > \mu$ which, from (43), is always the case when $\rho_0 < N$. Therefore, there are regions of increased and decreased production. Since I measures the width of the band in which production is decreased with the introduction of small exhaustibility, we see it is proportional to the inexhaustible production rate γ.

For the *linear* pricing function $P(Q) = \eta - Q$, $\rho_0 = 0$ and

$$\gamma = \eta/3, \quad A = -4\eta/9, \quad B = 2\eta/9, \quad \lambda = -2/3, \quad \mu = 1/3,$$

so player 1 increases production when $x > y + \frac{4\eta}{9r} \log 2$. See Figure 2 (left).

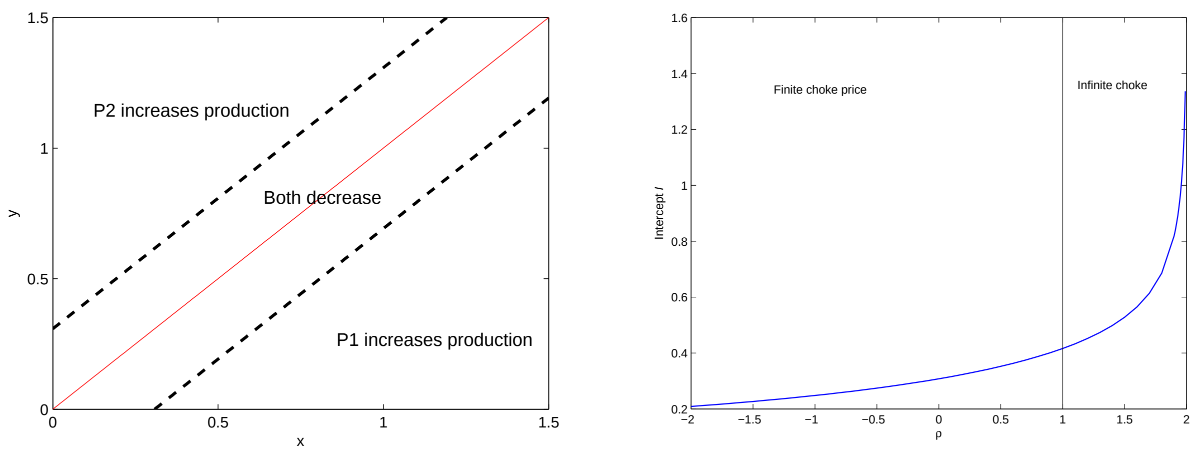

Figure 2: *Extraction increase over the inexhaustible case using the asymptotic approximation for small c. The left panel is for the linear pricing function ($\rho = 0$) with choke price $P(0) = \eta = 1$ and $r = 1$. The right panel shows the (half-)width of the band of decreased production I for constant prudence price curves against $\rho < 2$, again when $\eta = 1$ and $r = 1$.*

To understand the dependence on the curvature coefficient ρ_0, we consider the pricing functions (16) with *constant prudence* $\rho < 2$. We fix $\eta = 1$ so that for $\rho < 1$, when there is a finite choke price, it is given by $P(0) = (1 - \rho)^{-1}$. Recall that in all cases, η is the finite saturation point where $P(\eta) = 0$ and $P'(\eta) = -1$. Then it is easy to compute

$$\gamma = \begin{cases} \frac{1}{2}\left(\frac{2}{3-\rho}\right)^{\frac{1}{1-\rho}} & \rho \neq 1, \\ \frac{1}{2} e^{-1/2} & \rho = 1. \end{cases} \tag{59}$$

Figure 2 (right) shows how the width of the band in which both players slow down production relative to their inexhaustible rates increases (to infinity) with ρ.

Figure 3 shows the quantities (x, y) where $v^{(1)}$ and/or $w^{(1)}$ are positive for two different constant ρ pricing functions. Notice that introduction of some exhaustibility (small $c > 0$) can improve a firm's lifetime profit compared to the inexhaustible case, even when it is behind in resources. In other words, even in some part of $\{x < y\}$, player 1's overall profit can increase because the exhaustibility affects player 2 as well. However, once x and y are small enough, both value functions decrease.

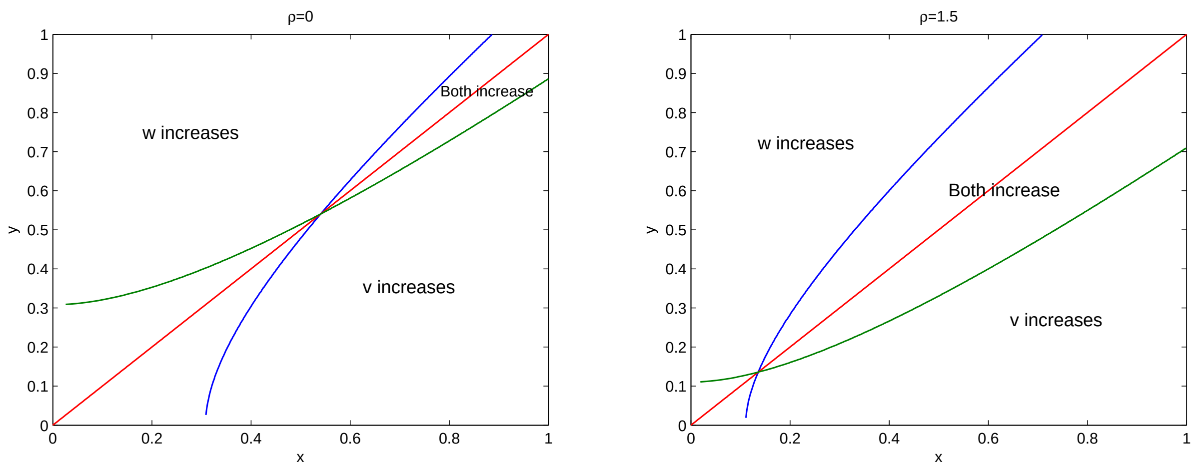

Figure 3: *Regions where $v^{(1)}, w^{(1)} > 0$: exhaustibility improves lifetime profit using the asymptotic approximation applied to the contant ρ price curves. In the left panel $\rho = 0$ (linear, finite choke price) and in the right, $\rho = 1.5$ (infinite choke price).*

Finally, we illustrate the evolution of two asymmetric games with different constant ρ pricing functions: $\rho_1 = 0$ and $\rho_2 = 1.5$. Using the formula (59) to compute γ, and then (57) and (58) for the parameters (A, B, λ, μ) that show up in the asymptotic approximations, we run the games in which the initial resource levels are $(x(0), y(0)) = (5, 2)$ and cost of the alternative technology is $c = 0.25$, using the approximate dynamics (50). The game paths are shown in Figure 4; the production rates and the ensuing market price are shown in Figure 5.

5 Duopoly Extraction Problem with the Linear Pricing Function

We now concentrate on a more detailed examination of the two-player game for the linear pricing function $P(Q) = 1 - Q$, and we no longer assume that c is small, but of course we do still assume that $c < P(0^+) = 1$. We use the notation for the two-player dynamic game introduced in Section 4.4, and clearly, by symmetry, we have $w(x, y) = v(y, x)$. We maintain our standing assumption that the value functions v and w are continuously differentiable up to the axes with $v_x, w_y < c$ in the interior $\{(x, y) \mid x > 0, y > 0\}$, so neither player would want to produce from the alternative technology while traditional supplies remain. Our numerical experiments (Section 5.3) suggest that the latter bound is obeyed, and switching in the interior does not occur.

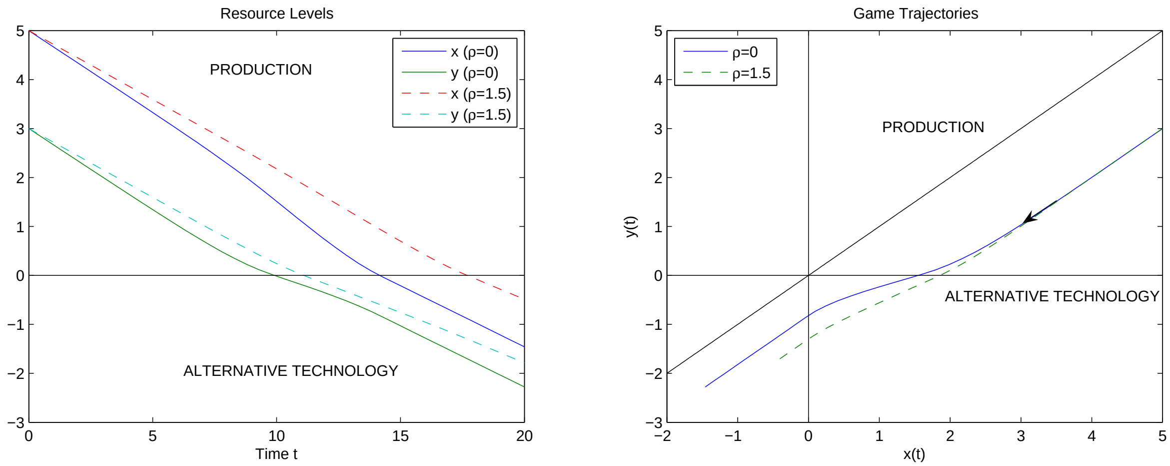

Figure 4: *Decline of the individual resource levels $x(t)$ and $y(t)$ over time (left), and the game paths $y(t)$ vs. $x(t)$ (right). Negative quantities correspond to production with the alternative technology. Production is much slower for the more prudent pricing curve $\rho = 1.5$.*

When both players have resources left, the candidate strategies are

$$q_1^*(v_x, w_y) = \frac{1}{3}(1 - 2v_x + w_y), \qquad q_2^*(v_x, w_y) = \frac{1}{3}(1 - 2w_y + v_x). \tag{60}$$

Clearly for $c \leq \frac{1}{2}$, and $v_x, w_y < c$, we have $q_1^*, q_2^* > 0$, and both players participate in the interior. In this case, the partial differential equations in $\{(x, y) \mid x > 0, y > 0\}$ are

$$\begin{aligned} \frac{1}{9}(1 - 2v_x + w_y)^2 - \frac{1}{3}(1 + v_x - 2w_y)v_y - rv &= 0, \\ \frac{1}{9}(1 + v_x - 2w_y)^2 - \frac{1}{3}(1 - 2v_x + w_y)w_x - rw &= 0. \end{aligned} \tag{61}$$

However, for larger costs of the alternative technology ($\frac{1}{2} < c < 1$), one player could in principle have a large enough shadow cost, that he would be better off not producing for a while and waiting for his competitor to run down some more of his resources. For the duopoly problem, this would mean that, while he was blockaded, the other player would have a monopoly, at least temporarily.

In Section 5.1, we study the reduced game on the axes, when one player is using the inexhaustible alternative technology and the value functions satisfy ordinary differential equations. We establish some specific conditions under which blockading occurs on the axes. We then make some remarks on the mathematical properties of the partial differential equations (which are highly nonstandard) and describe their numerical solution in the case without blockading. Analysis of the blockading case in the interior is beyond the scope of the current paper, and Section 5.2 describes the sort of complications that might arise.

5.1 The Axis Game & Blockading

When one player exhausts his resources, we can compute the value function, strategies and blockade regions explicitly. We state the results on $y = 0$ where player two has exhausted

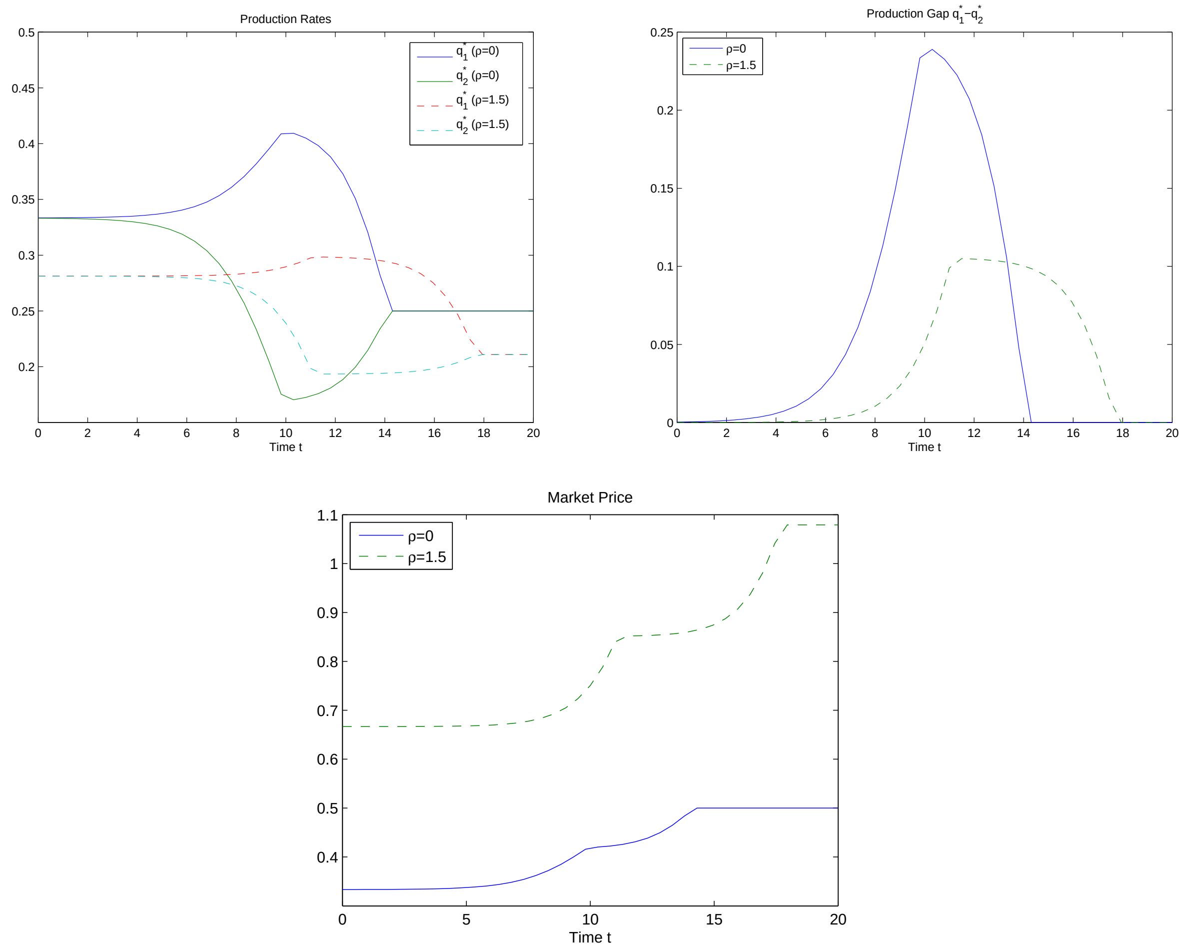

Figure 5: *Production rates over time (left), the difference in production between the two players (right), and the market price (bottom). Notice as player 2 approaches exhaustion, he rapidly ramps down production and player 1 ramps up production, but not as fast, so the market price spikes. The production gap is much smaller for the more prudent pricing curve* $\rho = 1.5$. *Also, the range of production rates is much narrower, but the variation in the market price is larger as the choke price is infinite in this case.*

his reserves and may produce from the alternative inexhaustible source at cost c; the results for the axis game on $x = 0$ follows analogously from symmetry.

The equations for $v(x, 0)$ and $w(x, 0)$ in the reduced game are (25) and (26), which here are

$$rv = (q_1^*(v', c))^2, \qquad rw = (q_2^*(v', c))^2 - q_1^*(v', c)w',$$

where $v' = v_x(x, 0)$ and $w' = w_x(x, 0)$.

If there is no blockading at x, we have

$$\begin{aligned} \frac{1}{9}(1 - 2v' + c)^2 - rv &= 0, \\ \frac{1}{9}(1 + v' - 2c)^2 - \frac{1}{3}(1 - 2c + v')w' - rw &= 0, \end{aligned} \tag{62}$$

and these ODEs have initial condition (27), specifically $v(0,0) = w(0,0) = (1-c)^2/(9r)$. Comparison of (61) and (62) along with the assumed continuity of the first derivatives of the value functions up to the axes led to the Neumann boundary conditions (55). However we will not use the Neumann conditions as we analyze the transition to the blockading case, since they do not apply there. If player two is blockaded at some $x > 0$, we have $q_2^* = 0$, and $q_1^* = \frac{1}{2}(1-v')$, so

$$rv = \frac{1}{4}(1-v')^2, \qquad rw = -\frac{1}{2}(1-v')w'. \tag{63}$$

Proposition 5.1. For the linear pricing function $P(Q) = 1 - Q$, and for $c \leq \frac{1}{2}$, there is no blockading when one player has exhausted his resources. On $y = 0$, the value function $v(x,0)$ for the player with resources is implicitly given by

$$\frac{1}{1+c}\left(3\sqrt{rv} - (1-c)\right) + \log\left(\frac{(1+c) - 3\sqrt{rv}}{2c}\right) = -\frac{9rx}{4(1+c)}. \tag{64}$$

It is concave, increasing and

$$\lim_{x\to\infty} v(x,0) = \frac{(1+c)^2}{9r}.$$

The value function $w(x,0)$ of the other player is given explicitly by

$$w = \frac{c(5c-2)}{3r}\left(\frac{v'}{c}\right)^{4/3} + \frac{(1-2c)^2 + 8(1-2c)v' - 2v'^2}{9r}. \tag{65}$$

Proof. From (62), we have

$$v' = \frac{1}{2}(1 + c - 3\sqrt{rv}), \qquad v(0,0) = \frac{1}{9r}(1-c)^2. \tag{66}$$

Integrating leads to the transcendental equation $\phi(z) = e^{-\theta x}$, where $\theta = 9r/4(1+c)$,

$$\phi(z) = \frac{1}{2c}e^{-\frac{(1-c)}{(1+c)}}e^{\frac{3\sqrt{r}}{(1+c)}z}(1 + c - 3\sqrt{r}\,z),$$

and the solution to the ODE is $v(x,0) = z^2$. Since it can be shown that ϕ is decreasing for $z \geq 0$ and

$$\phi(0) = \frac{(1+c)}{2c}e^{-\frac{(1-c)}{(1+c)}} > 1, \quad \text{for } 0 < c < 1,$$

it follows that, given $x > 0$, the equation $\phi(z) = e^{-\theta x}$ uniquely determines $v(x,0) = z^2 \in (0, \frac{(1+c)^2}{9r})$, which leads to the formula (64). From (66), we see $v' > 0$. Differentiating the ODE gives

$$v'' = -\frac{3}{4}\sqrt{\frac{r}{v}}\,v' < 0,$$

so v is strictly concave. Since $v_x(x,0) = c$, we have $0 \leq v_x(x,0) < c \leq \frac{1}{2}$ for $x > 0$, so the production rates $q_1^*, q_2^* > 0$, and there is no blockading. The limit $x \to \infty$ is easy to calculate.

Lastly, the second equation in (62) is readily solved by using v' as the independent variable. ∎

Proposition 5.2. When $c > \frac{1}{2}$, blockading occurs on $y = 0$ for $x \geq x_b$, where

$$x_b = \frac{4}{9r}\left(2(c-1) + (1+c)\log\left(\frac{c}{2c-1}\right)\right). \tag{67}$$

The value functions $v(x, 0)$ and $w(x, 0)$ are given by (64) and (65) for $0 \leq x < x_b$, and, for $x \geq x_b$, by

$$2\left(\sqrt{rv} - (1-c)\right) + \log\left(\frac{1-2\sqrt{rv}}{2c-1}\right) = -2r(x - x_b), \tag{68}$$

$$w = \frac{1}{r}\left(\frac{5c-2}{3}\left(\frac{2c-1}{c}\right)^{1/3} - (2c-1)\right)(1 - 2\sqrt{rv}). \tag{69}$$

Proof. Using the no-blockade solution (64) in the formula

$$q_2^* = \frac{1}{3}(1 - 2c + v')$$

shows that $q_2^* \leq 0$ for $x \geq x_b$ defined by (67). For $x \geq x_b$, player 1 has a monopoly: $q_2^* = 0$ and $q_1^* = \frac{1}{2}(1-v')$. These imply that at x_b, $v' = 2c-1$ and $q_1^* = 1-c$, so $v(x_b, 0) = \frac{1}{r}(1-c)^2$. Solving the first ODE in (63) with this boundary condition leads to (68). Solving the second ODE with the boundary condition $w(x_b, 0)$ coming from (65) leads to (69). $\square$

We remark that v' and w' are continuous at $x = x_b$, and that, as $x \to \infty$, $v \to \frac{1}{4r}$, independent of $c > \frac{1}{2}$. The value functions and strategies on the axis in a case with blockading are illustrated in Figure 6. We note also that x_b is decreasing in c, and $x_b \downarrow 0$ as $c \uparrow 1$, so the region in which player 2 is discouraged from participating with the alternative technology increases with the cost of that technology, as we would expect (see Figure 8). Formally, the limit $c = 1$ corresponds to the model where player 1 has a total monopoly when player 2 has exhausted his resources, and vice versa.

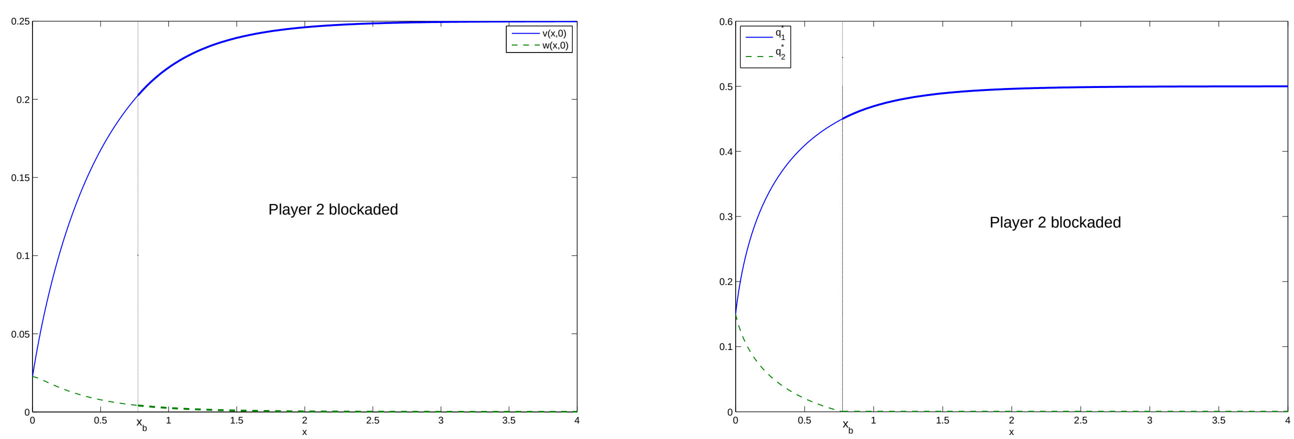

Figure 6: *Value functions and equilibrium production rates when $c = 0.55$ and there is blockading.*

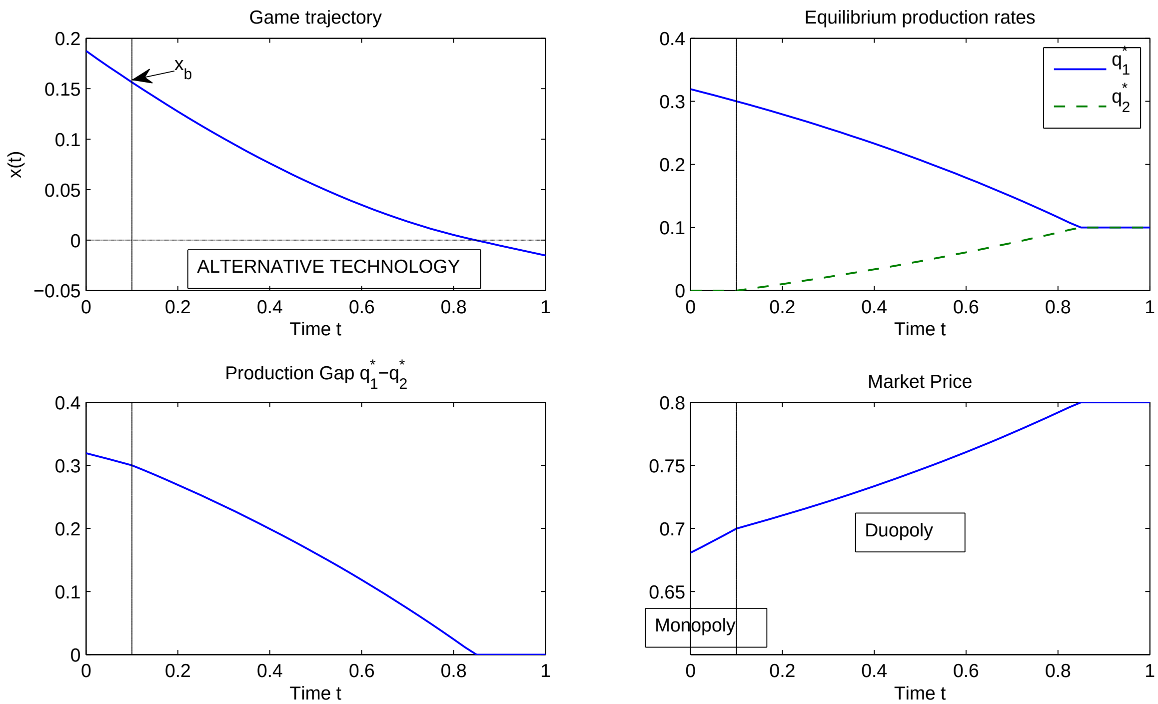

Figure 7: *Axis game when player two has run out of reserves for $c = 0.7$: game path (top left), equilibrium policies (top right), production gap (bottom left) and market price (bottom right) . The vertical line marks where blockading ends and player one no longer has a monopoly as player two re-enters.*

In Figure 7, we show the game trajectory and corresponding production rates and market price for $c = 0.7$ as firm two is initially blockaded and then re-enters with the alternative technology once firm one has sufficiently run down his reserves. We observe that when player one's reserves reach x_b, his monopoly becomes a duopoly again and the rate of increase of the market price slows dramatically.

Finally in this section, we give the formulas when player one has infinite resources, while player two has a finite resource y. The ODEs at $x = +\infty$ are obtained by setting all x derivatives to zero. The ODEs for $W(y) = w(\infty, y)$ and $V(y) = v(\infty, y)$ are

$$rW = \frac{1}{9}(1 - 2W')^2, \qquad rV = \frac{1}{9}(1 + W')^2 - \frac{1}{3}V'(1 - 2W').$$

As usual, there are two cases either side of $c = \frac{1}{2}$. Since for the edge case $c = \frac{1}{2}$, $x_b = \infty$, we will here put that case under blockading.

No blockading. For $c < \frac{1}{2}$, the boundary conditions at $y = 0$ are $V = (1 + c)^2/(9r)$, $W = (1 - 2c)^2/(9r)$. Then, as before, we solve to find

$$1 - 2c - 3\sqrt{rW} - \log\left(\frac{1 - 3\sqrt{rW}}{2c}\right) = \frac{9ry}{4},$$

and, with $\xi = 1 - 3\sqrt{rW}$,

$$V = \frac{c(c-2)}{3r}\left(\frac{\xi}{2c}\right)^{\frac{4}{3}} - \frac{1}{18r}(\xi^2 - 8\xi - 2).$$

Both V and W tend to $1/(9r)$ as $y \to \infty$ ($\xi \to 0$).

With blockading. For $c \geq \frac{1}{2}$, the equations are the same but the initial conditions are $V = 1/(4r)$ (the large-x behaviour of the monopoly solution on the x axis), $W = 0$. The solutions are

$$-3\sqrt{rW} - \log\left(1 - 3\sqrt{rW}\right) = \frac{9ry}{4},$$

and

$$V = -\frac{1}{4r}\xi^{\frac{4}{3}} - \frac{1}{18r}(\xi^2 - 8\xi - 2).$$

We note that they are independent of c and they agree with the previous case when $c = \frac{1}{2}$. In particular, as $q_2^* = \frac{1}{3}(1 - 2W') = \sqrt{rW}$, the solution $W > 0$ for $y > 0$ implies there is no blockading in the interior on $x = \infty$. This is not surprising since player two is so very far from being un-blockaded once he runs out, he has no incentive to hang on to reserves.

5.2 Type of the PDEs

With their full quadratic nonlinearity, the coupled HJ equations (61) are highly nonstandard. The middle terms, involving v_y in the first equation and w_x in the second, describe the impact of one player's resource level on the other's value function. These *externality terms* are intrinsic to PDE systems arising from game problems, and contribute significantly to their complexity. Despite the full nonlinearity of the PDEs, it is helpful to consider their general behaviour along the lines of the standard elliptic/parabolic/hyperbolic classification of quasilinear second-order equations in two variables.

The PDEs are conveniently written as

$$rv = q_1^2 - q_2 v_y, \qquad rw = q_2^2 - q_1 w_x, \tag{70}$$

where we have dropped the stars on the optimal strategies, and we use the shorthand

$$q_1 = \frac{1}{3}(1 - 2v_x + w_y), \qquad q_2 = \frac{1}{3}(1 - 2w_y + v_x).$$

We are assuming a solution with continuous first derivatives, but we enquire whether it is possible for the second derivatives v_{xx} etc. to have a jump $[v_{xx}]$ etc. across a curve $y = y(x)$. This is a standard approach for a quasilinear system. We call the system hyperbolic if there are two nontrivial[6] such directions hyperbolic, while with no real directions it is elliptic, and parabolic with coincident roots.

Suppose we have such a curve; by assumption, v_x and v_y are continuous across it. Thus differentiating $[v_x] = 0$ and $[v_y] = 0$ along the curve gives

$$[v_{xx}] + \lambda[v_{xy}] = 0, \qquad [v_{xy}] + \lambda[v_{yy}] = 0,$$

[6] That is, after discounting spurious roots introduced by differentiation.

where $\lambda = \mathrm{d}y/\mathrm{d}x$. Differentiating the first of (70) in x and subtracting across the curve gives

$$-2q_1[q_{1x}] + q_2[v_{xy}] + v_y[q_{2x}] = 0,$$

which is

$$-2q_1[-2v_{xx} + w_{xy}] + 3q_2[v_{xy}] + v_y[-2w_{xy} + v_{xx}] = 0,$$

namely

$$\alpha_1[v_{xx}] + 3q_2[v_{xy}] - \alpha_2[w_{xy}] = 0, \qquad \alpha_1 = 4q_1 + v_y, \quad \alpha_2 = 2(q_1 + v_y).$$

Doing the same for w, this time differentiating in y,[7] we have

$$[w_{xx}] + \lambda[w_{xy}] = 0. \qquad [w_{xy}] + \lambda[w_{yy}] = 0$$

and

$$\alpha_4[w_{yy}] + 3q_1[w_{xy}] - \alpha_3[v_{xy}] = 0, \qquad \alpha_4 = 4q_2 + w_x, \quad \alpha_3 = 2(q_2 + w_x).$$

These homogeneous equations only have a solution if the determinant of coefficients vanishes:

$$\begin{vmatrix} 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 \\ \alpha_1 & 3q_2 & 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \\ 0 & -\alpha_3 & 0 & 0 & 3q_1 & \alpha_4 \end{vmatrix} = 0.$$

The third and fourth columns can be removed with the second and fourth rows (they give two spurious roots due to differentiation) and after some manipulation and replacement of the α_i we arrive at

$$q_1(4q_1 + v_y)\lambda^2 + \lambda(v_y w_x - 7q_1 q_2) + q_2(4q_2 + w_x) = 0, \tag{71}$$

the number of real roots of which determines the type at each point. As expected for a quasilinear system (let alone a fully nonlinear one), the determination of type depends on the solution itself, and this places severe restrictions on what we are able to say with certainty.

Large x and y. When both x and y tend to infinity, the derivatives of v and w tend to zero and the strategies q_1 and q_2 both tend to $\frac{1}{3}$. Thus (71) becomes

$$4\lambda^2 - 7\lambda + 4 = 0,$$

which has complex roots. Hence the system is elliptic in this region.

[7] Other combinations of derivatives give the same result.

The x axis; $c \le \frac{1}{2}$. When $c \le \frac{1}{2}$, no blockading occurs and so $v_y = 0$ on the x axis. Then we consider

$$4q_1^2\lambda^2 - 7q_1q_2\lambda + 4q_2^2 + q_2w_x = 0,$$

which has discriminant $-q_1^2q_2^2(15 + 16w_x/q_2)$; hence the system is elliptic on the x axis if $-15/16 < w_x/q_2 < 0$ (recall $w_x < 0$) and it is readily verified numerically, using (65), that this is the case for $0 < c < \frac{1}{2}$. It is also possible to show that, when $c \uparrow \frac{1}{2}$, the discriminant tends to zero as $x \to \infty$, which has implications that we now explore.

The x axis; $c > \frac{1}{2}$. As shown above, when $\frac{1}{2} < c < 1$, blockading on the x axis of player 2 occurs for $x_b < x < \infty$, and we have $q_2 = 0$ for $x > x_b$. The quadratic then has the roots

$$\lambda = 0, -\frac{v_y w_x}{q_1(4q_1 + v_y)}, \qquad x > x_b,$$

and the system is *hyperbolic* unless $v_y = 0$ (and note that the x axis is a possible curve of discontinuity). For $x < x_b$ we have $v_y = 0$ so the quadratic (71) reduces as above to

$$4q_1^2\lambda^2 - 7q_1q_2\lambda + q_2(4q_2 + w_x) = 0,$$

where now

$$q_1 = \frac{1}{3}(1 + c - 2v_x), \qquad q_2 = \frac{1}{3}(1 - 2c + v_x).$$

The discriminant of this quadratic is (as above) $q_1^2q_2^2\,(-15 - 16w_x/q_2)$ and numerically it is seen to be positive on a non-empty interval $x_0(c) < x < x_b$; here x_0, which is such that $w_x(x_0(c), 0)/q_2(x_0, 0) = -\frac{15}{16}$, emanates from infinity as c increases from $\frac{1}{2}$. A graph of x_0 and x_b against c is shown in Figure 8. Indeed, as we approach $x_b(c)$ from below, $v_x \to 2c-1$ and, using $w_y = c$, $q_1 \to 1 - c$ and $q_2 \to 0$. Lastly $w_x = -rw/q_1$ there which is $\mathcal{O}(1)$ and negative. Thus, this discriminant is positive just to the left of $x_b(c)$ (because $|q_2| \ll |w_x|$) and crosses zero only at $x = x_0 < x_b(c)$. For $0 \le x < x_0$, however, we still have an elliptic system on the x axis.

Variation in y; $x \to \infty$. Lastly we consider the case when x is very large and v, w depend only on y. Then, the discriminant of the quadratic becomes $q_1q_2^2(-15q_1 - 16v_y)$. Again, numerical investigation of the explicit solution of the (ordinary) differential equations for $v(\infty, y)$ and $w(\infty, y)$ shows that, for $0 < c \le \frac{1}{2}$, the discriminant is negative (elliptic); for $\frac{1}{2} < c < 1$, however, we have the blockaded solution as $x \to \infty$ on $y = 0$, and the discriminant is positive on an interval $0 < y < y_\infty$, indicating a hyperbolic system. The blockaded solution is asymptotically independent of c, and the switch point y_∞ is also independent of c. It is found numerically to be approximately 0.003357; this value is remarkably small.

Discussion. This analysis can only give a very partial picture of the behaviour of the full system of PDEs and boundary conditions. It illustrates in particular the very strong influence of the boundary conditions on the global behaviour of the solutions. Even though our evidence is sketchy, we tentatively conjecture that for $0 < c \le \frac{1}{2}$, the system is elliptic everywhere, and this is supported by evaluation of (71) for the numerical solutions described in Section 5.3.

In contrast, for $c > \frac{1}{2}$, the axes calculations suggest that the PDEs may be of mixed type in the interior. The economic interpretation of this scenario remains wholly unclear.

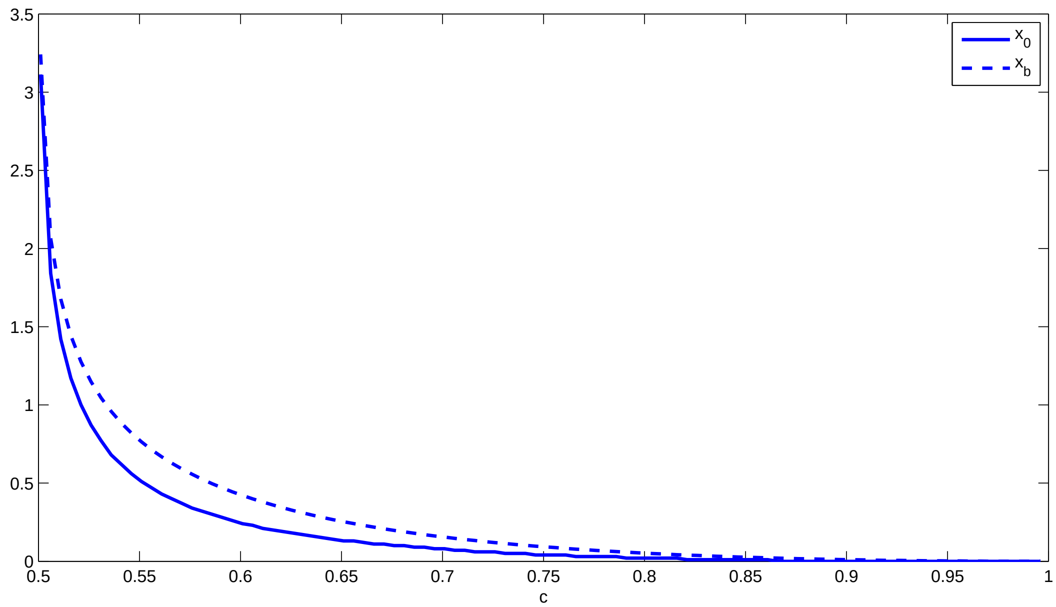

Figure 8: *Type transition point x_0 and blockade point x_b against c .*

5.3 Numerical Solutions

Numerical discretization of the system (61) is highly non-trivial given the complexity of the equations. Moreover, the change of type outlined above, for $\frac{1}{2} < c < 1$, is an enormous complication. Hence we limit our numerical illustrations to the case $0 < c \leq \frac{1}{2}$. While we do not go into the full details of our approach here, features of the method we have used and found to be effective are:

- Solve for the steady state of the time-dependent stochastic analogue of the differential game, namely add a small Laplacian term to the equations to eliminate grid-scale oscillations and obtain solutions with smooth (non-oscillatory) first derivatives, and hence optimal strategies. All approaches that did not use this regularization were found, without exception, to exhibit instabilities. This involves discretizing

$$v_t = G_1(v_x, w_y) - q_2^*(v_x, w_y)v_y - rv + \varepsilon \Delta v,$$

  and the analogue for the w equation, for a small diffusion coefficient ε of size $\mathcal{O}(\Delta x)$, where Δx is the typical grid size, and to large time t. The error is still measured with the residual of the original equations. In practice, we use an explicit time-stepping method, and we work with the scalar equation derived by substituting the non-local term $w(x, y) = v(y, x)$ and similarly for the derivatives.

- We subtract off the two-term asymptotic approximation $v = \frac{G_1(0,0)}{r} + cv^{(1)}$, where $v^{(1)}$ is given by (56), to solve for the residual.

- We use a non-uniform grid to put more resolution near the axes. We use standard central finite-differences, with higher-order one-sided difference at the axes for the

Neumann boundary conditions (55), and zero Neumann conditions at the far edges of the computational grid .

Although this method is slow, our numerical experiments indicate convergence at a rate $\mathcal{O}(\Delta x^{2/3})$. This is illustrated in Figure 9, showing the log-error decreasing with the number of grid points on a domain $0 < x, y < 5$. (The inf norm of the error in the boundary values on the axes, which can be checked against the exact solution of the ODEs in Proposition 5.1, shows similar convergence behaviour.)

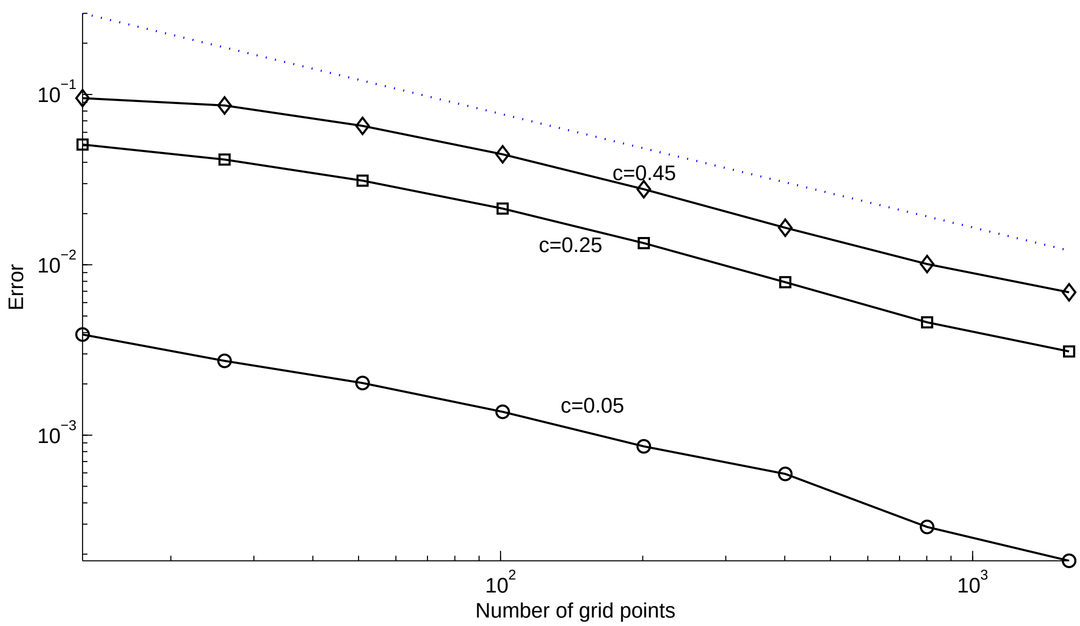

Figure 9: *Error vs. number of grid points plotted on log scales. The error is a measure of the residual of the discretized equations, scaled by the largest value taken by* v*. The dashed line has (log-log) slope* $-2/3$.

Numerically, we solve for the value functions (v, w) and check the shadow costs (v_x, w_y) (from finite differencing) do not exceed c. Throughout, we set $r = 1$. Figure 10 (left) shows the computed $v(x, y)$ surface for $c = 0.5$, and Figure 11 (left) shows the computed $v_x(x, y)$ surface. The shadow cost for player 1 increases up to c at $x = 0$. The right panel shows player 1's optimal strategy as a function of resource levels. Figure 10 (right) compares the game paths, starting at $(5, 2)$, when there is a small cost $c = 0.05$ and a much larger one $c = 0.5$. Figure 12 shows how shadow costs, production rates and the market price evolve over the course of the dynamic game. Even though initially player 2 drops production and player 1 increases production, the price for the consumer always rises, with particularly sharp increases as each player approaches exhaustion.

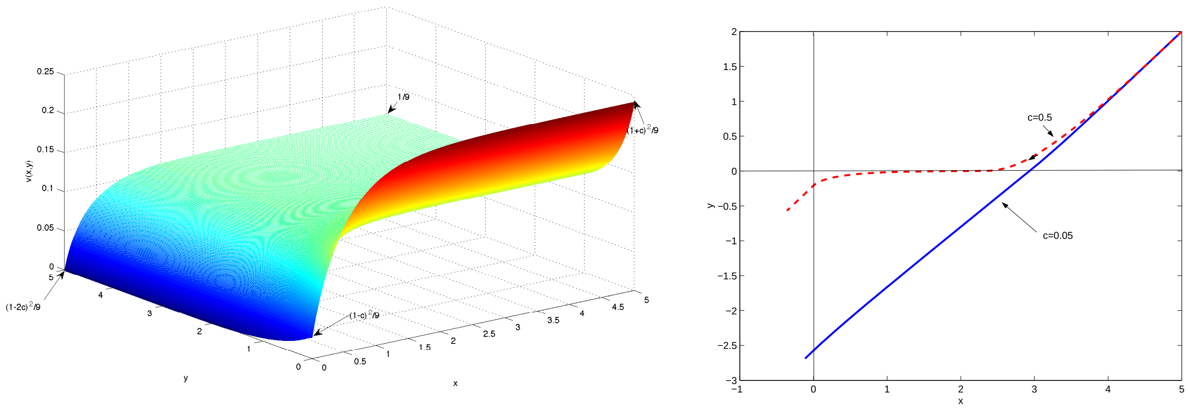

Figure 10: *Value function (left)* $v(x, y)$ *for* $c = 0.5$ *showing the theoretical values on the corners (with* $r = 1$*). The numerical solution on the truncated domain achieves those values to within* 3%*. The right plot shows a comparison between game paths from cheap to expensive alternative technology.*

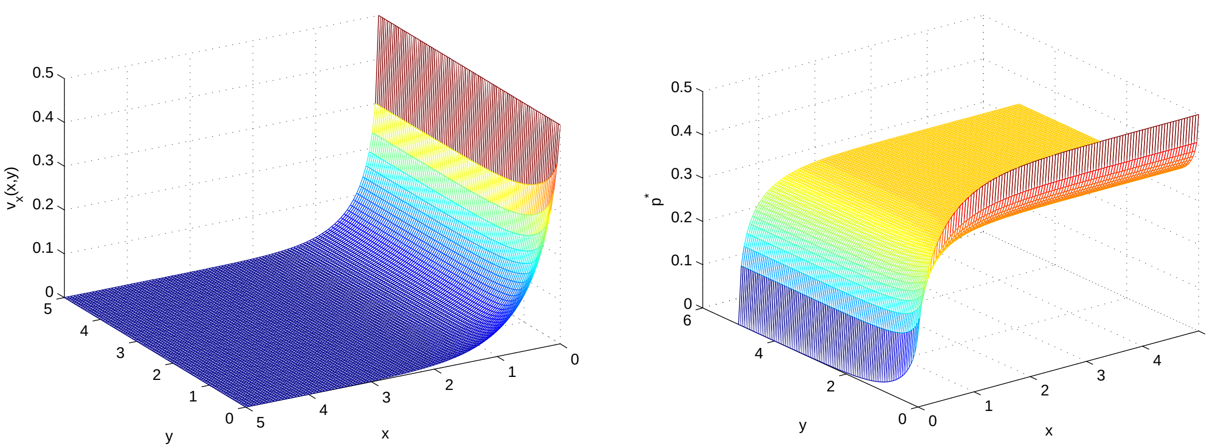

Figure 11: *Shadow cost and optimal production rate for player 1 when* $c = 0.5$.

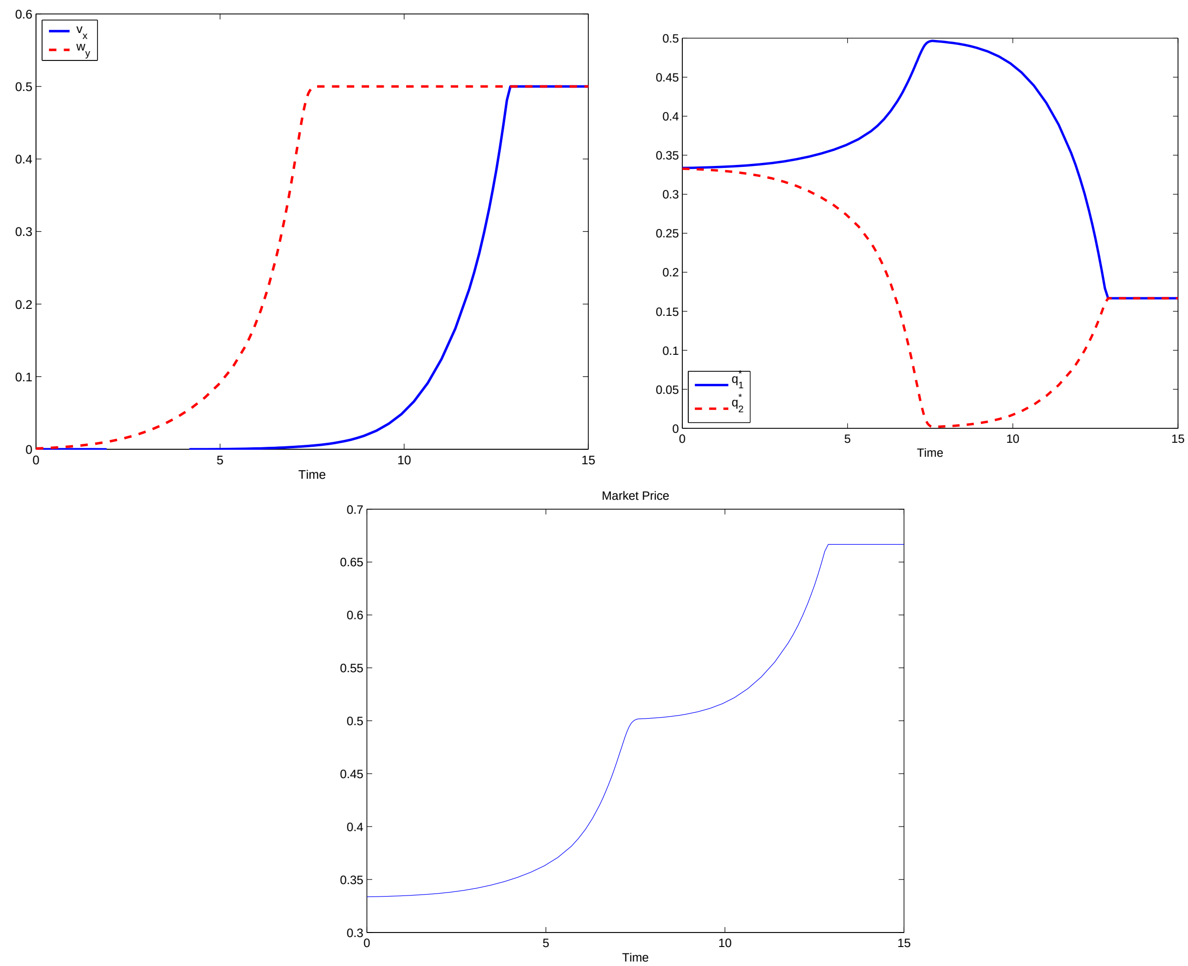

Figure 12: *Evolution of shadow costs (left), production rates (right) and market price (bottom) when* $c = 0.5$.

6 Conclusions

In this paper, we have initiated a study of Cournot differential games as applied to problems of exhaustible resources. These are characterized by systems of first-order nonlinear PDEs which may, in certain parameter regimes, be of mixed type. As with most nonzero-sum games in continuous time, existence, uniqueness and regularity of value functions remain very difficult open issues, while numerical solutions also pose a major challenge.

Our analysis so far has focused on the case where the cost of the alternative inexhaustible technology is not too large, and there is no blockading. Here we are able to construct asymptotic and numerical approximations that provide quantitative insight into how exhaustibility may affect production and prices in a Cournot market. For example, Figure 2 demonstrates that if producers alter their production to account for diminishing resources far enough in advance, their overall lifetime profits may actually increase.

The regime where the cost of the alternative resource is relatively high introduces the issue of some firms potentially being temporarily priced out of production by the dominance of others. We are able to study the reduced two-player game in this regime when one player has exhausted his traditional resources. The effect of his re-entry, when the monopoly becomes a duopoly again, is to slow the rise in prices as illustrated in Figure 7. The issue of blockading in the interior remains a numerical challenge we are pursuing.

There are many related problems and extensions of the current work we plan to consider. Uncertain and fluctuating price functions or cost structures lead to dynamic games played in a random environment. Costs may vary with resources remaining or past production, to model industries where learning-by-doing can cause costs to drop with experience. Stochastic differential games, where inventory levels are estimated with uncertainty (for example due to noisy seismic estimates of oil reserves) may lead to PDE problems whose numerical solution is simpler than for the corresponding ordinary differential game, for example if the noise sources are Brownian motions, adding a strongly elliptic second-order term to the PDEs. Such benefits may also be gained by considering problems over finite time horizons.

A structurally similar type of PDE problem to the ones considered here arises for Bertrand competitions [2] in which firms set prices and the market determines demand quantities. The framework could also be adapted to Kreps-Scheinkman competitions [14] in which firms first play a Cournot game to set quantity pre-commitments, and then play a Bertrand game to set prices and delivery amounts. Different markets lend themselves to different models of oligopolistic competition. For example, markets for consumer goods may best be described by Bertrand competition, whereas a Cournot model may be more suitable for commodities markets.

In terms of energy production and renewable resources, a major policy issue is how governments can impose costs or taxes to encourage (or nudge) firms to partially produce from greener technologies before oil has run out. This can be viewed as an inverse problem for a dynamic game in which a cost function ($c(x, y)$ in the two-player notation) incorporating a tax is imposed to force partial conversion to, say, solar energy in the interior. Design of such a regulatory structure to generate environmentally attractive Nash equilibria from profit-maximizing firms will rely on good numerical algorithms for the forward problem. Robustness to choice of the discount rate r and the price curve is another difficult and interesting area of investigation.

Finally, there are other applications of this type of model where there are choices between different energy sources of different costs and degrees of renewability: the costless product might be hydroelectricity (the resource level being the reserve of water) or geothermal energy (limited by the heat capacity of the rock), as opposed to generating electricity from bought-in fossil fuel; or extraction of natural gas compared with generation of gas from coal. In the spirit of Cournot's work on mineral water production, there are interesting problems related to extraction of a limited reservoir of underground water compared with desalination.

Appendix

We give here the calculation needed for the remaining case of infinite choke price in the proof of Lemma 2.3. From (10),

$$Q\, f_n'(Q) \leq (n+1-\overline{\rho})\, Q\, P'(Q) = (n+1-\overline{\rho})\, (f_n(Q) - n\, P(Q))$$

or $Q\, f_n'(Q) - \lambda\, f_n(Q) \leq -\lambda\, n\, P(Q)$, where $\lambda = n+1-\overline{\rho}$. Multiplying though by $Q^{-(\lambda+1)}$, integrating from $\varepsilon > 0$ to η, multiplying through by ε^{λ} and rearranging gives

$$\begin{aligned} f_n(\varepsilon) &\geq \left(\frac{\varepsilon}{\eta}\right)^{\lambda} f_n(\eta) + n \int_{\varepsilon}^{\eta} \lambda \left(\frac{Q}{\varepsilon}\right)^{-(\lambda+1)} P(Q)\, \frac{1}{\varepsilon}\, dQ \\ &= \left(\frac{\varepsilon}{\eta}\right)^{\lambda} f_n(\eta) + n \int_{1}^{\varepsilon^{-1}\eta} \lambda u^{-(\lambda+1)} P(\varepsilon u)\, du. \end{aligned}$$

As $P(\varepsilon u) = P(\eta) = 0$ when $u = \varepsilon^{-1}\eta$, putting $\widetilde{P} = \max\{P, 0\}$, we have

$$\int_{1}^{\varepsilon^{-1}\eta} \lambda\, u^{-\lambda-1}\, P(\varepsilon u)\, du = \int_{1}^{\infty} \lambda\, u^{-\lambda-1}\, \widetilde{P}(\varepsilon u)\, du.$$

Then $\widetilde{P}(\varepsilon u) \uparrow \widetilde{P}(0^+) = P(0^+)$ pointwise as $\varepsilon \downarrow 0$. Hence, using $\int_1^{\infty} \lambda\, u^{-(\lambda+1)} du = 1$,

$$\int_{1}^{\infty} \lambda\, u^{-\lambda-1}\, \widetilde{P}(\varepsilon\, u)\, du \to \widetilde{P}(0^+) = P(0^+).$$

by the monotone convergence theorem. Therefore $f_n(0^+) \geq nP(0^+) > A_n$, as required.

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