Accelerated Share Repurchases Under Stochastic Volatility

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ABSTRACT
Accelerated share repurchases (ASRs) are a type of stock buyback wherein the repurchasing firm contracts a financial intermediary to acquire the shares on its behalf. The intermediary purchases the shares from the open market and is compensated by the firm according to the average of the stock price over the repurchasing interval, whose end can be chosen by the intermediary. Hence, the intermediary needs to decide both how to minimize the cost of acquiring the shares, and when to exercise its contract to maximize its payment. Studies of ASRs typically assume a constant volatility, but the longer time horizon of ASRs, on the order of months, indicates that the variation of the volatility should be considered. We analyze the optimal strategy of the intermediary within the continuous-time framework of the Heston model for the evolution of the stock price and volatility, which is described by a free-boundary problem which we derive here. To solve this system numerically, we make use of deep learning. Through simulations, we find that the intermediary can acquire shares at lower cost and lower risk if it takes into account the stochasticity of the volatility.

KEYWORDS
Accelerated Share Repurchases; Stochastic Volatility; Deep Learning

1. Introduction

A stock buyback refers to when a firm purchases back its own shares. Firms choose to engage in buybacks for various reasons including signalling positive prospects (Vermaelen 1984), deterring takeovers (Sinha 1991), or to reduce free cash flow (Jensen 1986). Since the passage by the SEC of safe harbor rules in 1982, allowing stock buybacks with lesser regulatory scrutiny, stock buybacks have become increasingly popular, with companies announcing buybacks worth over a trillion dollars in 2018 (Palladino 2020).

One method to execute a stock buyback is an accelerated share repurchase (ASR), where the firm uses an intermediary, like an investment bank, to acquire shares on its behalf. ASRs have become an increasingly popular way to conduct stock buybacks, becoming the second most popular method after Open Market Repurchases (King and Teague 2021). There are several advantages for a firm to repurchase its shares with an ASR, as compared to other methods. Most importantly, an ASR grants the firm credibility that it will actually complete the announced repurchase, since the shares are delivered immediately (Bargeron, Kulchania, and Thomas 2011). An ASR also transfers the risk of the stock buyback from the firm to the intermediary, since it is

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the intermediary which has to actually repurchase the shares.

In this paper, we analyze an ASR contract with early exercise, and compute the optimal way for the intermediary to repurchase shares. To facilitate this, it is important to understand the specific features of ASRs, as well as what distinguishes them from other optimal execution problems such as stock liquidation. An ASR has the following steps: (i) The firm immediately receives the shares it wishes to repurchase from an intermediary, generally an investment bank, in exchange for a payment equal to the number of shares times the current price per share; the intermediary obtains these shares by borrowing from other financial institutions or traders; (ii) the intermediary settles its short position by repurchasing shares from the open market up to some specified maturity date; (iii) at any time up to this maturity date, the intermediary can exercise early; (iv) when the intermediary exercises (or the maturity date is reached), if the average of the stock price from the starting date to the exercise date is greater than the previous price per share, than the firm will compensate the intermediary the difference; if the average is lower then the intermediary will instead pay the firm the difference.

There are two main considerations for the intermediary in deciding how to repurchase shares from the market. Firstly, the intermediary wants to repurchase shares from the market at the lowest cost. This is essentially an optimal execution problem. In this type of problem, the seller wants to liquidate their position, balancing the incentive to trade quickly in order to minimize the risk of price uncertainty with the incentive to trade slowly in order to minimize the price impact of limited liquidity. This type of optimal selling was first explored in Almgren and Chriss (2001). There is a large literature on optimal execution. In Guéant (2014) the issue of block trading under exponential utility is studied. Optimal execution is extended to a basket with multiple assets in Schied, Schöneborn, and Tehranchi (2010). More complex models of price impact have also been studied, like Obizhaeva and Wang (2013) which explores optimal execution under dynamic supply and demand.

Secondly, the payoff is given by the average of the stock price up until the exercise time, making it an Asian Option with an American exercise provision. Several papers consider pricing exotic options with optimal stopping, all in a geometric Brownian motion/Black-Scholes model. In Shepp and Shiryaev (1993), a perpetual lookback option with American exercise, called a Russian option, is priced. The fair price of an option whose payoff is determined by the integral of the stock price up until the chosen exercise time is studied in Kramkov and Mordecki (1995). Most relevant for us is Adachi (2003) where an Asian option with infinite time horizon and American exercise time is considered, and the option price was found to solve a free boundary problem.

ASRs present a mixture of the incentives of optimal execution and those of optimal stopping with a complicated payout structure. On one hand, the intermediary wants to purchase the shares cheaply to minimize costs. On the other hand, the intermediary wants to be able to take advantage of the freedom to choose a stopping time by exercising the option when the average of stock price is relatively large to maximize its payout.

There is a small but growing literature on ASRs. In Guéant, Pu, and Royer (2015), a discrete space and time model of an ASR is developed. Sensitivity to risk is implemented by way of an exponential utility of the cumulative wealth. In turn, they are able to reduce the dimensionality of the problem and determine the optimal solution by solving the associated Bellman equations recursively, which can be done with a tree-based method. This analysis is extended in Guéant (2017) from the case of an

ASR with a fixed number of shares to the case of an ASR with fixed notional. A different type of dimensionality reduction is required, giving a more complicated approximation to the value function. Orbe (2018) analyzes a similar model of an ASR, but with the addition of a lookback option. In Guéant, Manziuk, and Pu (2020) neural networks are introduced to the problem. In contrast to other numerical methods, neural networks are able to handle more complicated dynamics and larger problems without succumbing to the curse of dimensionality.

Most important for our paper is Jaimungal, Kinzebulatov, and Rubisov (2017), which we extend to stochastic volatility. There, a continuous time model is considered with constant execution costs, a quadratic penalty on unpurchased shares, and a quadratic penalty on the remaining shares to be purchased when the option is exercised. In particular, they demonstrate that the problem can be modeled by a quasi-variational inequality whose dimension can be reduced. Subsequently, the optimal strategy of the intermediary, as well as the exercise time, depends on the ratio between the stock price and its average up to that time.

However, previous studies share the limitation that they only consider the case where the volatility of the stock price is constant. ASR contracts occur over longer periods than standard optimal execution problems, on the scale of weeks and months rather than hours and days. Hence, it is more realistic to incorporate stochastic volatility. It is well known that in the context of the Black-Scholes model, the assumption of constant volatility is contradicted by observed data. Indeed, Black and Scholes themselves noted when empirically evaluating their model that "there is evidence of non-stationarity in the variance" (Black and Scholes 1972). The same conclusion was found in Canina and Figlewski (1993), where implied volatility calculated from Black-Scholes was found to depart systematically from real-world volatility.

While ignored in the context of ASRs, optimal execution with stochastic volatility has been studied. A continuous time model with stochastic volatility and liquidity is looked at in Almgren (2012), while Cheridito and Sepin (2014) examines optimal liquidation in discrete time with stochastic volatility and liquidity. Optimal execution in a highly volatile market is explored in Criscuolo and Waelbroeck (2014). Our work seeks to study the implications of stochastic volatility when combined with the novel options pricing introduced by the ASR.

This paper is organized as follows. Section 2 details our model of the ASR and stochastic volatility. In Section 3, we solve the European version of the problem, where the intermediary can only exercise their option at the end of the time horizon. In Section 4, we examine the full American version, with variable exercise time permitted. To solve the American problem, we use deep learning, which is detailed in Section 5, along with a discussion of the results. Our numerical method is relatively flexible, so in Section 6 we solve the higher dimensional problem of an ASR with a fixed notional. Finally, in Section 7 we explore ASRs where the stock is driven by a local volatility model rather than stochastic volatility.

2. Model

In this section, we present our model for the intermediary to repurchase shares in an ASR. We consider a stock whose price S with expected growth rate μ and volatility y

which is given by the Heston model (Heston 1993)

$$dS_t = \mu S_t\,dt + \sqrt{y_t}S_t\,dW_t^{(1)} \qquad (1)$$
$$dy_t = \gamma(m - y_t)\,dt + \psi\sqrt{y_t}\,dW_t^{(2)}$$

where $W^{(1)}, W^{(2)}$ are standard Brownian motions whose increments are correlated as $d\langle W^{(1)}, W^{(2)}\rangle = \rho\,dt$. The square volatility y has long run mean $m > 0$ and reverts to the mean according to the rate $\gamma > 0$; $\psi > 0$ scales the volatility of the volatility, and so long as the Feller condition $\psi^2 < 2\gamma m$ is satisfied, the square volatility y will remain strictly positive almost surely.

Further, we assume that there is a market impact on trading, so that a faster rate of repurchase requires the intermediary to spend more. The execution price of a trade is given by $S_t + aS_t\nu_t$, where $a > 0$ is a parameter that specifies the strength of the market impact and ν_t is the intermediary's rate of trading the stock.

We also define the process A_t as the continuous average of the stock price up to time t:

$$A_t = \frac{1}{t}\int_0^t S_u\,du.$$

The task of the intermediary is to acquire n total shares over the time interval $[0, T]$. If it achieves this target and exercises its option at a time $\tau \leq T$, it is compensated at the average stock price over $[0, \tau)$; in other words it receives nA_τ.

Let $q_t \geq 0$ denote the remaining inventory yet to be acquired at time t:

$$q_t = n - \int_0^t \nu_u\,du.$$

Following Jaimungal, Kinzebulatov, and Rubisov (2017), we assume that once the option is executed at time τ, the remaining q_τ shares which need to be repurchased are purchased on the interval $[\tau, \tau + \epsilon]$. While the payout of nA_τ is not affected by repurchases after time τ, we strictly enforce that all outstanding shares must be repurchased by time $\tau + \epsilon$. The optimal strategy to repurchase these shares with this constraint is detailed in Appendix A. The cumulative cost of acquiring the shares is given by a quadratic penalty function

$$\ell(q, S) = S(\alpha_2 q^2 + \alpha_1 q),$$

for some $\alpha_1, \alpha_2 > 0$. Then the profit or loss received by the intermediary for executing the option at time $\tau \in [0, T]$ is

$$nA_\tau - \ell(q_\tau, S_\tau) - \int_0^\tau \nu_u(S_u + aS_u\nu_u)\,du.$$

Note that there may be an additional constant fee from the repurchasing firm to the intermediary or vice versa; we ignore this here since it does not affect the intermediary's repurchasing strategy.

We will also introduce running penalties to model the intermediary's urgency and aversion to risk. First, we introduce a running penalty proportional to Sq^2, the stock price times the remaining shares to be repurchased squared, as in Jaimungal, Kinzebulatov, and Rubisov (2017). This can be viewed both as a penalty to encourage urgency (especially when the stock price is high), and as a way to express uncertainty about the model (Cartea, Donnelly, and Jaimungal 2017). Using a parameter ϕ to control the strength of this penalty, the full penalty is given by $\phi S q^2$. Further discussion of this penalty is in Appendix B. However, since we want to incorporate stochastic volatility, this is not sufficient (for example the optimal strategy in Section 3 of Jaimungal, Kinzebulatov, and Rubisov (2017) is wholly independent of the constant volatility level).

In order to guarantee that the volatility plays a role in the trading strategy, we also introduce a penalty of the type in Gatheral and Schied (2011). To derive this additional penalty, following Fischer (2018), first note that the SDE for the stock price has the following (strong) solution:

$$S_t = S_0 \exp \int_0^t \left(\mu - \frac{y_s}{2}\right) ds + \int_0^t \sqrt{y_s}\, dW_s^{(1)}.$$

We consider a risk measure R which is positive homogeneous, meaning that for scalar $c > 0$ and random variable X, $R(cX) = cR(X)$. For a risk measure of this type, the risk of the profit and loss over a horizon $t_0 > 0$ is

$$R(q_t(S_{t+t_0} - S_t)) = q_t S_t R\left(\exp \int_t^{t+t_0} \left(\mu - \frac{y_s}{2}\right) ds + \int_t^{t+t_0} \sqrt{y_s}\, dW_s^{(1)} - 1\right).$$

Note that if $t_0 << 1$ then

$$R\left(\exp \int_t^{t+t_0} \left(\mu - \frac{y_s}{2}\right) ds + \int_t^{t+t_0} \sqrt{y_s}\, dW_s^{(1)} - 1\right)$$

is approximately a non-negative function of y_t alone. Hence, we can approximate the risk of the profit and loss by $q_t S_t \lambda(y_t)$ where λ is some non-negative function of y_t. Note that the volatility y is a mean-reverting process, and hence y generally stays in a small interval around its mean m. Since a function like λ can be reasonably well approximated by a linear function on this small interval, for simplicity's sake we will usually use $\lambda(y) = \theta y + \kappa$ for scalars $\theta, \kappa \geq 0$. This penalty acts to encourage rapid liquidation (especially with large stock price or large volatility), with θ determining the sensitivity to the volatility.

The full value function is given by

$$\begin{aligned} H(t, S, A, q, y) = \sup_{\nu \geq 0, \tau \leq T} \mathbb{E}_{t,S,A,q,y} \Bigg[& nA_\tau - \ell(q_\tau, S_\tau) \\ & - \int_t^\tau \left[\nu_u(S_u + aS_u\nu_u) + \phi S_u q_u^2 + \lambda(y_u) S_u q_u\right] du \Bigg], \end{aligned} \tag{2}$$

where we use the notation $\mathbb{E}_{t,S,A,q,y}[\cdot] = \mathbb{E}[\cdot \mid S_t = S, A_t = A, q_t = q, y_t = y]$. The intermediary chooses an optimal strategy ν and an exercise time τ to maximize the expected returns, subject to the running penalties we discussed before.

3. Stochastic Volatility Without Early Termination

We begin by imposing the requirement that the intermediary cannot exercise their option early. This is equivalent to fixing $\tau = T$. The value function is

$$H(t,S,A,q,y) = \sup_{\nu \geq 0} \mathbb{E}_{t,S,A,q,y}\Bigg[nA_T - \ell(q_T, S_T) - \int_t^T [\nu_u(S_u + aS_u\nu_u) + \phi S_u q_u^2 + \lambda(y_u)S_u q_u]\, du \Bigg].$$

It is convenient to introduce the infinitesimal generator $\mathcal{L}_{S,y}$ of the Markov process (S,Y) in (1):

$$\mathcal{L}_{S,y} = \frac{1}{2}yS^2\partial_{SS} + \mu S\partial_S + \rho\psi Sy\partial_{Sy} + \frac{1}{2}\psi^2 y\partial_{yy} + \gamma(m-y)\partial_y.$$

Dynamic programming arguments tell us that the value function solves the following PDE problem:

$$\begin{aligned}
&\partial_t H + \mathcal{L}_{S,y}H + \frac{S-A}{t}\partial_A H - \phi Sq^2 - \lambda(y)Sq + \sup_{\nu\geq 0}\{-\nu\partial_q H - (aS\nu + S)\nu\} = 0 \quad (3)\\
&H(T,S,A,q,y) = nA - \ell(q,S),
\end{aligned}$$

where $t \in [0,T]; A, S \in [0,\infty); q \in [0,n]; y \in (0,\infty)$. The first order condition to optimize over ν leads to the optimal feedback control

$$\nu = \frac{1}{2aS}(-\partial_q H - S)\mathbf{1}_{\{q>0\}}. \tag{4}$$

While we primarily rely on the price impact a to constrain the trading rate, some contracts might impose restrictions on the maximum trading rate permitted. It is straightforward to incorporate this by trading according to $\min(\nu, \nu^{max})$, where ν^{max} is the maximum permitted trading rate.

Substituting ν gives

$$\begin{aligned}
&\partial_t H + \mathcal{L}_{S,y}H + \frac{S-A}{t}\partial_A H - \phi Sq^2 - \lambda(y)Sq + \frac{1}{4aS}(\partial_q H + S)^2 = 0\\
&H(T,S,A,q,y) = nA - \ell(q,S).
\end{aligned}$$

The following change of variables in terms of the ratio of the average and the stock price reduces the dimension of the problem. Let $H(t,S,A,q,y) = Sh(t,z,q,y)$, where $z = A/S$. Then the reduced PDE is given by

$$\begin{aligned}
&\partial_t h + \mu h + \mathcal{L}_{z,y}h + \left(\frac{1-z}{t} - \mu z\right)\partial_z h - \phi q^2 - \lambda(y)q + \frac{1}{4a}(\partial_q h + 1)^2 = 0, \quad (5)\\
&h(T,z,q,y) = nz - \alpha_2 q^2 - \alpha_1 q,
\end{aligned}$$

where we define

$$\mathcal{L}_{z,y} = \frac{1}{2}yz^2\partial_{zz} + [\gamma m + (\rho\psi - \gamma)y]\partial_y + \frac{1}{2}\psi^2 y\partial_{yy}h - \rho\psi yz\partial_{zy}.$$

For affine $\lambda(y) = \theta y + \kappa$, motivated above, the polynomial ansatz

$$h(t,z,q,y) = f(t,z) - v(t)q^2 + D(t)yq + F(t)q + G(t)y^2 + J(t)y + K(t) \tag{6}$$

gives an explicit solution. We define $\beta = \sqrt{a^2\mu^2 + 4a\phi}$, and $r_{1,2}$ as the roots of the quadratic equation $v^2 - a\mu v - a\phi = 0$, namely $r_{1,2} = \frac{1}{2}(a\mu \pm \beta)$.

Lemma 3.1. *The solution to the PDE problem* (3) *is given by*

$$\begin{aligned} H(t,S,A,q,y) =& \frac{n}{T}\left(tA + S\frac{e^{\mu(T-t)}-1}{\mu}\right) \\ &+ S\left\{-v(t)q^2 + [D(t)y + F(t)]q + G(t)y^2 + J(t)y + K(t)\right\}, \end{aligned} \tag{7}$$

where

$$v(t) = \begin{cases} \frac{r_1 - Q(t)r_2}{1-Q(t)}, \quad Q(t) = \frac{\alpha_2 - r_1}{\alpha_2 - r_2}e^{-\frac{\beta}{a}(T-t)} & \phi \neq (\alpha_2^2/a) - \mu\alpha_2 \\ \alpha_2 & \phi = (\alpha_2^2/a) - \mu\alpha_2, \end{cases} \tag{8}$$

and D, F, G, J, K are solutions of linear ODEs given in the proof. The optimal acquisition rate is given by

$$\nu^* = \left[\frac{q}{a}v(t) - \frac{D(t)y + F(t) + 1}{2a}\right]\mathbf{1}_{\{q>0\}}.$$

Proof. If we substitute (6) into (5), and group all terms which are dependent on z we get the following PDE:

$$f_t + \mu f + \frac{1}{2}yz^2 f_{zz} + \left(\frac{1-z}{t} - \mu z\right)f_z = 0, \quad f(T,z) = nz,$$

which has unique solution

$$f(t,z) = \frac{n}{T}\left(tz + \frac{e^{\mu(T-t)}-1}{\mu}\right).$$

By grouping all terms of order q^2 we obtain the ODE

$$v_t + \mu v + \phi - \frac{1}{a}v^2 = 0, \quad v(T) = \alpha_2$$

which is solved by (8). Note that the second case there includes the edge cases $\alpha_2 = r_{1,2}$. Finally, if we group the terms of order yq, q, y^2, y, as well as constant terms, we derive the following linear ODEs:

$$\begin{aligned}
&D_t + \left(\mu + \rho\psi - \gamma - \frac{v}{a}\right) D = \theta, && D(T) = 0\\
&F_t + \left(\mu - \frac{v}{a}\right) F = \kappa - \gamma m D + \frac{v}{a}, && F(T) = -1\\
&G_t + (\mu + 2(\rho\psi - \gamma))\, G = -\frac{D^2}{4a}, && G(T) = 0\\
&J_t + (\mu + (\rho\psi - \gamma))\, J = -[2\gamma m + \psi^2]G - \frac{1}{2a} D(F+1), && J(T) = 0\\
&K_t + \mu K = -\gamma m J - \frac{1}{4a}(F+1)^2, && K(T) = 0.
\end{aligned}$$

Performing the change of variables $z = A/S$ and $H(t, S, A, q, y) = Sh(t, z, q, y)$ gives (7). Finally, we can derive the optimal acquisition rate by substituting H into our formula for the optimal control (4):

$$\nu^* = \frac{1}{2aS}(-\partial_q H - S)\mathbf{1}_{\{q>0\}} = \left[\frac{q}{a}v(t) - \frac{D(t)y + F(t) + 1}{2a}\right]\mathbf{1}_{\{q>0\}}.$$

□

The linear ODEs can be solved explicitly, but we omit their explicit solutions for brevity.

Figure 1 plots some sample paths of the ratio A/S, the trading rate ν, the volatility y, and the amount of shares to repurchase q, as well as plotting $v(t), D(t), F(t)$. We can see that both D and F are negative functions which encourage the intermediary to trade more rapidly. D is scaled by the volatility in the formula for ν, and the effect of this is to encourage faster trading when volatility is high. We can see this in the sample paths where the blue and green curves initially have a higher trading rate since they begin with a higher volatility, and accordingly they finish repurchasing the shares earlier. On the other hand, the orange curve starts with noticeably lower volatility, and hence it has a lower trading rate and finishes acquiring shares later. It is also notable that the ratio of A/S has no effect on trading, since the intermediary cannot exercise early to take advantage of a high A/S.

4. ASR With Early Termination

We now enable the intermediary to exercise early. We first analyze the case where there are no shares to repurchase before considering the full problem.

4.1. *American-Asian Option*

To analyze the case where there are no shares to repurchase (i.e. $q = 0$), we follow Appendix B of Jaimungal, Kinzebulatov, and Rubisov (2017). When there are no shares to repurchase, the only decision for the intermediary to make is the time to exercise. Hence, we consider an American option which pays the average of the stock price up to the exercise time (i.e. an option that pays A_τ when exercised at time τ). Define $p(t, S, A, y)$ to be the optimal expected payoff at time t of this option given

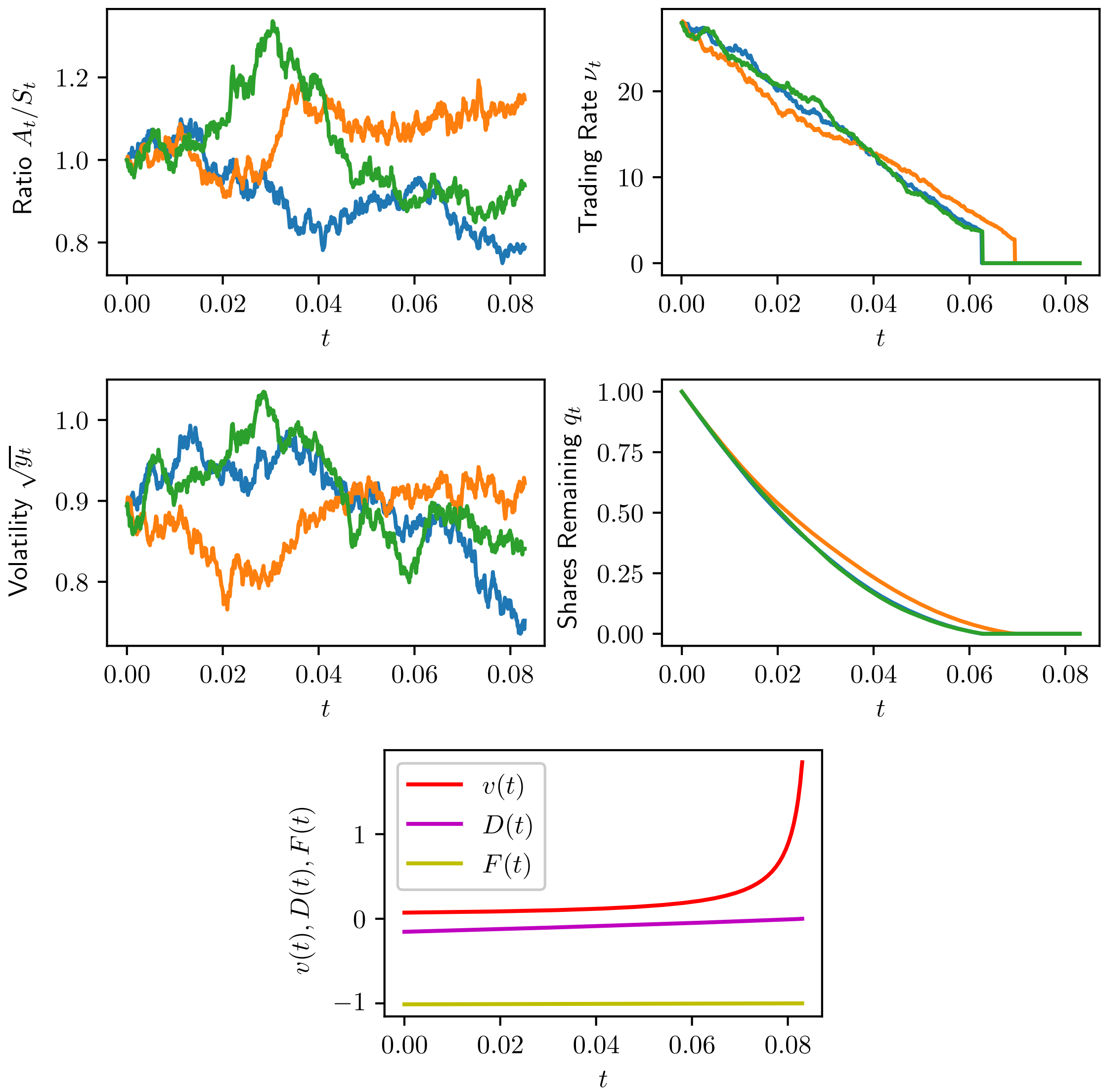

Figure 1. The top four plots are sample paths of ASR without early exercise, showing the ratio of the average process to the stock price A_t/S_t, the trading rate ν_t, the volatility y_t, and the shares remaining to purchase q_t. The fifth plot shows $v(t), D(t), F(t)$ in red, purple and yellow respectively. Parameters are $a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \psi = 1, \theta = 4, n = 1, T = 0.083$.

that $S_t = S, A_t = A, y_t = y$ (where S_t, A_t, y_t are the same stock price, averaging, and volatility processes from before):

$$p(t,S,A,y) = \sup_{\tau \le T} \mathbb{E}_{t,S,A,y}\left[A_\tau\right].$$

Then dynamic programming suggests that

$$\begin{aligned} p(t,S,A,y) &\ge A \\ \partial_t p + \mathcal{L}_{S,y} p + \frac{S-A}{t}\partial_A p &\le 0, \end{aligned}$$

with terminal condition $p(T,S,A,y) = A$ and with equality in at least one of the inequalities at all times.

We can substitute in the ansatz $p(t,S,A,y) = Sw(t,z,y)$ where $z = A/S$ to obtain

$$\begin{aligned} w(t,z,y) &\ge z \\ \partial_t w + \mu w + \mathcal{L}_{z,y} w + \left(\frac{1-z}{t} - \mu z\right)\partial_z w &\le 0, \end{aligned}$$

with terminal condition $w(T,z,y) = z$. We'll see in the next subsection that we need $w(t,z,y)$ to solve the full problem. Hence, we begin by determining appropriate boundary conditions and characterizing $w(t,z,y)$ in terms of a free boundary problem which we can solve numerically.

To begin, we have $0 < t < T$ and a terminal condition $w(T,z,y) = z$ since the option must be exercised by time T. We truncate z to fall in the interval $[\underline{z}, \overline{z}]$ where $\underline{z}, \overline{z}$ is sufficiently small (respectively large) enough. If $z = A/S$ is large, this indicates that the stock price has fallen, and hence that the option should be exercised before the average drops as well. It follows that the option should be exercised immediately, giving $w(t,\overline{z},y) = \overline{z}$.

On the other hand, if A/S is small, this indicates that the stock price has surged, and so we should wait to exercise until the average catches up with the stock price, which will increase the value of A/S. We set a reflecting boundary condition $\partial_z w(t,\underline{z},y) = 0$.

Finally, since we choose the parameters of the volatility process to obey the Feller condition, our volatility y is always positive and so it generally falls in an interval $(0,\overline{y}]$ where $\overline{y}$ is sufficiently large. Since the volatility y is mean-reverting, we choose reflecting boundary conditions at both boundaries so that $\partial_y w(t,z,0) = \partial_y w(t,z,\overline{y}) = 0$.

The dimensionally reduced QVI is characterized by a free boundary problem, where the intermediary should exercise early when $z = A/S$ is sufficiently large. The free boundary $z^*(t,y)$ divides the domain into portions so that the continuation region, stopping region, and exercise boundaries are given by the following sets respectively:

$$\begin{aligned} \{(t,S,A,y) : p(t,S,A,y) > A\} &= \{(t,S,A,y) : A/S < z^*(t,y)\} \\ \{(t,S,A,y) : p(t,S,A,y) < A\} &= \{(t,S,A,y) : A/S > z^*(t,y)\} \\ \{(t,S,A,y) : p(t,S,A,y) = A\} &= \{(t,S,A,y) : A/S = z^*(t,y)\}. \end{aligned}$$

The optimal stopping time τ^* is then given by

$$\tau^* = \inf\{t : A_t/S_t \geq \min(\overline{z}, z^*(t, y_t))\} \wedge T.$$

4.2. *Full Problem*

We now consider the full problem with the intermediary permitted to exercise early. The associated value function for the problem is given by (2). Dynamic programming arguments suggest that the value function satisfies the following QVI in the viscosity sense:

$$\min\{-\partial_t H - \mathcal{L}_{S,y} H - \frac{S-A}{t}\partial_A H + \phi S q^2 + \lambda(y) S q - \frac{1}{4aS}(\partial_q H + S)^2, \\ H(t, S, A, q, y) - nA + \ell(q, S)\} = 0,$$

with terminal condition $H(T, S, A, q, y) = nA - \ell(q, S)$. Similar to the case without early exercise, we can substitute the ansatz

$$H(t, S, A, q, y) = S h(t, z, q, y), \quad z = A/S.$$

Then

$$\min\{-\partial_t h - \mu h - \mathcal{L}_{z,y} h - \left(\frac{1-z}{t} - \mu z\right)\partial_z h + \phi q^2 + \lambda(y) q - \frac{1}{4a}(\partial_q h + 1)^2, \\ h(t, z, q, y) - nz + \alpha_2 q^2 + \alpha_1 q\} = 0.$$

We next consider the appropriate domain and associated boundary conditions. The boundaries for z, y are essentially the same as for the American-Asian option. To begin, $t \in [0, T]$ and the terminal condition is given by

$$h(T, z, q, y) = nz - \alpha_2 q^2 - \alpha_1 q$$

since the intermediary must exercise at time T. Again $z \in [\underline{z}, \overline{z}]$ and if $z = A/S$ is large, this indicates that the stock price has fallen, and corresponds to a large payoff for the intermediary. It follows that the intermediary should exercise immediately, giving

$$h(t, \overline{z}, q, y) = n\overline{z} - \alpha_2 q^2 - \alpha_1 q.$$

If A/S is small, this indicates that the intermediary should wait to exercise until the average catches up with the stock price, which will increase the value of A/S. We set a reflecting boundary condition $\partial_z h(t, \underline{z}, q, y) = 0$.

Unlike the American-Asian option, we also need a boundary for our inventory q, which falls in the interval $[0, n]$. At the boundary $q = 0$ the intermediary only needs to decide the best time to exercise in order to receive a payoff of n American-Asian options. This exactly corresponds to the case discussed in Section 4.1 so the appropriate boundary condition is $h(t, z, 0, y) = nw(t, z, y)$.

Finally, since the volatility y is mean-reverting, we again choose reflecting boundary conditions at both boundaries so that $\partial_y h(t, z, q, 0) = \partial_y h(t, z, q, \overline{y}) = 0$.

The dimensionally reduced QVI is characterized by a free boundary problem, where the intermediary should exercise early when $z = A/S$ is sufficiently large. The free boundary $z^*(t,q,y)$ divides the domain into portions so that the continuation region, stopping region, and exercise boundaries are given by the following sets respectively:

$$\begin{aligned}
\{(t,S,A,q,y) : H(t,S,A,q,y) > nA - \ell(q,S)\} &= \{(t,S,A,q,y) : A/S < z^*(t,q,y)\}\\
\{(t,S,A,q,y) : H(t,S,A,q,y) < nA - \ell(q,S)\} &= \{(t,S,A,q,y) : A/S > z^*(t,q,y)\}\\
\{(t,S,A,q,y) : H(t,S,A,q,y) = nA - \ell(q,S)\} &= \{(t,S,A,q,y) : A/S = z^*(t,q,y)\}.
\end{aligned}$$

The optimal stopping time τ^* is then given by

$$\tau^* = \inf\{t : A_t/S_t \geq \min(\overline{z}, z^*(t,q_t,y_t))\} \wedge T.$$

Hence, the intermediary should trade according to the feedback control of (4) until time τ^*, at which point they should exercise, and acquire the remaining shares according to the strategy of Appendix A.

4.3. *Constant Volatility*

The full problem can be reduced to the constant volatility case by taking volatility of the volatility $\psi = 0$ and fixing the volatility to be $y_t = \sigma^2, t \in [0,T]$. In this case, the value function is defined as

$$H(t,S,A,q) = \sup_{\nu\geq 0,\tau\leq T} \mathbb{E}_{t,S,A,q}\left[nA_\tau - \ell(q_\tau,S_\tau) - \int_t^\tau [\nu_u(S_u + aS_u\nu_u) + \phi S_u q_u^2 + \lambda S_u q_u]\,du\right],$$

where $\lambda = \lambda(\sigma^2)$ is a constant. Similarly, the associated QVI is given by

$$\min\{-\partial_t H - \mu S\partial_S H - \frac{1}{2}\sigma^2 S^2 \partial_{SS}H - \frac{S-A}{t}\partial_A H + \phi S q^2 + \lambda S q - \frac{1}{4aS}(\partial_q H + S)^2,$$
$$H(t,S,A,q) - nA + \ell(q,S)\} = 0,$$

with terminal condition $H(T,S,A,q) = nA - \ell(q,S)$. The dimensionally reduced QVI with $H(t,S,A,q) = Sh(t,z,q), z = A/S$ is

$$\min\{-\partial_t h - \mu h - \frac{1}{2}\sigma^2 z^2 \partial_{zz}h - \left(\frac{1-z}{t} - \mu z\right)\partial_z h + \phi q^2 + \lambda q - \frac{1}{4a}(\partial_q h + 1)^2,$$
$$h(t,z,q) - nz + \alpha_2 q^2 + \alpha_1 q\} = 0.$$

All of the rules for the boundary and terminal conditions discussed in the case of stochastic volatility are equally valid here, so the appropriate boundary and terminal conditions are

$$\begin{aligned}
h(T,z,q) &= nz - \alpha_2 q^2 - \alpha_1 q & \qquad h(t,z,0) &= nw(t,z)\\
h(t,\overline{z},q) &= n\overline{z} - \alpha_2 q^2 - \alpha_1 q & \qquad \partial_z h(t,\underline{z},q) &= 0.
\end{aligned}$$

Again, the intermediary should exercise early when $z = A/S$ is sufficiently large, and so the free boundary $z^*(t,q)$ divides the domain into portions. The continuation region, stopping region, and exercise boundaries for constant volatility are given by the following sets respectively:

$$\{(t,S,A,q) : H(t,S,A,q) > nA - \ell(q,S)\} = \{(t,S,A,q) : A/S < z^*(t,q)\}$$
$$\{(t,S,A,q) : H(t,S,A,q) < nA - \ell(q,S)\} = \{(t,S,A,q) : A/S > z^*(t,q)\}$$
$$\{(t,S,A,q) : H(t,S,A,q) = nA - \ell(q,S)\} = \{(t,S,A,q) : A/S = z^*(t,q)\}$$

and the optimal stopping time τ^* is given by

$$\tau^* = \inf\{t : A_t/S_t \geq \min(\overline{z}, z^*(t,q_t))\} \wedge T.$$

The intermediary should trade according to the feedback control of (4) until time τ^*, and then exercise and acquire the remaining shares according to the strategy of Appendix A.

5. Numerical Solution

The free boundary problem derived in Section 4.2 has a four dimensional domain in (t,z,q,y). Deep learning has recently been used to great success in solving high dimensional PDEs (see E, Han, and Jentzen (2020) for a review of techniques for solving high-dimensional PDEs) and more specifically in free boundary problems. For example Sirignano and Spiliopoulos (2018) used neural networks to solve high-dimensional free boundary PDEs related to pricing an American option of many stocks. Not only could very high-dimensional problems be solved, but the resulting neural network solutions compared favorably in terms of error to semi-analytic solutions for the problem. Indeed, neural networks have even been applied to ASRs with Guéant, Manziuk, and Pu (2020) demonstrating how to solve various high dimensional ASR problems in discrete time using neural networks. The use of neural networks allowed for more sophisticated contracts to be priced by limiting the effect of dimensionality. For us, the most important result is in Wang and Perdikaris (2021), which uses two simple fully-connected neural networks to solve the free-boundary Stefan problem. Their numerical solutions had low error when compared to exact problems, even with their straightforward implementation. We will use the same approach to numerically solve our free boundary problem for ASR with stochastic volatility.

The general idea is to use two distinct neural networks, one to model the value function of interest on the interior of the continuation region, and another to model the free exercise boundary. Both neural networks will be fully connected networks, with 3 hidden layers and 200 hidden units. The parameters will be initialized through Xavier initialization (Glorot and Bengio 2010), where the weights are initialized as normal random variables with variances chosen to ensure the weights neither blowup nor vanish. The only difference between the two networks will be the activation functions: the network for the value function will use only hyperbolic tangent activation functions for the hidden layers, while the network for the free boundary will use hyperbolic tangent activation functions for the hidden layers, and an exponential activation for the output layer since in our case the free boundary must be non-negative.

Training will be done using the Adam optimizer (Kingma and Ba 2017) with default settings and a learning rate of 0.001. Adam is frequently used in deep learning as an optimizer, since it improves on stochastic gradient descent by incorporating learning rates for the various parameters to improve fitting. There will be 160000 iterations of batch size $N = 10000$ points each. We will go into detail for our two cases: (i) the American-Asian option (ii) the full problem. We will also discuss how our analysis generalizes to the simpler case of constant volatility.

5.1. *American-Asian Option*

For the American-Asian option, we will model the value function $w(t, z, y)$ with one neural network, $v_\Gamma(t, z, y)$, and we will approximate the corresponding free boundary $z^*(t, y)$ with another neural network $r_\Delta(t, y)$ where Γ and Δ are the respective parameters for the networks. We then seek to minimize the following objective

$$\mathcal{L}^w(\Gamma, \Delta) = \Xi_1 \mathcal{L}^w_{interior}(\Gamma) + \Xi_2 \mathcal{L}^w_T(\Gamma) + \Xi_3 \mathcal{L}^w_z(\Gamma) + \Xi_4 \mathcal{L}^w_y(\Gamma) + \Xi_5 \mathcal{L}^w_{boundary}(\Gamma, \Delta)$$

where we define the linear operator $\mathcal{M}$ as

$$\mathcal{M}[w] = \partial_t w + \mu w + \mathcal{L}_{z,y} w + \left(\frac{1 - z}{t} - \mu z \right) \partial_z w$$

and the losses as

$$\begin{aligned}
\mathcal{L}^w_{interior}(\Gamma) &= \sum_{i=1}^{N} |\mathcal{M}[v_\Gamma](t^i, z^i, y^i)|^2 \\
\mathcal{L}^w_T(\Gamma) &= \sum_{i=1}^{N} |v_\Gamma(T, z^i, y^i) - z^i|^2 \\
\mathcal{L}^w_z(\Gamma) &= \sum_{i=1}^{N} |v_\Gamma(t, \overline{z}, y^i) - \overline{z}|^2 + \sum_{i=1}^{N} |\frac{\partial v_\Gamma}{\partial z}(t^i, \underline{z}, y^i)|^2 \\
\mathcal{L}^w_y(\Gamma) &= \sum_{i=1}^{N} |\frac{\partial v_\Gamma}{\partial y}(t^i, z^i, \overline{y})|^2 + \sum_{i=1}^{N} |\frac{\partial v_\Gamma}{\partial y}(t^i, z^i, 0)|^2 \\
\mathcal{L}^w_{boundary}(\Gamma, \Delta) &= \sum_{i=1}^{N} |v_\Gamma(t^i, r_\Delta(t^i, y^i), y^i) - r_\Delta(t^i, y^i)|^2.
\end{aligned}$$

The first requires that the neural network satisfy the differential operator. The second, third, and fourth terms are the boundary conditions discussed in Section 4. The fifth term specifies that the value function is equal to z across the free boundary, ensuring continuity. The weights $\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5$ allow us to prioritize various parts of the loss. N is the batch size and (t^i, z^i, y^i) are collocation points sampled uniformly at random on $[0, T] \times [\underline{z}, \overline{z}] \times [0, \overline{y}]$. In each of the 160000 iterations, we sample $N = 10000$ collocation points, compute the loss $\mathcal{L}^w(\Gamma, \Delta)$, and then minimize it using the Adam optimizer.

5.2. *Full Problem*

The same principle can be used for the full problem, using one neural network to model the value function $h(t, z, q, y)$ on the interior of the continuation region, and another to model the exercise boundary $z^*(t,q,y)$. More specifically, we will approximate $h(t,z,q,y)$ with the neural network $u_\Lambda(t,z,q,y)$ and we will approximate $z^*(t,q,y)$ with the neural network $s_\beta(t,q,y)$ where Λ and β are the respective parameters for the networks. We then seek to minimize the following objective

$$\begin{aligned}\mathcal{L}(\Lambda,\beta) = \xi_1\mathcal{L}_{interior}(\Lambda) + \xi_2\mathcal{L}_T(\Lambda) + \xi_3\mathcal{L}_z(\Lambda) + \xi_4\mathcal{L}_q(\Lambda)\\ + \xi_5\mathcal{L}_y(\Lambda) + \xi_6\mathcal{L}_{boundary}(\Lambda,\beta) + \xi_7\mathcal{L}_{s_0}(\beta)\end{aligned}$$

where we define the nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}[h] = -\mathcal{M}[h] + \phi q^2 + \lambda(y)q - \frac{1}{4a}(\partial_q h + 1)^2$$

and the losses as

$$\begin{aligned}
\mathcal{L}_{interior}(\Lambda) &= \sum_{i=1}^{N} |\mathcal{N}[u_\Lambda](t^i,z^i,q^i,y^i)|^2\\
\mathcal{L}_T(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(T,z^i,q^i,y^i) - nz^i + \alpha_2(q^i)^2 + \alpha_1 q^i|^2\\
\mathcal{L}_z(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(t^i,\overline{z},q^i,y^i) - n\overline{z} + \alpha_2(q^i)^2 + \alpha_1 q^i|^2 + \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial z}(t^i,\underline{z},q^i,y^i)|^2\\
\mathcal{L}_q(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(t^i,z^i,0,y^i) - nv_\Gamma(t^i,z^i,y^i)|^2\\
\mathcal{L}_y(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial y}(t^i,z^i,q^i,\overline{y})|^2 + \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial y}(t^i,z^i,q^i,0)|^2\\
\mathcal{L}_{boundary}(\Lambda,\beta) &= \sum_{i=1}^{N} |u_\Lambda(t^i,s_\beta(t^i,q^i,y^i),q^i,y^i) - ns_\beta(t^i,q^i,y^i) + \alpha_2(q^i)^2 + \alpha_1 q^i|^2\\
\mathcal{L}_{s_0}(\beta) &= \sum_{i=1}^{N} |s_\beta(t^i,0,y^i) - r_\Delta(t^i,y^i)|^2.
\end{aligned}$$

The first term requires that the value function satisfies the nonlinear differential operator. The next four terms are the boundary conditions discussed in Section 4. The sixth term specifies that the value function is equal to $nz - \alpha_2 q^2 - \alpha_1 q$ across the free boundary, and the final term specifies that the free boundary is equal to the boundary of the American-Asian option when $q = 0$, as we would expect. It is critical that we have already computed v_Γ and r_Δ as numerical solutions to the American-Asian option, so that we can use them when computing the losses $\mathcal{L}_q$ and $\mathcal{L}_{s_0}$.

The weights $\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7$ allow us to prioritize various parts of the loss. N is the batch size and (t^i,z^i,q^i,y^i) are collocation points sampled uniformly at random on $[0,T]\times[\underline{z},\overline{z}]\times[0,n]\times[0,\overline{y}]$. In each of the 160000 iterations, we sample $N = 10000$ collocation points, compute the loss $\mathcal{L}(\Lambda,\beta)$, and then minimize it using the Adam optimizer.

Finally, we note that the constant volatility problem can be solved in much the same way as the full problem with two important changes. Firstly, since the volatility is constant, the problem is reduced from four dimensions in (t, z, q, y) to three dimensions in (t, z, q). Correspondingly, the American-Asian option is reduced in dimension from three dimensions in (t, z, y) to two dimensions in (t, z). Secondly, because of this, there are no boundary conditions for the square volatility y. Otherwise, the approach for constant volatility is the same as for stochastic volatility, namely to use one neural network as the value function and another as the exercise boundary. The minimized loss is then the sum squared of the boundary conditions for t, z, q and the corresponding nonlinear operator for the interior of the continuation region.

5.3. *Results*

Figure 2 shows sample paths for ASR with early exercise. We can see several important features from these paths. The free boundary decreases as a function of time. This can be explained by the reduced shares left to repurchase. Since the penalty for early exercise is $\alpha_1 q + \alpha_2 q^2$, when q decreases as shares are repurchased, the penalty for early exercise becomes less of a factor, while the running penalties on inventory cause a greater impact, incentivizing the intermediary to exercise early. We can also see how important A/S is in the exercise time by comparing the orange and blue curves. Even though the orange curves have more shares remaining, the uptick in A/S for orange, and the downturn for blue, lead to orange exercising early.

As expected, a higher volatility corresponds to a faster trading rate. The orange curve has a noticeably lower volatility, and as such the corresponding trading rate is lower leaving more shares remaining. The final plot shows the evolution of the free boundary $z^*(t, q_t, y_t)$ as a function of time. Since the orange curve trades less quickly, it has more inventory, corresponding to a larger penalty for early exercise. Hence, the corresponding free boundary is larger, since it requires a larger value of z to justify exercising early.

We plot more sample paths of the trading rate ν_t in Fig. 3. Trading rates for $n = 0.5, 2, 5$ are shown for the same stock and volatility sample paths in Fig. 2. We can see that starting with a lower number of initial shares n leads to earlier exercise, since the penalty for exercise is monotone increasing in the number of shares to repurchase q. We can also see that for $n = 0.5$, the volatility has relatively more of an effect, owing to the shorter time horizon. On the other hand, for $n = 2, 5$, the volatility has less of an effect, since the trading is more impacted by the quadratic urgency term ϕq^2 for large q.

To demonstrate the benefit of incorporating stochastic volatility, in Fig. 4 we plot the expectation and standard deviation of the intermediary's profit or loss as a result of facilitating an ASR with early exercise. Recall that the intermediary's profit or loss for executing an ASR with strategy ν and exercise time τ is given by

$$nA_\tau - \ell(q_\tau, S_\tau) - \int_0^\tau \nu_u(S_u + aS_u\nu_u)\,du,$$

where the first term is the payoff for exercising at time τ, the second term is the penalty function for exercising with q_τ shares remaining, and the third term is the cost of acquiring the shares.

For each dot in the figure we simulate 10^6 iterations of each trading strategies and calculate both the average profit or loss, as well as the risk as measured by the

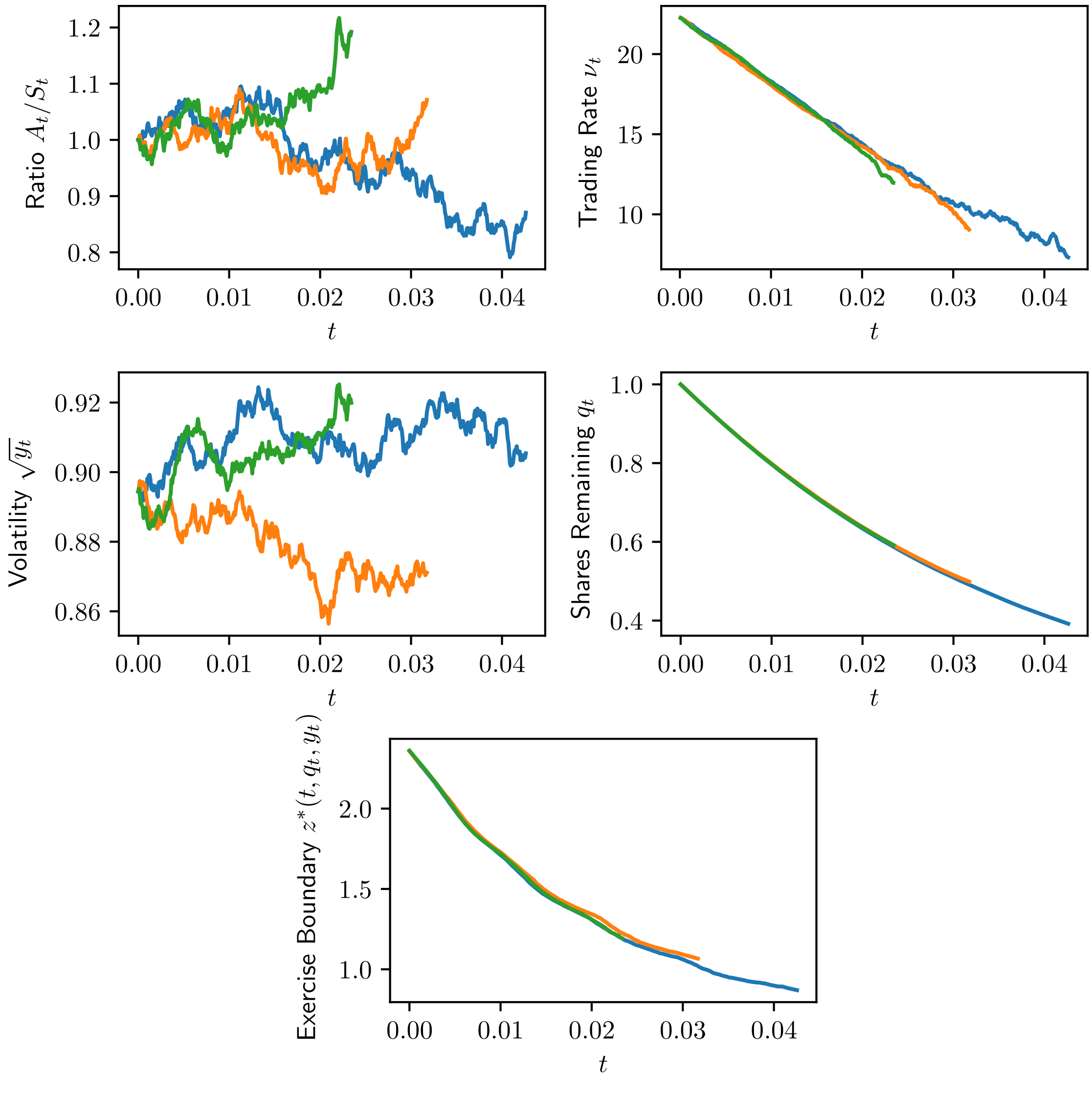

Figure 2. Sample paths of ASR with early exercise, showing the ratio of the average process to the stock price A_t/S_t, the trading rate ν_t, the volatility $\sqrt{y_t}$, the shares remaining to purchase q_t, and the corresponding exercise boundary. Paths are plotted until the intermediary exercises their option. Parameters are $a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \psi = 1, \theta = 4, n = 1, T = 0.083$

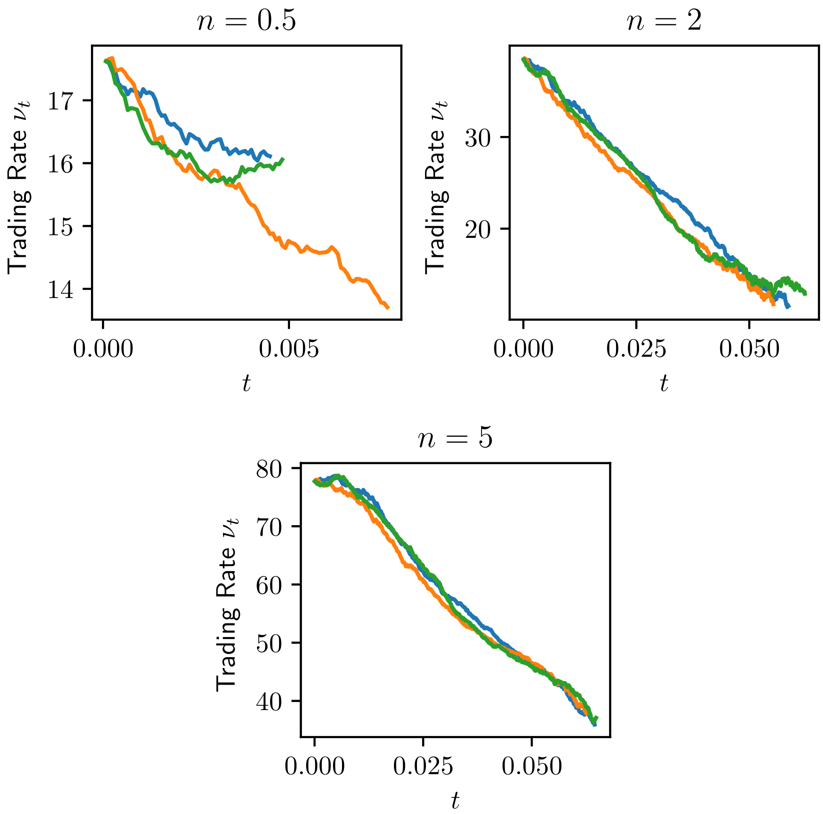

Figure 3. Trading rates ν_t of ASR with early exercise, for $n = 0.5, 2, 5$. Paths are plotted until the intermediary exercises their option. Stock and volatility trajectories are the same as in Fig. 2. Parameters are $a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \psi = 1, \theta = 4, T = 0.083$

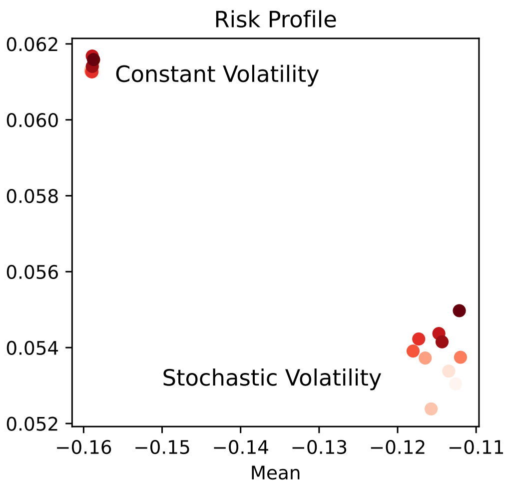

Figure 4. The expected returns and standard deviation of ASR with early exercise is plotted. The bottom right dots are generated by a strategy incorporating stochastic volatility, while the top left dots are generated by ignoring stochastic volatility. Points are plotted for $\psi = 0.3, 0.4, ..., 1.2$, where the darker the dot, the higher the corresponding value of ψ. Parameters are $a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \theta = 4, n = 1, T = 0.083$

standard deviation. The stock and volatility are given by the Heston model (1), which we simulate for $\psi = 0.3, 0.4, ..., 1.2$. For the dots in the bottom right of the plot, we simulate (using neural networks) a strategy for the intermediary which incorporates stochastic volatility, by solving the problem from Section 4.2. For the dots in the top left of the plot, we use a reduced strategy which ignores the stochasticity of the volatility and treats it as constant, by solving the problem from Section 4.3. In particular, we can see that incorporating the stochastic volatility yields better expected returns and less risk for all values of ψ. We can also see that in general a larger value of ψ corresponds to a greater risk in the case of stochastic volatility. Note that the mean returns are negative; hence an intermediary might charge a fee for executing this ASR.

5.4. *Validation*

We have seen in previous sections that we can use neural networks to solve the QVI modeling an ASR with early exercise, and that the resulting solution has qualitative features we desire out of our model. To validate our model, we would like some method of determining quantitative performance of our solution compared to some benchmark. In general, there is no straightforward way of doing this, since the QVIs we solve with neural networks lack explicit solutions. However, in a very specific parameter case we will see that we can derive an explicit solution to the QVI, which we can use to compare to our neural network solution.

In what follows, we set the drift $\mu = 0$, we assume the square volatility y is constant and equal to its mean m, and we set the urgency parameters $\theta = \kappa = 0$. In this case, the penalty function for early exercise reduces to $q(1 + \alpha q)$ for a single constant α. In order to derive an explicit solution we will fix $\alpha = \sqrt{\phi a}$. Given all of these restrictions, consider the American-Asian Option from Section 4.1. Recall that we want to derive the price of an option with the following pricing structure

$$p(t, S, A) = \sup_{\tau \leq T} \mathbb{E}_{t,S,A} [A_\tau]$$

where we suppress dependence on the volatility since it is constant in this section. With all of our restrictions to the parameter space, the QVI for this problem reduces to

$$\begin{aligned} p(t, S, A) &\geq A \\ \partial_t p + \frac{1}{2} m S^2 \partial_{SS} p + \frac{S - A}{t} \partial_A p &\leq 0 \end{aligned}$$

with terminal condition $p(T, S, A) = A$ and with equality in at least one of the inequalities at all times. We can substitute in the ansatz $p(t, S, A) = S w(t, z)$ where $z = A/S$ to obtain

$$\begin{aligned} w(t, z) &\geq z \\ \partial_t w + \frac{1}{2} m z^2 \partial_{zz} w + \frac{1 - z}{t} \partial_z w &\leq 0, \end{aligned}$$

with terminal condition $w(T, z) = z$. The explicit form for $w(t, z)$ is given by

$$w(t,z) = \max\{\frac{T + tz - t}{T}, z\} = \begin{cases} z & z \geq 1 \\ \frac{T+tz-t}{T} & z < 1. \end{cases}$$

Though we are primarily concerned with $w(t, z)$ we can also reverse the change of variables to derive that

$$p(t,S,A) = \begin{cases} A & A \geq S \\ \frac{At+S(T-t)}{T} & A < S. \end{cases}$$

Recall that the dimensionally reduced QVI for the ASR is given by

$$\begin{aligned} \min\{-\partial_t h - \frac{1}{2} m z^2 \partial_{zz} h - \frac{1-z}{t} \partial_z h + \phi q^2 - \frac{1}{4a}(\partial_q h + 1)^2, \\ h(t,z,q) - nz + q(1 + \sqrt{\phi a} q)\} = 0, \end{aligned} \tag{9}$$

where we have substituted in our restricted parameters for this section. Noting the similarity between the QVI for the ASR, and the QVI for the American-Asian option, we find the solution

$$h(t,z,q) = nw(t,z) - q(1 + \sqrt{\phi a} q). \tag{10}$$

Note that this solution crucially relies on the relationship $\alpha = \sqrt{\phi a}$ which enables $h(t, z, q)$ to have its z and q dependence split into separate terms.

For comparison's sake, we can also solve the QVI in (9) using a neural network. Fig. 5 plots the explicit solution (10) and the relative error between the neural network solution and the explicit solution at $t = 0.01$. We can see that the neural network is indeed a good approximation for the true solution, and this provides some numerical justification for our use of neural networks throughout this paper. Looking at the plot of the error, we can see that the worst difference is when the explicit solution is around zero; this is expected since the relative error is the difference between the numerical solution and the true solution divided by the true solution, and hence when the magnitude of the true solution is near zero, the relative error blows up. In general, the error decreases as t increases, since the neural networks are increasingly constrained by the terminal conditions. On the other hand the worst error is for small t, since the numerical solutions have the greatest freedom here, being relatively unconstrained by the terminal condition. The average relative error is approximately 0.9%.

6. Fixed Notional

We now consider another common form of ASR, where instead of requiring an intermediary to purchase a fixed number of shares, the intermediary is instead provided a fixed amount of cash F to repurchase a variable number of shares. This form of ASR has the following steps: (i) The firm provides the intermediary with a fixed amount F. In exchange, the intermediary borrows some number of shares, for example 80% of $\frac{F}{S}$ for stock price S, from other traders or financial institutions, and provides them to the firm; (ii) the intermediary settles its short position and acquires an additional

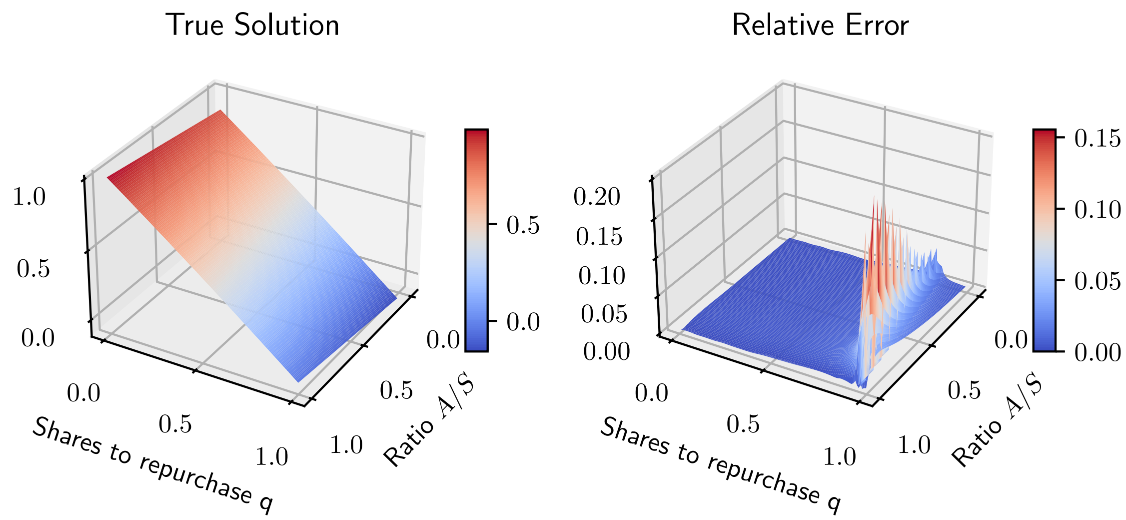

Figure 5. Contour plots of the solution to the QVI (9) at $t = 0.01$ as a function of q and $z = A/S$. The first panel shows the explicit solution (10), while the second shows the relative error between the numerical solution and the explicit solution. The average relative error is 0.9%. Parameters are $a = 0.005, \phi = 0.5, \alpha = 0.005, m = 0.8, \kappa = 0, \theta = 0, n = 1, T = 0.083$

number of shares so that it can provide a total of $\frac{F}{A}$ shares where A is the average stock price; (iii) at any time up to the maturity date the intermediary can execute the contract early; (iv) when the intermediary exercises (or the maturity date is reached), the remaining shares are provided to the repurchasing firm so that the total number of shares repurchased is $\frac{F}{A}$.

A discrete time version of this type of contract was analyzed by Guéant (2017) with exponential utility. Here, we analyze this problem in continuous time in a similar framework to the previous sections. In Section 6.1 we define the model and derive the associated control problem. The numerical solution using neural networks is presented in Section 6.2.

6.1. *Model*

In an ASR with a fixed notional, the repurchasing firm provides the intermediary a cash amount F. The task of the intermediary is to acquire F/A_τ shares over the time interval $[0, T]$, which it then provides to the repurchasing firm. Since there is no longer a fixed inventory to be acquired, it is convenient to define c_t to be the inventory acquired at time t:

$$c_t = \int_0^t \nu_u du.$$

The analogous quantity to q_t, the inventory yet to be acquired at time t, is

$$\frac{F}{A_t} - c_t.$$

The profit or loss received by the intermediary for exercising the option at time $\tau \in [0, T]$ is

$$F - \ell(\frac{F}{A_\tau} - c_\tau, S_\tau) - \int_0^\tau \nu_u(S_u + aS_u\nu_u)du.$$

The first two terms are the payout and the penalty for any shares which have yet to be acquired at exercise time, while the third time gives the cost to acquire the shares. The full value function is given by

$$H(t, S, A, c, y) = \sup_{\nu \geq 0, \tau \leq T} \mathbb{E}_{t,S,A,c,y}\Bigg[F - \ell(\frac{F}{A_\tau} - c_\tau, S_\tau) - \int_t^\tau [\nu_u(S_u + aS_u\nu_u) + \phi S_u(\frac{F}{A_u} - c_u)^2 + \lambda(y_u)S_u(\frac{F}{A_u} - c_u)]\, du\Bigg].$$

Note that we include running penalties on the difference between the inventory acquired c_t and the number of shares which need to be repurchased $\frac{F}{A_t}$. Like (2) this serves to model the intermediary's urgency and aversion to risk.

Dynamic programming tells us that the value function satisfies the following QVI in the viscoscity sense:

$$\min\{-\partial_t H - \mathcal{L}_{S,y}H - \frac{S-A}{t}\partial_A H + \phi S(\frac{F}{A} - c)^2 + \lambda(y)S(\frac{F}{A} - c) - \sup_{\nu \geq 0}\{\nu\partial_c H - (aS\nu + S)\nu\},$$
$$H(t, S, A, c, y) - F + \ell(\frac{F}{A} - c, S)\} = 0,$$

with terminal condition $H(T, S, A, c, y) = F - \ell(\frac{F}{A} - c, S)$. The first order condition to optimize over ν yields

$$\nu = \frac{1}{2aS}(\partial_c H - S)\mathbf{1}_{\{c < \frac{F}{A}\}}, \tag{11}$$

which when substituted yields

$$\min\{-\partial_t H - \mathcal{L}_{S,y}H - \frac{S-A}{t}\partial_A H + \phi S(\frac{F}{A} - c)^2 + \lambda(y)S(\frac{F}{A} - c) - \frac{1}{4aS}(\partial_c H - S)^2,$$
$$H(t, S, A, c, y) - F + \ell(\frac{F}{A} - c, S)\} = 0.$$

Notice that the change of variables $z = A/S$ which we used before to simplify the problem is no longer available, so the problem is 5-dimensional in (t, S, A, c, y). Next, let us consider the domain and boundary conditions. We have $t \in [0, T]$ and terminal condition

$$H(T, S, A, c, y) = F - \ell(\frac{F}{A} - c, S).$$

For c, we will bound it from above so that $c \in [0, \bar{c}]$. If the number of shares acquired is sufficiently high, it follows that the intermediary has likely acquired all the shares it needs, and so should exercise immediately. Hence

$$H(t, S, A, \overline{c}, y) = F.$$

We will also bound $A \in [\underline{A}, \overline{A}]$, $S \in [\underline{S}, \overline{S}]$ and $y \in [0, \overline{y}]$. We will choose to apply reflecting boundary conditions at these boundaries so that

$$\begin{aligned} \partial_A H(t, S, \overline{A}, c, y) &= 0 \\ \partial_S H(t, \overline{S}, A, c, y) = \partial_S H(t, \underline{S}, A, c, y) &= 0 \\ \partial_y H(t, S, A, c, 0) = \partial_y H(t, S, A, c, \overline{y}) &= 0. \end{aligned}$$

The QVI is characterized by a free boundary problem, where the intermediary should exercise early when it has acquired sufficiently many shares i.e. when c is sufficiently large. The free boundary $c^*(t, S, A, y)$ divides the domains into portions so that the continuation region, stopping region, and exercise boundaries are given by the following sets respectively:

$$\begin{aligned} \{(t, S, A, c, y) : H(t, S, A, c, y) > F - \ell(\frac{F}{A} - c, S)\} &= \{(t, S, A, c, y) : c < c^*(t, S, A, y)\} \\ \{(t, S, A, c, y) : H(t, S, A, c, y) < F - \ell(\frac{F}{A} - c, S)\} &= \{(t, S, A, c, y) : c > c^*(t, S, A, y)\} \\ \{(t, S, A, c, y) : H(t, S, A, c, y) = F - \ell(\frac{F}{A} - c, S)\} &= \{(t, S, A, c, y) : c = c^*(t, S, A, y)\} \end{aligned}$$

and the optimal stopping time τ^* is given by

$$\tau^* = \inf\{t : c_t \geq \min(\overline{c}, c^*(t, S_t, A_t, y_t))\} \wedge T.$$

The intermediary should trade according to the feedback control (11) until time τ^*, and then exercise and acquire the remaining shares according to the strategy of Appendix A.

6.2. *Numerical Solution*

The ASR with fixed notional can be analogously simulated to the ASR with fixed shares as in Section 5.2. We represent the value function as one neural network, the free boundary as another, and then attempt to minimize a constituent loss made up of applying a nonlinear operator over the interior, as well as the various boundary conditions. Full details are in Appendix C.1.

Sample paths of an ASR with fixed notional are shown in Figure 6. Notably, we can see that it is not advantageous to exercise early, in contrast to the ASR with fixed number of shares. For an ASR with a fixed number of shares, it is critical to exercise when A is relatively large, since the payoff function nA entirely depends on exercising optimally. On the other hand, for an ASR with fixed notional, the payoff F is independent of exercise time, so early exercise serves only to reduce the penalty from the number of shares remaining. Hence, it is more optimal to simply purchase shares normally, making use of the whole time period before exercising.

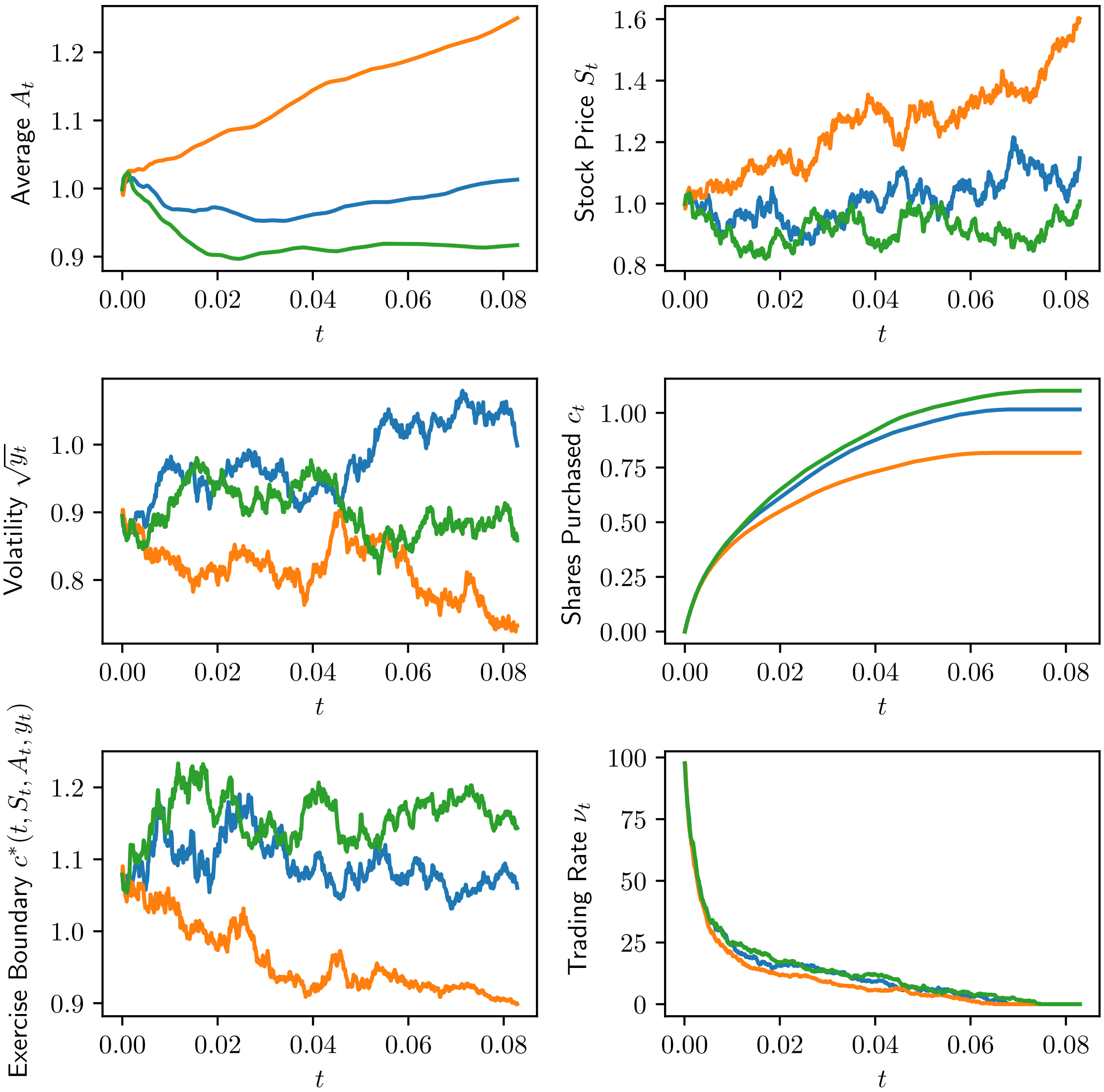

Figure 6. Sample paths of ASR with fixed notional and early exercise, showing the average process A_t, the stock price S_t, the trading rate ν_t, the volatility $\sqrt{y_t}$, the shares purchase c_t, and the corresponding exercise boundary. Paths are plotted until the intermediary exercises their option. Parameters are $a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \psi = 1, \theta = 4, F = 1, T = 0.083$

We can see that lower volatility corresponds to a lower trading rate and fewer share purchased. Also, we can observe that as the average A increases, the optimal exercise boundary decreases, since the number of shares which need to be repurchased is $\frac{F}{A}$. Since stock price and volatility are inversely correlated, there are two relevant trends. On one hand, the lower volatility implies higher stock prices and a higher average, which in turn implies that fewer shares need to be repurchased. On the other hand, lower volatility also implies a lower trading rate, which means fewer shares are repurchased. The combination of these two implies that when there are fewer shares to be repurchased the intermediary repurchases even fewer shares, which leads to the intermediary not exercising early.

7. Local Volatility

While the Heston model used throughout this paper is a popular model of stochastic volatility, an alternative is to use local volatility, where the volatility of the stock price is given by some function $\sigma(S,t)$ which depends purely on the current stock price and time. The evolution of the stock price is then given by

$$dS_t = \mu S_t dt + \sigma(S,t) dW_t.$$

Choosing $\sigma(S,t)$ to be constant recovers arithmetic Brownian motion, while $\sigma(S,t) = \sigma S$ recovers geometric Brownian motion. Local volatility is attractive since models can be easily fit using data from the options market.

In the following section, we will analyze an ASR with local volatility, specifically implementing the constant elasticity of variance model introduced by Cox (1975) where $\sigma(S,t) = \sigma S_t^{\delta/2}$. In this framework, the stock price evolves as

$$dS_t = \mu S_t dt + \sigma S_t^{\delta/2} dW_t$$

with volatility $\sigma S_t^{\delta/2-1}$. In particular, if $\delta < 2$, then the volatility is inversely related to the stock price. This model of an ASR is one dimension lower, since the volatility no longer varies independently. However, we will see that the dimensional reduction via change of variables $z = A/S$ is no longer possible, and hence the final problem formulation remains 4-dimensional in (t, S, A, q). We begin by analyzing the American-Asian option with no shares to repurchase in Section 7.1. The full problem is then studied in Section 7.2.

7.1. *American-Asian Option*

When there are no shares to repurchase, the intermediary holds an option with an Asian payout (i.e. it pays the average of the stock price) and an American exercise time. Define $p(t,S,A)$ to be the optimal expected payoff at time t given $S_t = S, A_t = A$. Then

$$p(t,S,A) = \sup_{\tau \le T} \mathbb{E}_{t,S,A}[A_\tau].$$

Dynamic programming tells us that p satisfies the following inequalities with equality in at least one of the inequalities at all times:

$$p(t,S,A) \geq A$$
$$\partial_t p + \mu S \partial_S H + \frac{1}{2}\sigma^2 S^\delta \partial_{SS} H + \frac{S-A}{t}\partial_A p \leq 0$$

with terminal condition $p(T,S,A) = A$. To solve for this numerically, we need to set boundary conditions. We already have a terminal condition, so all that is needed are boundaries in S, A. We truncate the domain so that $S \in [\underline{S}, \overline{S}]$ and $A \in [\underline{A}, \overline{A}]$. We then set reflecting boundary conditions so that

$$\partial_S p(t,\underline{S},A) = \partial_S p(t,\overline{S},A) = \partial_A p(t,S,\overline{A}) = 0.$$

The QVI is then characterized by a free boundary problem. A rapid decline in S presages a decline in the payout A, and hence the intermediary should exercise if the stock price is sufficiently small. Hence, the free boundary $S^*(t,A)$ divides the domain into a continuation, stopping, and exercise boundary region, given respectively by the following sets:

$$\{(t,S,A) : p(t,S,A) > A\} = \{(t,S,A) : S > S^*(t,A)\}$$
$$\{(t,S,A) : p(t,S,A) < A\} = \{(t,S,A) : S < S^*(t,A)\}$$
$$\{(t,S,A) : p(t,S,A) = A\} = \{(t,S,A) : S = S^*(t,A)\}.$$

The optimal stopping time τ^* is then given by

$$\tau^* = \inf\{t : S_t \leq \max(\underline{S}, S^*(t,A_t))\} \wedge T.$$

The American-Asian option can be simulated like in Section 5.1. We represent the interior as one neural network, the free boundary as another, and then minimize a constituent loss made up of applying a differential operator to the interior, as well as the boundary conditions. Full details are in Appendix C.2.

7.2. ***Full Model***

We now analyze the full model. The value function in this case is almost identical to (2), and is given by

$$H(t,S,A,q) = \sup_{\nu \geq 0, \tau \leq T} \mathbb{E}_{t,S,A,q}\Bigg[nA_\tau - \ell(q_\tau, S_\tau) - \int_t^\tau [\nu_u(S_u + aS_u\nu_u) + \phi S_u q_u^2 + \lambda(\sigma^2 S_u^{\delta-2}) S_u q_u]\, du\Bigg].$$

Importantly, the volatility penalization is now given by $\lambda(\sigma^2 S_u^{\delta-2})$, since the square volatility is now given by $\sigma^2 S_u^{\delta-2}$. The QVI can be derived through dynamic programming as

$$\min\{-\partial_t H - \mu S\partial_S H - \frac{1}{2}\sigma^2 S^\delta \partial_{SS} H - \frac{S-A}{t}\partial_A H + \phi S q^2 + \lambda(\sigma^2 S^{\delta-2}) S q - \frac{1}{4aS}(\partial_q H + S)^2,$$

$$H(T,S,A,q) - nA + \ell(q,S)\} = 0,$$

with terminal condition $H(T,S,A,q) = nA - \ell(q,S)$. In addition to the terminal condition, we impose a similar domain truncation and boundary conditions as with the American-Asian option:

$$\partial_S H(t,\underline{S},A,q) = \partial_S H(t,\overline{S},A,q) = \partial_A H(t,S,\overline{A},q) = 0.$$

At the $q = 0$ boundary, there are no shares left to repurchase, and so the value function should be equal to that of the American-Asian option:

$$H(t,S,A,0) = p(t,S,A).$$

The full problem is also characterized by a free boundary problem in S. A lower value of S anticipates a decline in the payout A, and also corresponds to a lower penalty for exercising. The intermediary should exercise when the stock price is below the free boundary, so $S < S^*(t,A,q)$. The continuation, stopping, and exercise boundary regions are given by

$$\begin{aligned}
\{(t,S,A,q) : H(t,S,A,q) > nA - \ell(q,S)\} &= \{(t,S,A,q) : S > S^*(t,A,q)\} \\
\{(t,S,A,q) : H(t,S,A,q) < nA - \ell(q,S)\} &= \{(t,S,A,q) : S < S^*(t,A,q)\} \\
\{(t,S,A,q) : H(t,S,A,q) = nA - \ell(q,S)\} &= \{(t,S,A,q) : S = S^*(t,A,q)\}.
\end{aligned}$$

The optimal stopping time τ^* is

$$\tau^* = \inf\{t : S_t \le \max(\underline{S}, S^*(t,A_t,q_t)\} \wedge T.$$

The full problem can be simulated similarly to Section 5.2. We represent the free boundary and value function as separate neural networks. The model is fitted by constructing a constituent loss made up of applying a differential operator to the interior, as well as contributions from boundary conditions. All details are in Appendix C.3.

Some sample paths for an ASR with local volatility are plotted in Figure 7. We can see that a lower volatility corresponds to a lower trading rate, with the green curve having a noticeably lower rate of repurchase and lower volatility. We can see that it is optimal to exercise early, typically with a peak in A coupled with a steep reduction in the stock price. Volatility and stock price are inversely related, as desired. The exercise boundary increases over time, since as shares are repurchased, the penalty $\ell(q,S)$ becomes lesser. A larger average A also corresponds to a larger exercise boundary, since this implies a larger payout.

8. Conclusion

We have extended the model of accelerated share repurchases from Jaimungal, Kinzebulatov, and Rubisov (2017) to include stochastic volatility. This inclusion is important

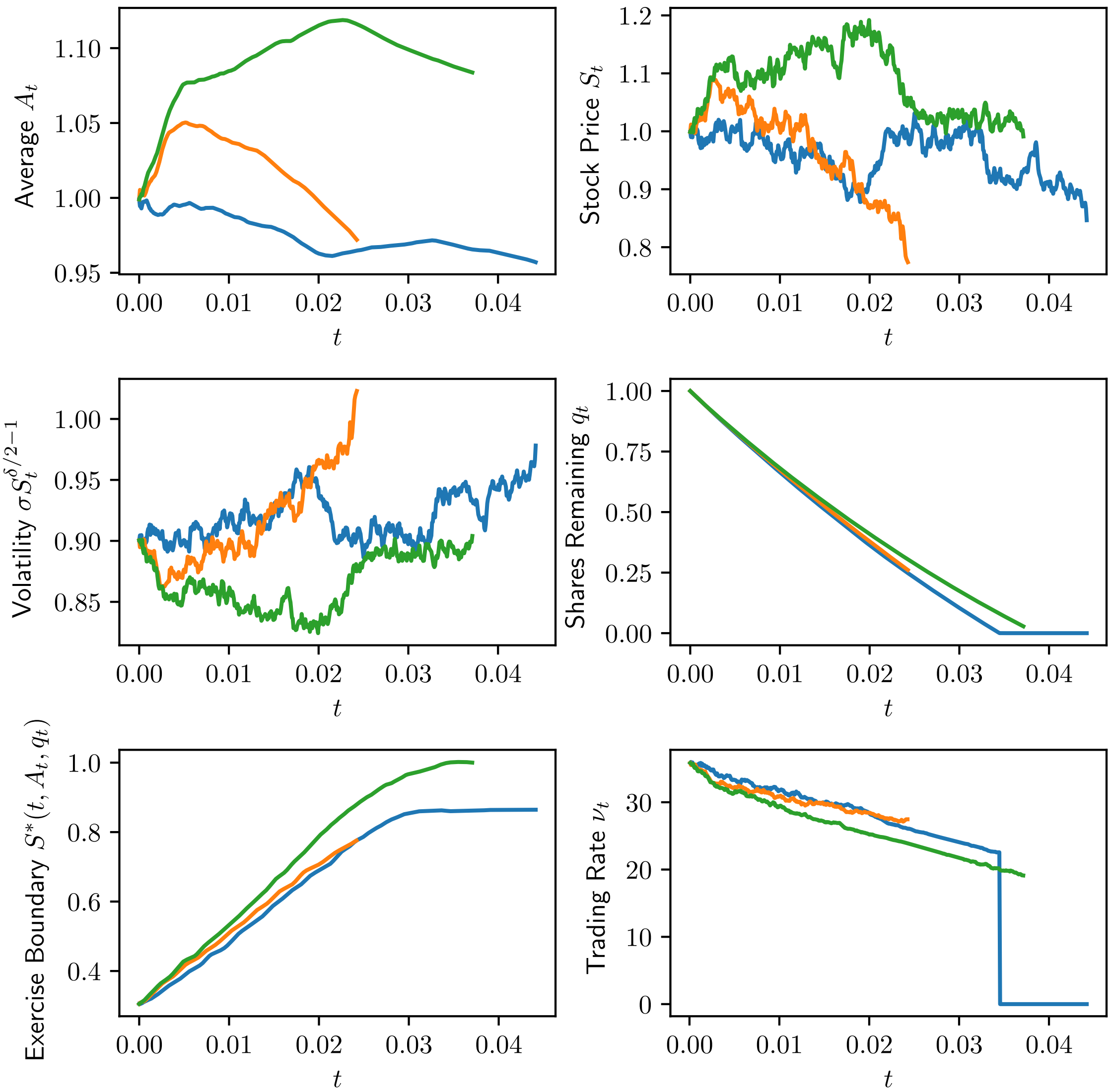

Figure 7. Sample paths of ASR with local volatility and early exercise, showing the average process A_t, the stock price S_t, the trading rate ν_t, the volatility $\sqrt{y_t}$, the shares remaining to repurchase q_t, and the corresponding exercise boundary. Paths are plotted until the intermediary exercises their option. Parameters are $\delta = 0.5, \sigma = 0.9, a = 0.005, \phi = 0.5, \alpha_1 = 1.00, \alpha_2 = 1.85, \kappa = 0.1, \theta = 4, n = 1, T = 0.083$

since ASRs are typically conducted over longer time horizons than standard optimal execution problems, so that the variability of the volatility has a more pronounced effect. We derived the associated free boundary problem which describes the intermediary's optimal trading strategy and exercise time.

By solving this problem numerically, we demonstrated that the intermediary is encouraged to trade more rapidly when the volatility is greater, and that the optimal exercise time occurs when the ratio of the average stock price to the current stock price crosses the free boundary. We have also illustrated that it is beneficial to include stochastic volatility. Simulations of the intermediary's profit demonstrate that strategies which incorporate stochastic volatility have greater average returns and lower average risk. Our numerical solution uses deep learning to solve the high-dimensional free boundary problem. By comparing our numerical solution to an analytical solution which we can derive for a special choice of parameters, a low numerical error is observed. The numerical method is capable of solving other models of ASRs including with fixed notional and local volatility. We find that in the case of fixed notional, it is optimal to make use of the full trading interval, rather than exercising early.

Disclosure Statement

No potential conflict of interest was reported by the author(s).

References

Adachi, Takashi. 2003. "The Value of the Perpetual American Call on the Time-Average of the Stock." *Interdisciplinary Information Sciences* 9 (2): 243–257.

Almgren, Robert. 2012. "Optimal Trading with Stochastic Liquidity and Volatility." *SIAM Journal on Financial Mathematics* 3 (1): 163–181.

Almgren, Robert, and Neil Chriss. 2001. "Optimal execution of portfolio transactions." *Risk* 3 (2): 5–40.

Bargeron, Leonce, Manoj Kulchania, and Shawn Thomas. 2011. "Accelerated share repurchases." *Journal of Financial Economics* 101 (1): 69–89. https://www.sciencedirect.com/science/article/pii/S0304405X11000377.

Black, Fischer, and Myron Scholes. 1972. "The Valuation of Option Contracts and a Test of Market Efficiency." *The Journal of Finance* 27 (2): 399–417. http://www.jstor.org/stable/2978484.

Canina, Linda, and Stephen Figlewski. 1993. "The Informational Content of Implied Volatility." *The Review of Financial Studies* 6 (3): 659–681. https://doi.org/10.1093/rfs/5.3.659.

Cartea, Álvaro, Ryan Donnelly, and Sebastian Jaimungal. 2017. "Algorithmic Trading with Model Uncertainty." *SIAM Journal on Financial Mathematics* 8 (1): 635–671.

Cheridito, Patrick, and Tardu Sepin. 2014. "Optimal Trade Execution Under Stochastic Volatility and Liquidity." *Applied Mathematical Finance* 21 (4): 342–362.

Cox, John C. 1975. "Notes on Option Pricing 1: Constant Elasticity of Variance Diffusions." *Unpublished* .

Criscuolo, Adriana M., and Henri Waelbroeck. 2014. "Effect of Volatility Fluctuations on Optimal Execution Schedules." *The Journal of Trading* 9 (4): 82–90. https://jot.pm-research.com/content/9/4/82.

E, Weinan, Jiequn Han, and Arnulf Jentzen. 2020. "Algorithms for Solving High Dimensional PDEs: From Nonlinear Monte Carlo to Machine Learning." .

Fischer, Paul. 2018. "Optimal Liquidation In Stochastic Volatility Models." Master's thesis, Humboldt-Universität zu Berlin, Germany.

Gatheral, Jim, and Alexander Schied. 2011. “Optimal Trade Execution Under Geometric Brownian Motion in the Almgren and Chriss Framework.” *International Journal of Theoretical and Applied Finance* 14 (03): 353–368.

Glorot, Xavier, and Yoshua Bengio. 2010. “Understanding the difficulty of training deep feedforward neural networks.” In *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, edited by Yee Whye Teh and Mike Titterington, Vol. 9 of *Proceedings of Machine Learning Research*, Chia Laguna Resort, Sardinia, Italy, 13–15 May, 249–256. PMLR. https://proceedings.mlr.press/v9/glorot10a.html.

Guéant, Olivier. 2017. “Optimal execution of accelerated share repurchase contracts with fixed notional.” *Risk* 19 (05): 77–99.

Guéant, Olivier, Jiang Pu, and Guillaume Royer. 2015. “Accelerated Share Repurchase: Pricing and Execution Strategy.” *International Journal of Theoretical and Applied Finance* 18 (03): 1550019.

Guéant, Olivier. 2014. “Optimal execution and block trade pricing: a general framework.” .

Guéant, Olivier, Iuliia Manziuk, and Jiang Pu. 2020. “Accelerated share repurchase and other buyback programs: what neural networks can bring.” *Quantitative Finance* 20 (8): 1389–1404.

Heston, Steven L. 1993. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.” *The Review of Financial Studies* 6 (2): 327–343. http://www.jstor.org/stable/2962057.

Jaimungal, Sebastian, Damir Kinzebulatov, and D.H. Rubisov. 2017. “Optimal accelerated share repurchases.” *Applied Mathematical Finance* 24 (3): 216–245.

Jensen, Michael C. 1986. “Agency Costs of Free Cash Flow, Corporate Finance, and Takeovers.” *The American Economic Review* 76 (2): 323–329. http://www.jstor.org/stable/1818789.

King, Tao-Hsien Dolly, and Charles E. Teague. 2021. “Accelerated share repurchases: value creation or extraction.” *Review of Quantitative Finance and Accounting* https://doi.org/10.1007/s11156-021-00989-y.

Kingma, Diederik P., and Jimmy Ba. 2017. “Adam: A Method for Stochastic Optimization.” .

Kramkov, D. O., and E. Mordecki. 1995. “Integral Option.” *Theory of Probability & Its Applications* 39 (1): 162–172. https://doi.org/10.1137/1139007.

Obizhaeva, Anna A., and Jiang Wang. 2013. “Optimal trading strategy and supply/demand dynamics.” *Journal of Financial Markets* 16 (1): 1–32. https://www.sciencedirect.com/science/article/pii/S1386418112000328.

Orbe, Easton. 2018. “Pricing Accelerated Share Repurchases with Lookback Options.” Master’s thesis, Princeton University, USA. http://arks.princeton.edu/ark:/88435/dsp01b8515r137.

Palladino, Lenore. 2020. “Do corporate insiders use stock buybacks for personal gain?” *International Review of Applied Economics* 34 (2): 152–174. https://doi.org/10.1080/02692171.2019.1707787.

Schied, Alexander, Torsten Schöneborn, and Michael Tehranchi. 2010. “Optimal Basket Liquidation for CARA Investors is Deterministic.” *Applied Mathematical Finance* 17 (6): 471–489. https://doi.org/10.1080/13504860903565050.

Shepp, Larry, and A. N. Shiryaev. 1993. “The Russian Option: Reduced Regret.” *The Annals of Applied Probability* 3 (3): 631 – 640. https://doi.org/10.1214/aoap/1177005355.

Sinha, Sidharth. 1991. “Share Repurchase as a Takeover Defense.” *Journal of Financial and Quantitative Analysis* 26 (2): 233–244.

Sirignano, Justin, and Konstantinos Spiliopoulos. 2018. “DGM: A deep learning algorithm for solving partial differential equations.” *Journal of Computational Physics* 375: 1339–1364. http://dx.doi.org/10.1016/j.jcp.2018.08.029.

Vermaelen, Theo. 1984. “Repurchase Tender Offers, Signaling, and Managerial Incentives.” *Journal of Financial and Quantitative Analysis* 19 (2): 163–181.

Wang, Sifan, and Paris Perdikaris. 2021. “Deep learning of free boundary and Stefan problems.” *Journal of Computational Physics* 428: 109914.

http://dx.doi.org/10.1016/j.jcp.2020.109914.

Appendix A. Acquisition After Exercise

Following Appendix A of Jaimungal, Kinzebulatov, and Rubisov (2017) we will derive the penalty function discussed in Section 2 for purchasing outstanding shares once the intermediary exercises their option. Once the intermediary exercises at time τ their payout of nA_τ is no longer affected. However, there may still be shares left to acquire which must be done on the time interval $[\tau, \tau+\epsilon]$. We strictly enforce that all outstanding shares are purchased so that $q_{\tau+\epsilon} = 0$. The value function will be given by

$$H(t,S,q,y) = \inf_{\nu \geq 0} \mathbb{E}_{t,S,q,y}\Bigg[\int_t^{\tau+\epsilon} S_u(1+a\nu_u)\nu_u + \phi_1 S_u q_u^2 \, du \\ + q_{\tau+\epsilon}(S_{\tau+\epsilon} + \phi_2 S_{\tau+\epsilon} q_{\tau+\epsilon})\Bigg]. \tag{A1}$$

The first term in the integral is the cost of acquiring the shares and the second term in the integral is to encourage urgency. The final term is the cost of acquiring all of the shares at the final time with a penalty corresponding to ϕ_2. Since we are interested in enforcing full repurchase, we will consider the limit as $\phi_2 \to \infty$. Dynamic programming suggests that the value function satisfies the following PDE:

$$\partial_t H + \mathcal{L}_{S,y} H + \phi_1 S q^2 + \min_{\nu \geq 0}[-\nu \partial_q H + \nu(S + aS\nu)] = 0$$
$$H(\tau+\epsilon, S, q, y) = q(S + \phi_2 S q).$$

Using first order conditions, the optimal acquisition strategy is given by the feedback control

$$\nu^* = \frac{\partial_q H - S}{2aS}\mathbf{1}_{\{q>0\}}.$$

Substituting this into the PDE gives

$$\partial_t H + \mathcal{L}_{S,y} H + \phi_1 S q^2 - \frac{(\partial_q H - S)^2}{4aS} = 0.$$

Consider the ansatz $H(t,S,q,y) = S(q^2 h_2(t) + q h_1(t) + h_0(t))$. Then

$$\partial_t h_0 + \mu h_0 - \frac{(h_1 - 1)^2}{4a} = 0, h_0(\tau+\epsilon) = 0$$
$$\partial_t h_1 + \mu h_1 - \frac{h_2(h_1 - 1)}{a} = 0, h_1(\tau+\epsilon) = 1$$
$$\partial_t h_2 + \mu h_2 - \frac{1}{a} h_2^2 + \phi_1 = 0, h_2(\tau+\epsilon) = \phi_2.$$

Then

$$\lim_{\phi_2\to\infty} h_2(t) = \frac{1}{2}\left(a\mu + \sqrt{a^2\mu^2 + 4a\phi_1}\coth(\frac{(\tau+\epsilon-t)\sqrt{\mu^2 a + 4\phi_1}}{2\sqrt{a}})\right).$$

By substituting this into the ODES for h_0, h_1, we can also solve for them explicitly, though we omit this for brevity's sake. Note that

$$-\frac{dq^*(t)}{dt} = \nu^*(t, q^*(t)) = \frac{2q^*(t)h_2(t) + h_1(t) - 1}{2a}\mathbf{1}_{\{q^*(t)>0\}} \tag{A2}$$

is independent of the volatility y and the stock price S, and is hence deterministic. Then, if we trade according to ν^* we have $q_{\tau+\epsilon} = 0$. This allows us to rewrite (A1) as

$$\begin{aligned} H(\tau, S, q, y) &= \mathbb{E}\int_0^\epsilon S_{u+\tau}(1 + a\nu^*_{u+\tau})\nu^*_{u+\tau} + \phi_1 S_{u+\tau} q^{*2}_{u+\tau}\, du \\ &= S\int_0^\epsilon e^{\mu u}(1 + a\nu^*_{u+\tau})\nu^*_{u+\tau} + \phi_1 q^{*2}_{u+\tau}\, du \end{aligned} \tag{A3}$$

where in the second step we can exchange order of integration by Tonelli's Theorem, since the integrand is non-negative. From the formulas for h_0, h_1, h_2 we can see that $H(\tau, S, q, y)$ is in fact independent of the value of τ.

Note that since ν^* depends linearly on q except for the indicator function, $H(\tau, S, q, y)$ is almost quadratic with respect to q. Hence, we will approximate $H(\tau, S, q, y)$ by a function of the form $\alpha_2 q^2 + \alpha_1 q$ as follows: (i) calculate q^*, ν^* by numerically solving the ODE in (A2) (ii) numerically solve the integral in (A3) (iii) repeat steps (i) and (ii) for various initial conditions q_τ, and use least squares to fit the coefficients α_1, α_2. The penalty function $\ell(q, S)$ is then given by

$$\ell(q, S) = H(\tau, S, q, y) = S(\alpha_2 q^2 + \alpha_1 q).$$

Figure A1 shows on the left a plot of the shares to repurchase q as a function of time. As expected, the number of shares reaches zero as $t \to \tau + \epsilon$. On the right Figure A1 shows $H(\tau, S = 1, q, y)$ and the least squares quadratic $\alpha_2 q^2 + \alpha_1 q$. We can see that there is good agreement between the actual function and our quadratic approximation, justifying the use of the approximation throughout the paper.

Appendix B. Quadratic Penalty

A referee pointed out that the quadratic penalty can lead to some misaligned incentives for the intermediary. The argument is as follows: we consider the European constant volatility problem with $\lambda(y) = 0$ and drift $\mu = -m/2$ so that S is a martingale, and no market impact i.e. $a = 0$. The profit and loss is given by

$$nA_T - \int_0^T \nu_u S_u du.$$

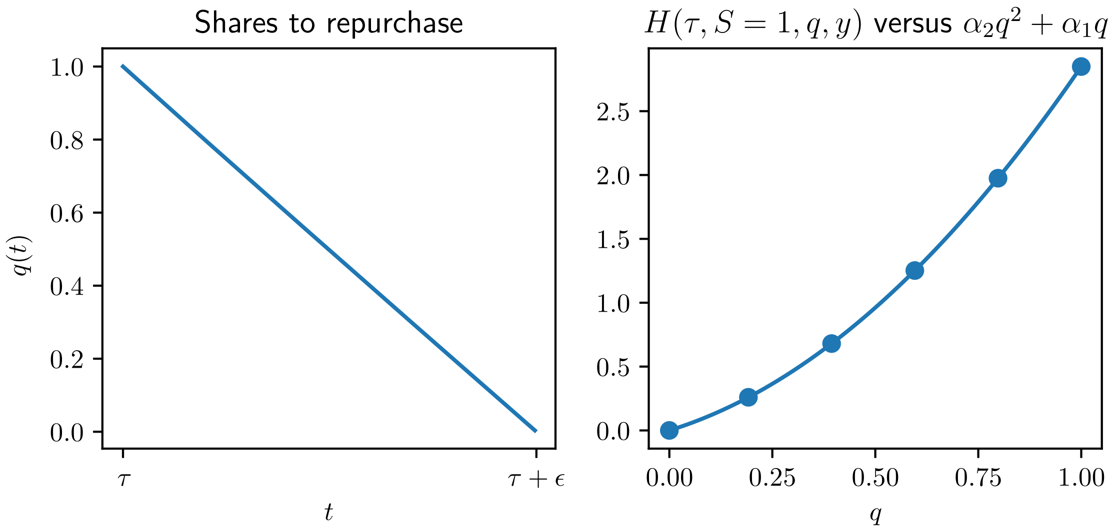

Figure A1. The left plot shows the shares remaining $q(t)$ for $t \in [\tau, \tau+\epsilon]$ with $q_\tau = 1$. The right plot shows $H(\tau, S = 1, q, y)$ as dots for $q = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ as well as the corresponding least squares quadratic $\alpha_2 q^2 + \alpha_1 q$. The parameters are $a = 0.005, \mu = 0.1, \phi_1 = 37.5, \epsilon = 1/365$ yielding $\alpha_1 = 1.00, \alpha_2 = 1.85$

If the intermediary purchases shares at a constant rate, so that $\nu_u = \frac{n}{T}$, then the profit and loss is zero, and the value of the strategy (2) is

$$-\mathbb{E}\left[\int_0^T \phi S_u n^2 (1 - \frac{u}{T})^2 du\right] < 0.$$

On the other hand, if the intermediary buys all the shares immediately, then the profit and loss is $n(A_T - s_0)$ with expectation zero and positive variance, but the value of the strategy is 0 (which is greater). Hence, for $\phi > 0$ the value function incentivizes the intermediary to adopt a strategy with a random profit and loss over a deterministic strategy.

The penalty $\phi S q^2$ is introduced to model the intermediary's risk aversion and induce urgency. While the running penalties act to encourage rapid trading, the market impact $1 + a\nu_t$ counters this, since a faster trading rate increases the execution price for the repurchase. The above argument demonstrates that if the market impact is negligible, then the quadratic penalty can lead to undesirable behavior, and hence it is an important consideration when calibrating parameters. Our model can straightforwardly accommodate this, since we simply need to set the parameter $\phi = 0$ to eliminate any potential concern.

To demonstrate the impact of the quadratic penalty, Figure B1 shows an identical plot to Figure 2, but with the quadratic penalty zeroed out. We can see very similar trading profiles and boundaries. Without the quadratic penalty, the intermediary trades for slightly longer, and there is greater effect of the volatility on the trading rates, since the volatility penalty $Sq\lambda(y)$ is the only running penalty.

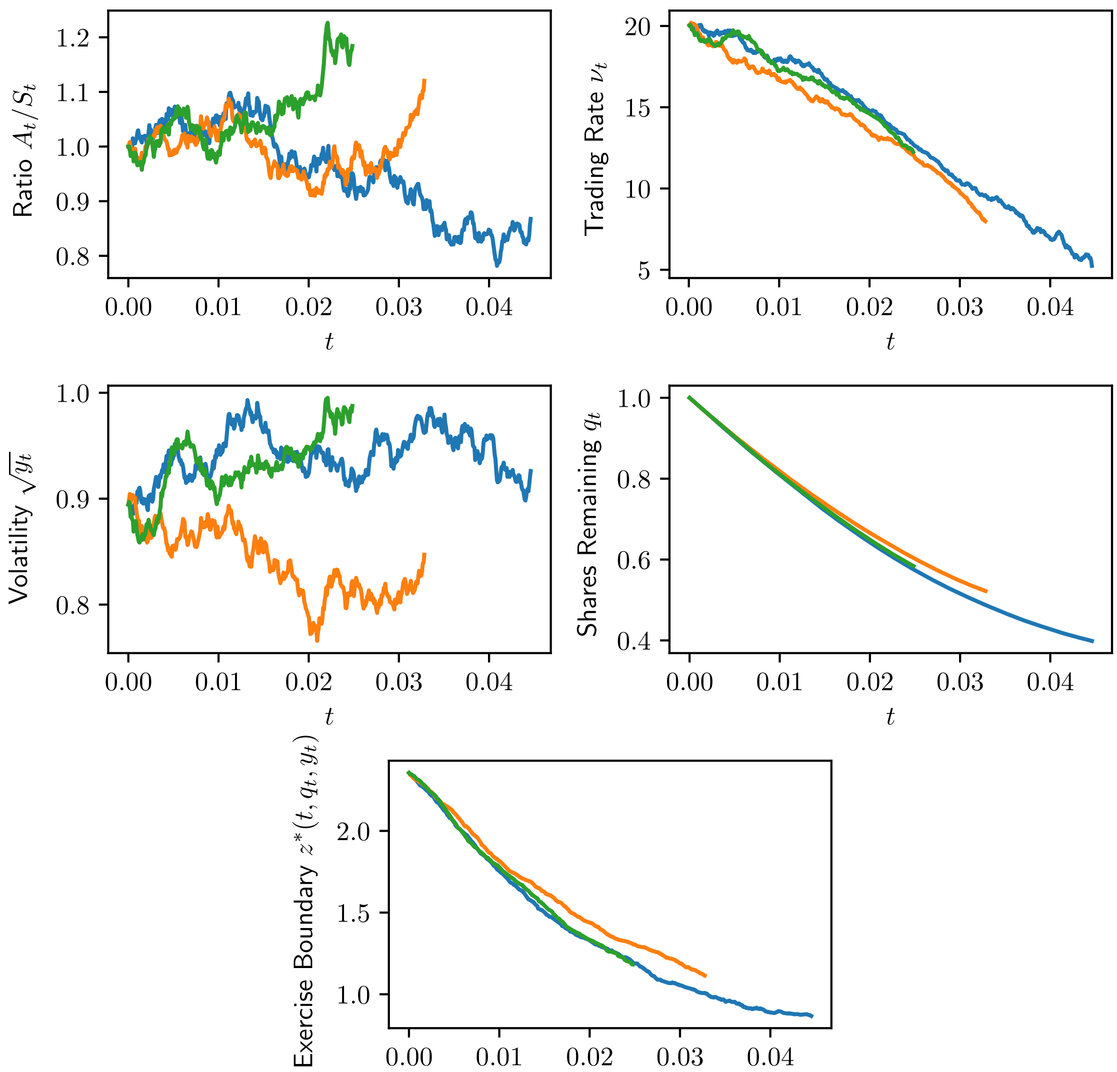

Figure B1. Sample paths of ASR with early exercise, showing the ratio of the average process to the stock price A_t/S_t, the trading rate ν_t, the volatility $\sqrt{y_t}$, the shares remaining to purchase q_t, and the corresponding exercise boundary. Paths are plotted until the intermediary exercises their option. Parameters are $a = 0.005, \phi = 0, \alpha_1 = 1.00, \alpha_2 = 1.85, \rho = -0.7, \gamma = 1, m = 0.8, \kappa = 0.1, \psi = 1, \theta = 4, n = 1, T = 0.083$

Appendix C. Numerical Details

We provide the details of the deep learning methods in the case of the ASR with fixed notional, as well as the ASR with fixed shares and local volatility. The general strategy is the same as in Section 5, namely that we represent that value function with one neural network, and the free boundary with another neural network. The hyper-parameters are tuned by constructing a constituent loss $\mathcal{L}$ made up of the sum of a nonlinear operator over the interior and losses associated with the boundary conditions. At each iteration, we sample a batch of collocation points uniformly across the truncated domain, compute the loss, and then minimize it using the Adam optimizer.

C.1. *Fixed Notional*

We can simulate the fixed notional ASR by approximating $H(t, S, A, c, y)$ with the neural network $u_\Lambda(t, S, A, c, y)$ and approximating $c^*(t, S, A, y)$ with the neural network $s_\beta(t, S, A, y)$ where Λ and β are the respective parameters for the networks. We then seek to minimize the following objective

$$\begin{aligned}\mathcal{L}(\Lambda, \beta) = \xi_1 \mathcal{L}_{interior}(\Lambda) + \xi_2 \mathcal{L}_T(\Lambda) + \xi_3 \mathcal{L}_S(\Lambda) + \xi_4 \mathcal{L}_A(\Lambda)\\ + \xi_5 \mathcal{L}_c(\Lambda) + \xi_6 \mathcal{L}_y(\Lambda) + \xi_7 \mathcal{L}_{boundary}(\Lambda, \beta)\end{aligned}$$

where we define the nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}[h] = \partial_t H + \mathcal{L}_{S,y} H + \frac{S-A}{t}\partial_A H - \phi S(\frac{F}{A} - c)^2 - \lambda(y) S(\frac{F}{A} - c) + \frac{1}{4aS}(\partial_c H - S)^2$$

and the losses as

$$\begin{aligned}
\mathcal{L}_{interior}(\Lambda) &= \sum_{i=1}^{N} |\mathcal{N}[u_\Lambda](t^i, S^i, A^i, c^i, y^i)|^2\\
\mathcal{L}_T(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(T, S^i, A^i, c^i, y^i) - F + \ell(\frac{F}{A^i} - c^i, S^i)|^2\\
\mathcal{L}_S(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial S}(t^i, \overline{S}, A, c^i, y^i)|^2 + \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial S}(t^i, \underline{S}, A, c^i, y^i)|^2\\
\mathcal{L}_A(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial A}(t^i, S^i, \overline{A}, c^i, y^i)|^2\\
\mathcal{L}_c(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(t^i, S^i, A^i, \overline{c}, y^i) - F|^2\\
\mathcal{L}_y(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial y}(t^i, S^i, A^i, c^i, \overline{y})|^2 + \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial y}(t^i, S^i, A^i, c^i, 0)|^2\\
\mathcal{L}_{boundary}(\Lambda, \beta) &= \sum_{i=1}^{N} |u_\Lambda(t^i, S^i, A^i, s_\beta(t^i, S^i, A^i, y^i), y^i) - F + \ell(\frac{F}{A^i} - s_\beta(t^i, S^i, A^i, y^i), S_i)|^2.
\end{aligned}$$

The first term requires that the value function satisfies the nonlinear differential operator. The next five terms are the boundary conditions discussed in Section 6.1. The final term specifies that the value function is equal to $F - \ell(\frac{F}{A} - c, S)$ across the free boundary.

The weights $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7$ allow us to prioritize various parts of the loss. N is the batch size and $(t^i, S^i, A^i, c^i, y^i)$ are collocation points sampled uniformly at random on $[0, T] \times [\underline{S}, \overline{S}] \times [\underline{A}, \overline{A}] \times [0, \overline{c}] \times [0, \overline{y}]$. In each of the 160000 iterations, we sample $N = 10000$ collocation points, compute the loss $\mathcal{L}(\Lambda, \beta)$, and then minimize it using the Adam optimizer.

C.2. *American-Asian Option with Local Volatility*

For the American-Asian option, we will model the value function $w(t, S, A)$ with one neural network, $v_\Gamma(t, S, A)$, and we will approximate the corresponding free boundary $S^*(t, A)$ with another neural network $r_\Delta(t, A)$ where Γ and Δ are the respective parameters for the networks. We then seek to minimize the following objective

$$\mathcal{L}^p(\Gamma, \Delta) = \Xi_1 \mathcal{L}^p_{interior}(\Gamma) + \Xi_2 \mathcal{L}^p_T(\Gamma) + \Xi_3 \mathcal{L}^S_z(\Gamma) + \Xi_4 \mathcal{L}^p_A(\Gamma) + \Xi_5 \mathcal{L}^p_{boundary}(\Gamma, \Delta)$$

where we define the linear operator $\mathcal{M}$ as

$$\mathcal{M}[p] = \partial_t p + \mu S \partial_S H + \frac{1}{2}\sigma^2 S^\delta \partial_{SS} H + \frac{S - A}{t} \partial_A p$$

and the losses as

$$\begin{aligned}
\mathcal{L}^p_{interior}(\Gamma) &= \sum_{i=1}^N |\mathcal{M}[v_\Gamma](t^i, S^i, A^i)|^2 \\
\mathcal{L}^p_T(\Gamma) &= \sum_{i=1}^N |v_\Gamma(T, S^i, A^i) - A^i|^2 \\
\mathcal{L}^p_S(\Gamma) &= \sum_{i=1}^N |\frac{\partial v_\Gamma}{\partial S}(t^i, \underline{S}, A^i)|^2 + \sum_{i=1}^N |\frac{\partial v_\Gamma}{\partial S}(t^i, \overline{S}, A^i)|^2 \\
\mathcal{L}^p_A(\Gamma) &= \sum_{i=1}^N |\frac{\partial v_\Gamma}{\partial A}(t^i, S^i, \overline{A})|^2 \\
\mathcal{L}^p_{boundary}(\Gamma, \Delta) &= \sum_{i=1}^N |v_\Gamma(t^i, r_\Delta(t^i, A^i), A^i) - A^i|^2.
\end{aligned}$$

The first requires that the neural network satisfy the differential operator. The second, third, and fourth terms are the boundary conditions discussed before. The fifth term specifies that the value function is equal to A across the free boundary, ensuring continuity. The weights $\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5$ allow us to prioritize various parts of the loss. N is the batch size and (t^i, z^i, y^i) are collocation points sampled uniformly at random on $[0, T] \times [\underline{S}, \overline{S}] \times [\underline{A}, \overline{A}]$. In each of the 160000 iterations, we sample $N = 10000$ collocation points, compute the loss $\mathcal{L}^p(\Gamma, \Delta)$, and then minimize it using the Adam optimizer.

C.3. *ASR with Local Volatility*

To simulate the full problem numerically, we will approximate $H(t, S, A, q)$ with the neural network $u_\Lambda(t, S, A, q)$ and we will approximate $S^*(t, A, q)$ with the neural network $s_\beta(t, A, q)$ where Λ and β are the respective parameters for the networks. We then seek to minimize the following objective

$$\begin{aligned}\mathcal{L}(\Lambda, \beta) = \xi_1 \mathcal{L}_{interior}(\Lambda) + \xi_2 \mathcal{L}_T(\Lambda) + \xi_3 \mathcal{L}_S(\Lambda) + \xi_4 \mathcal{L}_A(\Lambda) \\ + \xi_5 \mathcal{L}_q(\Lambda) + \xi_6 \mathcal{L}_{boundary}(\Lambda, \beta) + \xi_7 \mathcal{L}_{s_0}(\beta)\end{aligned}$$

where we define the nonlinear operator $\mathcal{N}$ as

$$\mathcal{N}[H] = \mathcal{M}[H] - \phi S q^2 - \lambda(\sigma^2 S^{\delta-2}) S q + \frac{1}{4aS}(\partial_q H + S)^2$$

and the losses as

$$\begin{aligned}
\mathcal{L}_{interior}(\Lambda) &= \sum_{i=1}^{N} |\mathcal{N}[u_\Lambda](t^i, S^i, A^i, q^i)|^2 \\
\mathcal{L}_T(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(T, S^i, A^i, q^i) - nA^i + \ell(q^i, S^i)|^2 \\
\mathcal{L}_S(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial S}(t^i, \overline{S}, A, q^i)|^2 + \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial S}(t^i, \underline{S}, A, q^i)|^2 \\
\mathcal{L}_A(\Lambda) &= \sum_{i=1}^{N} |\frac{\partial u_\Lambda}{\partial A}(t^i, S^i, \overline{A}, q^i)|^2 \\
\mathcal{L}_q(\Lambda) &= \sum_{i=1}^{N} |u_\Lambda(t^i, S^i, A^i, 0) - n v_\Gamma(T, S^i, A^i)|^2 \\
\mathcal{L}_{boundary}(\Lambda, \beta) &= \sum_{i=1}^{N} |u_\Lambda(t^i, s_\beta(t^i, A^i, q^i), A^i, q^i) - nA^i + \ell(q^i, s_\beta(t^i, A^i, q^i))|^2 \\
\mathcal{L}_{s_0}(\beta) &= \sum_{i=1}^{N} |s_\beta(t^i, A^i, 0) - r_\Delta(t^i, A^i)|^2.
\end{aligned}$$

The first term requires that the value function satisfies the nonlinear differential operator. The next four terms are the boundary conditions discussed before. The sixth term specifies that the value function is equal to $nA - \ell(q, S)$ across the free boundary, and the final term specifies that the free boundary is equal to the boundary of the American-Asian option when $q = 0$. Necessarily, we have already computed v_Γ and r_Δ as numerical solutions to the American-Asian option, so that we can use them when computing the losses $\mathcal{L}_q$ and $\mathcal{L}_{s_0}$.

The weights $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7$ allow us to prioritize various parts of the loss. N is the batch size and (t^i, S^i, A^i, q^i) are collocation points sampled uniformly at random on $[0, T] \times [\underline{S}, \overline{S}] \times [\underline{A}, \overline{A}] \times [0, n]$. In each of the 160000 iterations, we sample $N = 10000$ collocation points, compute the loss $\mathcal{L}(\Lambda, \beta)$, and then minimize it using the Adam optimizer.