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When uncertainty and volatility are disconnected: Implications for asset pricing and portfolio performance[☆]

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A B S T R A C T

We analyze an environment where the uncertainty in the equity market return and its volatility are both stochastic and may be potentially disconnected. We solve a representative investor's optimal asset allocation and derive the resulting conditional equity premium and risk-free rate in equilibrium. Our empirical analysis shows that the equity premium appears to be earned for facing uncertainty, especially high uncertainty that is disconnected from lower volatility, rather than for facing volatility as traditionally assumed. Incorporating the possibility of a disconnect between volatility and uncertainty significantly improves portfolio performance, over and above the performance obtained by conditioning on volatility only.

> *"You would think if uncertainty was high, you'd have a bit more volatility."*
>
> [William Dudley, New York Fed President, February 15, 2017.]

1. Introduction

Although the notions of uncertainty and volatility are often used interchangeably, the two concepts are inherently different: volatility measures the dispersion of short-term shocks around a long-term mean, while uncertainty measures the difficulty to forecast the distribution of returns, including its long-term mean. Fig. 1 contains a scatter plot of volatility and uncertainty at the weekly frequency between January 1986 and December 2020, proxied respectively by realized volatility computed from high-frequency

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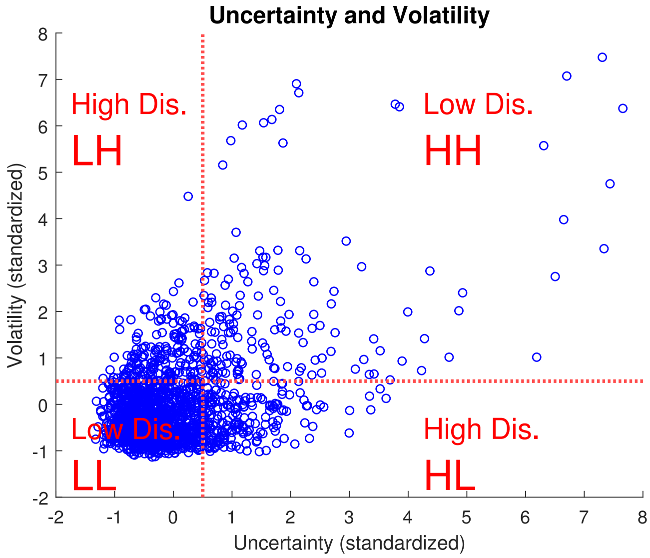

Fig. 1. Uncertainty and volatility regimes.
Notes: The figure shows a scatter plot of standardized uncertainty (proxied by the economic policy uncertainty index EPU_t) and volatility (proxied by realized volatility). Both are sampled at the weekly frequency from January 1986 until December 2020. The threshold values for volatility and uncertainty are given by their mean plus one half of their standard deviation. We say that volatility and uncertainty are high (respectively, low), when they are above (respectively, below) their threshold values. High disconnect occurs when either uncertainty is high while volatility is low (denoted "HL") or when uncertainty is low while volatility is high (denoted "LH"). In the other two quadrants, uncertainty and volatility are in sync and disconnect is low.

data and the economic policy uncertainty index (EPU) of Baker et al. (2016). The figure shows that the two variables, although generally positively correlated, are far from being perfect substitutes for one another. The degree of connection between uncertainty and volatility appears to vary across periods.

It is natural a priori to expect uncertainty and volatility to be in general positively correlated, as the quote above implies. For example, the theoretical model of Pastor and Veronesi (2013) implies such a positive relationship;[1] while Amengual and Xiu (2018) show that the resolution of monetary policy uncertainty is generally associated with declines in volatility. Yet, there have been several episodes in which either volatility was high and uncertainty was significantly lower or vice versa. For instance, the US 2016 presidential election generated some uncertainty about long-term economic and other policies, but was surprisingly characterized by very low levels of stock market volatility. Similarly, the UK's exit from the EU (Brexit) involved substantial uncertainty about trade, growth, and immigration policies for the UK and the EU, but had a barely noticeable impact on short-term volatility in their respective stock markets.[2] These events are examples of situations in which a disconnect between the two variables appeared because uncertainty was substantially higher than volatility. By contrast, the stock market dynamics during the financial crisis in 2008 and the initial stock market reaction to the diffusion of the Covid-19 pandemic in the spring of 2020 are examples of situations in which a disconnect occurred due to a higher increase in volatility compared to the rise in uncertainty. Interestingly, in the months following the stock market crash in March 2020, volatility declined much faster than uncertainty, leading to a switch in the nature of disconnect, characterized instead by uncertainty being higher than volatility.

Separately, the empirical evidence regarding the risk-return trade-off, with risk measured by volatility, is mixed: while many papers find a positive relationship,[3] as predicted by standard theory, equally many find a negative one,[4] depending on the sample period and methodology, whether total or idiosyncratic volatility is considered, and other distinctions.[5] This suggests that a model where investors worry only about volatility is perhaps too parsimonious.

Motivated by these empirical observations, we propose a model in which uncertainty and volatility are two separate stochastic processes, whose degree of connection is stochastic. The representative investor in our model can be averse to both volatility and uncertainty, with possibly different degrees of aversion to each. Going back to the simple regimes exhibited in Fig. 1, when we solve the model, we obtain a different equity premium, risk-free rate, and portfolio strategy in regimes in which uncertainty and volatility are high or low at the same time (connected), compared to regimes in which one of them is significantly higher than the

[1] At least if the precision of political signals in their model is constant over time. If it is allowed to vary then it is possible for the relationship to be reversed.

[2] See Bialkowski et al. (2022) for a related discussion.

[3] See, e.g., Harvey (1989), Campbell and Hentschel (1992), Harrison and Zhang (1999), Ghysels et al. (2005) and Ludvigson and Ng (2007).

[4] See, e.g., Campbell (1987), Breen et al. (1989), Nelson (1991), Glosten et al. (1993) and Brandt and Kang (2004). See also (Blitz et al., 2020) for an overview of the low-risk anomaly, including the empirical finding of a negative risk-return trade off and a discussion of potential theoretical explanations for this empirical result.

[5] See, e.g., Merton (1980), French et al. (1987), Goyal and Santa-Clara (2003), Ang et al. (2009) and Campbell et al. (2018).

other (disconnected). We show that incorporating this potential disconnect leads to substantially improved forecasts of the equity premium and portfolio performance.

To allow for a distinction between uncertainty and volatility, the decision maker should be uncertain about the correct probability model to use. To model this, we rely on the robust control approach pioneered by Hansen and Sargent (2001).[6] The representative investor recognizes that he or she is unable to know exactly the true underlying model, and considers instead a set of models that are statistically difficult to distinguish from one another, seeking consumption and investment policies that perform well across that full set of models. Uncertainty measures the radius of the set of potential models: when uncertainty is high, the range of potential models around the true model expands. How much the investor responds to this expansion is controlled by a parameter, which we think of as the investor's coefficient of uncertainty aversion.

We then represent uncertainty in the form of the product of stochastic volatility and a "disconnect" stochastic process, that drives a wedge between volatility and uncertainty. An investor interested in policies that are robust across the set of alternative models optimally evaluates his or her policies under the worst-case alternative in the set of models under consideration. For that reason, the dynamics of volatility and disconnect endogenously have an impact on asset prices and the associated optimal policies of the investor.

We first solve the model in partial equilibrium, computing in closed form a representative log-utility investor's optimal policies for consumption and portfolio choice. We show that the investor's optimal equity holding consists of the standard myopic term, which is inversely proportional to stock return variance, and a new (yet still myopic as befits a logarithmic investor) uncertainty correction term, which is the optimal response to the potential disconnect between volatility and uncertainty. We find that the contemporaneous interaction of volatility and uncertainty plays an important role in determining optimal portfolios. In particular, the sensitivity of portfolio weights to changes in volatility depends on the level of model uncertainty. Accordingly, the trajectory of portfolio weights in a given period depends on the joint dynamics of volatility and uncertainty, including whether they are connected or disconnected.

Given the optimal asset allocation by a representative investor, we solve for the equilibrium equity premium and risk-free rate. We obtain explicit formulae, which are nonlinear functions of both stochastic volatility and disconnect. They predict that the uncertainty term embedded in disconnect generates a flight-to-quality-like correlation among asset returns. In high-uncertainty periods, investors require a high equity premium to hold the risky asset and are willing to accept a low risk-free rate to hold the safe asset. The presence of stochastic volatility may amplify or diminish these effects depending on whether it is connected or disconnected from uncertainty, respectively. The interaction of volatility with a possibly disconnected uncertainty means that our model can generate a high equity premium in a low volatility environment, whenever the disconnect with uncertainty is high. These empirical patterns were observed in the period surrounding the US 2016 election, among other disconnected episodes.

Our model's results can help explain the challenges faced by previous empirical studies trying to establish a risk-return trade-off using volatility alone as a measure of risk. As noted above, while most theoretical asset pricing models imply a positive relationship between return and risk, the empirical evidence for such a trade-off is in reality mixed or inconclusive. Although our model also implies a positive relationship between return and volatility, consistent with the assumption of traditional risk (i.e., volatility) aversion, it adds a new component in the equity premium associated with the disconnect between volatility and uncertainty, which is theoretically positive. Including this component allows our model to more accurately reproduce asset pricing patterns. We find that uncertainty is a strong positive predictor of the equity premium, so even in periods when the traditional volatility-return trade off fails to materialize, or has the wrong sign, the predicted equity premium in our model is increasing in uncertainty due to the uncertainty-return trade off we find. Our results show that allowing for uncertainty to be disconnected from volatility makes uncertainty a much better variable than volatility in terms of generating a trade-off with expected returns: it appears from our results that the equity premium is earned mainly for facing uncertainty, especially high uncertainty that is disconnected from lower volatility, rather than for facing volatility per se.[7]

We then go on to evaluate the practical value of the equity premium formula that emerges from the model, by examining the portfolio performance of a reference investor who predicts future excess returns using our estimated relation between stock excess return, volatility, and uncertainty.[8] We find that our model significantly improves portfolio performance relative to both existing

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[6] See also (Anderson et al., 2003), Hansen et al. (2006), Hansen and Sargent (2008), and Hansen and Sargent (2011). An axiomatic justification for the approach is provided by Strzalecki (2011). Alternative frameworks can be employed to capture robust decision making. Ambiguity aversion is one such formulation: agents there prefer options whose probabilities are known with certainty over those whose probabilities are uncertain. Info-gap models of uncertainty are another approach: there, decision-makers choose actions that minimize the maximum possible regret, where regret is the difference between the payoff obtained and the payoff that would have been obtained had the decision-maker known the true probability distribution. In the maxmin expected utility approach, decision-makers maximize their expected utility using the worst-case probability distribution, rather than a specific probability distribution that is known to be correct; see Gilboa and Schmeidler (1989), and Kochov (2015) for an axiomatic justification. For a discussion and comparison of maxmin expected utility and robust control, see Hansen and Sargent (2001), Hansen et al. (2006) and Chen and Epstein (2002).

[7] The relation between uncertainty and returns has been considered at least since Keynes' General Theory (Keynes, 1936, p. 148): "It would be foolish, in forming our expectations, to attach great weight to matters which are very uncertain. It is reasonable, therefore, to be guided to a considerable degree by the facts about which we feel somewhat confident, even though they may be less decisively relevant to the issue than other facts about which our knowledge is vague and scanty. (...) The state of long-term expectation, upon which our decisions are based, does not solely depend, therefore, on the most probable forecast we can make. It also depends on the confidence with which we make this forecast—on how highly we rate the likelihood of our best forecast turning out quite wrong." We provide empirical validation for this relation, in a setup in which volatility also drives returns.

[8] We deliberately keep our empirical analysis simple given the fact that econometric inference of non-linear conditional asset pricing model with unknown parameters, which over model belongs to, is challenging due to miss specification or weak identification. A general discussion of this issue can be found in Antoine et al. (2020).

unconditional and conditional asset pricing models, including those that time volatility but do not account for its potential disconnect from uncertainty, and those that allow for the presence of uncertainty but not for its disconnect from volatility. These results are valid when back-testing the model in-sample but also out-of-sample.

The main contribution of the paper is therefore the construction of a simple and intuitive model where a representative investor can be averse to potentially disconnected volatility and uncertainty, which can be solved explicitly and results in equilibrium formulae for the equity premium and risk-free rate that perform well empirically.

The rest of the paper is organized as follows. We start with a brief discussion of the related literature in Section 2. Section 3 sets up the model, in which volatility and uncertainty are potentially disconnected. Section 4 solves for the optimal consumption and portfolio allocation taking prices as given. Section 5 solves for the equilibrium asset prices' dynamics, including the conditional equity premium and risk-free rate. Section 6 describes the data and our empirical analysis. Section 7 evaluates the performance of our model's implied portfolio strategy in the full sample and in specific high-disconnect regimes. Section 8 shows that our results also hold out-of-sample. Section 9 concludes. The Appendix contains technical material and proofs of the propositions in the paper.

2. Related literature

Our work is related to a growing literature that analyzes the effect of uncertainty on asset prices. While we do not model the source of uncertainty, different approaches can be employed to justify the presence of a disconnect between uncertainty and volatility. The model of Pastor and Veronesi (2012) and Pastor and Veronesi (2013) implies that political uncertainty shocks command a risk premium and stocks become more correlated and volatile in periods of elevated political uncertainty. If the constant signal precision in their model was generalized to be stochastic, a disconnect would arise. Alternatively, Barroso and Detzel (2021) show that the performance of the volatility-managed portfolios in Moreira and Muir (2017) is increasing in a sentiment variable, consistent with prior theory by citeyuyuan11 showing that sentiment traders are expected to under-react to volatility. A stochastic sentiment can be a source of disconnect as well. Bidder and Dew-Becker (2016) show that a model with recursive preferences and uncertainty about the dynamics of consumption is consistent with a large and time-varying equity premium. On the empirical side, Brogaard and Detzel (2015) show that uncertainty positively forecasts excess returns and innovations in uncertainty carry a significantly negative risk premium, while Bali et al. (2017) find that the difference between returns on portfolios with the highest and lowest uncertainty beta is negative and highly significant. In our paper, the fact that the equity premium responds to both volatility and uncertainty is not assumed, however. It occurs endogenously in the model in equilibrium because the representative investor is averse to volatility (risk-averse in the standard sense) but also averse to uncertainty (by seeking robustness to a range of potential models that may generate the observed empirical realizations).

In existing models that do not account for model uncertainty, time-varying volatility drives the optimal policies of a risk-averse agent, but there is no mechanism to account for the uncertainty attached to the assumptions of the model (see, for example, Chacko and Viceira (2005), Liu (2007), and Drechsler and Yaron (2011)). On the other hand, in models in which the agent is averse to model uncertainty but volatility is constant, the optimal portfolio strategy is to reduce the exposure to risky assets when uncertainty increases (see, for example, Trojani and Vanini (2000), Maenhout (2004), Maenhout (2006), and Illeditsch (2011)). Since volatility and returns tend to be negatively correlated, the resulting conservative asset allocation makes the investor forgo much of the upside of asset markets in situations in which volatility fails to materialize despite levels of uncertainty above or close to average, such as the aftermaths of the US 2016 election, Brexit, and the months following the stock market crash of March 2020 driven by the pandemic. We show that including uncertainty and volatility separately allows the investor to take advantage of such situations. The volatility-managed portfolios in Moreira and Muir (2017), which take less risk when volatility is high, produce large alphas and Sharpe ratios. We show that incorporating disconnect as an additional variable achieves even higher portfolio performance.

Liu et al. (2005) consider a setup with constant diffusive volatility but in which the stock price is subject to jump risk and the investor is averse to uncertainty with respect to jumps. They show that their model is consistent with several empirical patterns of option prices. Jahan-Parvar and Liu (2014) show that a production-based asset-pricing model with regime-switching productivity, constant volatility in each regime and ambiguity aversion can reproduce, among other empirical patterns, predictability of excess return by investment-capital, price-dividend, and consumption-wealth ratios.[9] In contrast to these papers, in our model both stochastic volatility and uncertainty predict excess returns, and we show empirically that this predictability generates high portfolio performance, especially in periods of high uncertainty that is disconnected from low volatility.

Similar to our work, other papers have investigated the differential impact of volatility and uncertainty on equity returns. In particular, in line with the results found in our paper, Anderson et al. (2009) find stronger empirical evidence for an uncertainty - return trade-off rather than for the traditional risk-return trade-off.[10] Bekaert et al. (2009) find that the equity risk premium is primarily driven by time-varying risk aversion, while uncertainty is the main driver of counter-cyclical volatility of asset returns, suggesting that uncertainty affects second rather than first return moments.[11] In contrast to these papers, however, we device a

[9] Gallant et al. (2019) show that in several model specifications, including the one used by Jahan-Parvar and Liu (2014), having model uncertainty as a feature of the model provides a better empirical asset pricing fit.

[10] While Anderson et al. (2009) also employ a robust control setup similar to ours to model time varying uncertainty and volatility, they make use of different empirical proxies for both uncertainty and volatility: First, their measure of uncertainty is based on forecaster's disagreement, while we use the economic policy uncertainty (EPU) index as our empirical measure for uncertainty. Second, we use a non-parametric estimator for our measure of conditional volatility, while they construct their conditional volatility proxy via a Mixed Data Sampling, or MIDAS, approach.

[11] In their model, uncertainty is proportional to the conditional volatility of dividend growth, while time-varying risk aversion is introduced via an external habit stock in the utility function of the representative investor.

dynamic portfolio strategy that not only exploits the fact that uncertainty and volatility have a differential impact on equity returns, but simultaneously takes advantage of the different time series properties of volatility and uncertainty.

The theoretical framework in this paper is related to the one presented in Sbuelz and Trojani (2008), which also considers a general equilibrium model with an ambiguity averse agent and stochastically varying uncertainty. However, while in their framework the uncertainty or ambiguity set is constrained by the stochastic state of the economy, we allow for a more flexible specification in which time-varying uncertainty is driven by both stochastic volatility and a disconnect process. Drechsler (2013) considers a general framework with stochastic model uncertainty, stochastic volatility and jumps. While his focus is on reproducing the empirical properties of index options and the variance premium, ours is on stock excess return predictability and the characterization of asset prices, especially by focusing on the importance of periods of high disconnect between uncertainty and volatility. Faria and Correia-da Silva (2016) study optimal asset allocation for an investor subject to stock return stochastic volatility and constant ambiguity uncertainty. Like us, they are interested in the impact of model uncertainty on optimal portfolios in a setup with stochastic volatility. However, because they assume constant model uncertainty, their setup is not suited to study the joint dynamics of volatility, uncertainty, and their disconnect on the optimal portfolio decision. Moreover, they do not study asset prices in equilibrium and their main focus is on the marginal impact of model uncertainty on investors' hedging demands, which they find to be very low. Brenner and Izhakian (2018) study the joint impact of risk and (ambiguity) uncertainty on the equity premium. While the research question they ask is related to ours, they consider a very different framework. First, their measure of uncertainty is independent of risk by construction. Accordingly, their setup would not allow to study time-varying levels of disconnect between volatility and uncertainty, which is central in our paper. Second, while we let the representative investor be averse to both risk and uncertainty, Brenner and Izhakian (2018) allow for the investor to be ambiguity loving in some states of nature. Lastly, while in our framework uncertainty and volatility are exogenous, a theoretical foundation for the disconnect process may be obtained in models in which the representative agent learns about the true model driving the economy.[12]

3. A framework with volatility and uncertainty possibly disconnected

We consider an infinite-horizon expected-utility maximization problem where a representative investor chooses his or her consumption level as well as allocates his or her funds between a risk-less and a risky asset, which exhibits stochastic volatility. For this purpose, the investor employs a benchmark or reference model that represents his or her best estimate of the risky asset's dynamics. However, the investor fears that the model he or she uses is potentially incorrect, and worries that the true model could lie in a set of alternative models that are statistically difficult to distinguish from the reference model. To cope with model uncertainty, the investor chooses optimal consumption and portfolio holdings that are robust across the set of alternative models.

The size of the set of alternative models is our proxy for how much uncertainty the representative investor faces. When the radius of the set is large, alternatives that are statistically far from the reference model will be considered, and the investor will face high uncertainty about the reference model and vice versa. We generalize the radius of the set of alternative models relative to the classical robust control literature. We first let stochastic volatility affect it, so that model uncertainty is ceteris paribus higher when there is higher volatility and vice versa. To avoid perfect correlation between model uncertainty and volatility, however, we introduce an additional stochastic process which drives the degree of connection between uncertainty and volatility. This setup allows us to study market scenarios in which, simultaneously, uncertainty can be potentially high while volatility is low and vice versa, such as the "HL" and "LH" regimes characterized in Fig. 1. Furthermore, we allow the investor to react differently to the same amount of uncertainty, through a coefficient of uncertainty aversion. An investor with a higher degree of uncertainty aversion would seek robustness against a larger set of models than an investor with a lower one, for the same amount of overall uncertainty. In the rest of this section, we formalize this modeling framework.

3.1. From the reference to the robust model

Assume a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual assumptions, where $\mathbb{P}$ denotes the reference probability measure that represents the investor's best estimate of the risky-asset dynamics. We consider a Lucas-tree economy (see Lucas, 1978) with a single risky asset with price S_t and a risk-free asset with price B_t. The risky asset is a perpetual claim on the stream of aggregate dividends D_t. The dynamics of dividends and asset prices under the reference probability measure are

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D v_t dW_t^D, \quad D_0 > 0, \tag{1}$$

$$\frac{dB_t}{B_t} = r_{f,t} dt, \quad B_0 > 0, \tag{2}$$

$$\frac{dS_t}{S_t} = \left(\mu_{S,t} - \frac{D_t}{S_t}\right) dt + \sigma_S v_t dW_t^S, \quad S_0 > 0, \tag{3}$$

[12] See for instance Epstein and Schneider (2007), Leippold et al. (2007) and Epstein and Schneider (2008), for frameworks in which the agent learns about the fundamentals of the economy under ambiguity and the quality of the information may be high or low. In Dew-Becker and Nathanson (2019), agents trying to learn about the fundamentals of the economy might attenuate the most severe effects of uncertainty (or ambiguity). Thus, translated to our setup, an agent who acquires information about the true dynamics of the stock price process may reduce the uncertainty about its fundamentals provided that the quality of information is sufficiently high and therefore affects the degree of disconnect between uncertainty and volatility.

$$dv_t = \mu_{v,t}dt + \sigma_{v,t}\left(\sqrt{1-\rho^2_{S,v}}dW_t^v + \rho_{S,v}dW_t^S\right), \quad v_0 > 0, \tag{4}$$

where W_t^D, W_t^S and W_t^v are Brownian motions, μ_D and σ_D are the drift and volatility of dividend growth, $r_{f,t}$ is the risk-free rate, $\mu_{S,t}$ is the expected total return of the risky asset and σ_S is a constant scaling stock return volatility.[13] The stochastic volatility of stock return and dividends growth, v_t, is a general stochastic process with drift $\mu_{v,t}$ and diffusion $\sigma_{v,t}$ under the reference probability measure.[14] The term $\rho_{S,v}$ captures the correlation between the stock return and its volatility.

In order to formally introduce model uncertainty, we define by $\mathbb{P}^\vartheta$ the robust probability measure, where ϑ_t is the change from the reference measure $\mathbb{P}$ to the robust measure $\mathbb{P}^\vartheta$. This implies that a robust investor considers alternative models of the form:[15]

$$\frac{dS_t}{S_t} = \left(\mu_{S,t} - \frac{D_t}{S_t} + \sigma_S v_t h_t\right)dt + \sigma_S v_t dW_t^{S,\vartheta}, \tag{5}$$

where $W_t^{S,\vartheta}$ is also a Brownian motion, but now under the robust probability measure $\mathbb{P}^\vartheta$. Importantly, the drift of the stock return is now perturbed by the new term $\sigma_S v_t h_t$, driven by both stochastic volatility v_t and the drift perturbation function h_t which will be optimally chosen by the representative agent.[16] As can be seen from Eq. (5), the higher the perturbation function h_t becomes, the larger becomes the drift distortion of the risky asset and hence, the higher is the investor's demand for robustness against potential model miss-specification. Conversely, if $h_t = 0$, then the representative investor has full confidence in his or her reference model.

3.2. The size of the alternative set of models and the role of disconnect

While a robust investor considers a variety of alternative models, not all of them are plausible, i.e. they may be too distinct to be considered as reasonably close to the reference model. To discipline the size of the alternative model set, we make use of relative entropy which is a convenient measure of the distance between the reference model and the alternative models. The growth in entropy of $\mathbb{P}^\vartheta$ relative to $\mathbb{P}$ over the time interval $[t, t+\Delta t]$ is defined as

$$G(t, t+\Delta t) \equiv \mathbb{E}_t^\vartheta\left[\log\left(\frac{\vartheta_{t+\Delta t}}{\vartheta_t}\right)\right], \tag{8}$$

and the instantaneous growth rate of relative entropy at time t is then given by

$$\mathcal{R}(\vartheta_t) \equiv \lim_{\Delta t \to 0} \frac{G(t, t+\Delta t)}{\Delta t} = \frac{1}{2}h_t^2, \tag{9}$$

where the last equality is proven in Appendix A.1. When $h_t = 0$, the relative entropy growth rate is zero, which implies that the two probability measures are equivalent. As h_t increases, so does the distance between the reference model and the alternative models. Next, we restrict the set of alternative models under consideration by the investor in the form of an upper bound on this distance. We assume that set of admissible alternative models is bounded from above as follows

$$\left\{\vartheta_t : \mathcal{R}(\vartheta_t) \le \frac{\epsilon^2}{2}\mathcal{U}_t^2\right\}, \tag{10}$$

where $\mathcal{U}_t$ denotes the stochastic model uncertainty and the constant parameter ϵ measures the investor's degree of uncertainty aversion. When $\epsilon = 0$ the set of alternative models is empty and the investor has full confidence in the reference model. By contrast, an investor with higher ϵ expands the set of alternative models to include models that are statistically further away from the reference model.

Finally, in order to formally introduce disconnect, we posit the following functional form for modeling uncertainty

$$\mathcal{U}_t = \eta_t v_t. \tag{11}$$

[13] We will show, that in equilibrium, the stock price is proportional to dividends, so that $W_t^D = W_t^S$, $\forall t \ge 0$. Therefore, instead of specifying the correlation between dividends D_t and our state variables in the model, we specify the correlations with respect to the stock price process directly.

[14] Given logarithmic preferences, the solution is independent of the specific choice of a drift and diffusion process for volatility. The investor is myopic with regards to the dynamics of volatility: he or she cares only about the current value of the state variables. However, the drift and volatility of volatility, $\mu_{v,t}$ and $\sigma_{t,v}$, are not fully unrestricted since we require the volatility process to remain stationary and positive. Specific restrictions to be imposed are model-dependent. As an example, Feller's square-root process in which the drift term is linearly mean reverting, i.e. $\mu_{v,t} := \kappa\left(\theta - v_t\right)$ with $\theta > 0$ and $\kappa > 0$ and the volatility is given by $\sigma_{v,t} := \sigma\sqrt{v_t}$, $\sigma > 0$ requires that the parameters satisfy $\kappa\theta > \frac{\sigma^2}{2}$ for the process v_t to be precluded from reaching zero.

[15] A similar perturbed equation for the dividend process is omitted for brevity. Since it turns out that the stock price will be proportional to dividends in equilibrium, adding that equation here is not necessary.

[16] This result follows immediately from an applications of Girsanov's theorem which states that for any $\mathcal{F}_t$-measurable random variable Z and $T > t$ we can write

$$\mathbb{E}^\vartheta\left[Z_T \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\frac{\vartheta_T}{\vartheta_t}Z_T \middle| \mathcal{F}_t\right]. \tag{6}$$

so that the Brownian Motions are related by $dW_t^S = dW_t^{S,\vartheta} + h_t dt$ where ϑ_t is an exponential martingale, with dynamics

$$\frac{d\vartheta_t}{\vartheta_t} = h_t dW_t^S. \tag{7}$$

where we refer to η_t as our stochastic disconnect process.[17] This specification is motivated by the empirical relation between volatility and uncertainty observed in Fig. 1, showing that while volatility and uncertainty are in general positively correlated, there are situations in which one of them is significantly higher than the other, i.e., they are disconnected. The stochastic process η_t measures the degree of disconnect, is positive and normalized to have mean one. This normalization is for ease of interpretation of the level of disconnect relative to its mean: when η_t is far away from one (either above or below) there is high disconnect between uncertainty and volatility, and when η_t is close to one disconnect is low. Rewriting Eq. (11) we have

$$\eta_t = \frac{\mathcal{U}_t}{\upsilon_t}, \tag{12}$$

so high levels of η_t capture situations in which the economy is in the "HL" regime, characterized by high uncertainty and low volatility, while low levels of η_t are observed when the economy is in the "LH" regime, characterized by low uncertainty and high volatility. By contrast, in the two connected regimes "HH" and "LL", η_t tends to be away from extreme values, and closer to its normalized mean value set to 1. Of course, the variables in Eq. (12) are continuous so the notion (and granularity) of any set of discrete regimes is simply a convenient low-dimensional representation of the investment environment that is useful for interpretation and aggregation, but plays no formal role in the analysis of the model. To fully specify the model, we assume that η_t is a positive process that, like υ_t, is a general stochastic process with drift $\mu_{\upsilon,t}$ and diffusion $\sigma_{\upsilon,t}$ under the reference probability measure.[18]

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}\left(\sqrt{1-\rho_{S,\eta}^2}dW_t^{\eta} + \rho_{S,\eta}dW_t^S\right), \quad \eta_0 > 0. \tag{13}$$

The term $\rho_{S,\eta}$ captures the correlation between the disconnect and stock price process. Furthermore, since volatility is correlated with the stock price, it follows that disconnect and the volatility are also correlated with correlation parameter equal to the product of $\rho_{S,\eta}$ and $\rho_{S,\upsilon}$.

4. Optimal portfolio allocation in partial equilibrium

To solve the investor's objective problem, we employ dynamic programming. We solve the resulting robust Hamilton–Jacobi–Bellman (HJB) equation under inequality constraints using the Lagrangian method. We then derive the optimal robust solution to the investor's investment and consumption problem in closed form. Let X_t denote the investor's wealth and ω_t be the percentage of wealth (or portfolio weight) invested in the risky asset; with $1-\omega_t$ is invested in the risk-free asset. The investor consumes at an instantaneous rate C_t and assumes the risky asset evolves according to the dynamics specified in Eq. (5). Accordingly, defining $\mu_{S,t}^h := \mu_{S,t} + \sigma_S\upsilon_t h_t$, the dynamics of investor's wealth X_t follow

$$\begin{aligned} dX_t &= \omega_t X_t\left(\frac{dS_t + D_t dt}{S_t}\right) + \left(1-\omega_t\right)X_t\frac{dB_t}{B_t} - C_t dt \\ &= \left(X_t\left[r_{f,t} + \omega_t\left(\mu_{S,t}^h - r_{f,t}\right)\right] - C_t\right)dt + \omega_t X_t \sigma_S \upsilon_t dW_t^{S,\vartheta}, \end{aligned} \tag{14}$$

starting from an initial endowment $X_0 > 0$. Under the robust probability measure $\mathbb{P}^{\vartheta}$, the investor derives logarithmic utility from consumption with subjective discount rate $\beta > 0$, and solves the infinite horizon problem[19]:

$$\sup_{\{C_s,\omega_s\}_{t\le s<\infty}} \inf_{\{h_s\}_{t\le s<\infty}} \mathbb{E}_t^{\vartheta}\left[\int_t^{\infty} e^{-\beta s}\log(C_s)ds\right], \tag{15}$$

subject to the entropy growth constraint in Eq. (10) and the wealth dynamics in Eq. (14). In partial equilibrium, the investor solves this problem taking the dynamics of asset prices as given. The investor's desire for robustness against model uncertainty is incorporated by evaluating the future evolution of the economy under the worst-case alternative model within the admissible set specified in Eq. (10). In order to solve the investor's optimization problem, we make use of standard robust dynamic programming techniques. To this end, we define the value function

$$V(t, X_t, \upsilon_t, \eta_t) = \sup_{\{C_s,\omega_s\}_{t\le s<\infty}} \inf_{\{h_s\}_{t\le s<\infty}} \mathbb{E}_t^{\vartheta}\left[\int_t^{\infty} e^{-\beta s}\log(C_s)ds\right], \tag{16}$$

[17] This approach to modeling a disconnect between uncertainty and volatility is not the only one possible. Other specifications, such as, for instance, an additive formulation can also be implemented and allow for explicit solutions within our framework provided that the investor has logarithmic utility. However, this simple multiplicative definition of uncertainty has two advantages: First, since disconnect is unobserved, it can only be identified once we have a proxy for volatility and uncertainty. By defining uncertainty $\mathcal{U}_t$ as the product of both volatility and disconnect, we have a simple and a consistent (within our model) way of extracting an empirical measure for what we defined to be the disconnect process η_t. Second, by imposing that η_t is a strictly positive process, we do not have to be concerned with, for instance in the case that uncertainty decomposes additive into volatility and disconnect, how to interpret negative disconnect η_t, and further technical issues as to whether the robust utility maximization problem is still well defined in the case when η_t is allowed to change signs.

[18] As was the case with volatility, the equilibrium is independent of the specific choice of the drift and diffusion for the disconnect process in the case of a logarithmic investor, who is myopic with respect to their specification. However, for the disconnect process to remain stationary and positive, just as in the case for the volatility process in Eq. (4), functional form and/or parameter restrictions on its drift and diffusive function are imposed.

[19] Appendix A.4 derives the equilibrium when the investor has CRRA preferences using the martingale approach.

associated with the optimal stochastic robust control problem in Eq. (15). Then, as above, we define $\mu^h_{v,t} := \mu_{v,t} + \rho_{S,v}\sigma_{v,t}h_t$ and similarly $\mu^h_{\eta,t} := \mu_{\eta,t} + \rho_{S,\eta}\sigma_{\eta,t}h_t$, the perturbed Hamilton–Jacobi–Bellman (HJB) equation characterizing the optimal solution is

$$\begin{aligned}0 = \sup_{\{C_t,\omega_t\}} \inf_{\{h_t\}} \Bigg\{ & e^{-\beta t}\log(C_t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}\left(X_t\left[r_{f,t} + \omega_t\left(\mu^h_{S,t} - r_{f,t}\right)\right] - C_t\right) \\ & + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\omega_t^2 X_t^2 \sigma_S^2 v_t^2 + \frac{\partial V}{\partial v}\mu^h_{v,t} + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2_{v,t} + \frac{\partial V}{\partial \eta}\mu^h_{\eta,t} + \frac{1}{2}\frac{\partial^2 V}{\partial \eta^2}\sigma^2_{\eta,t} \\ & + \frac{\partial^2 V}{\partial X\partial v}\omega_t X_t \sigma_S v_t \rho_{S,v}\sigma_{v,t} + \frac{\partial^2 V}{\partial X \partial \eta}\omega_t X_t \sigma_S v_t \rho_{S,\eta}\sigma_{\eta,t} + \frac{\partial^2 V}{\partial \eta \partial v}\rho_{S,\eta}\rho_{S,v}\sigma_{v,t}\sigma_{\eta,t}\Bigg\},\end{aligned} \tag{17}$$

subject to the relative entropy growth constraint[20]

$$\mathcal{R}(\vartheta_t) = \frac{h_t^2}{2} \leq \frac{\epsilon^2}{2}\mathcal{U}_t^2. \tag{18}$$

The robust optimal control problem is solved in two steps. First, the investor solves the inner optimization problem, deriving the optimal worst-case drift perturbation h_t^*. Second, the investor solves the outer problem, selecting the optimal consumption and portfolio holdings that maximize his or her expected utility of consumption under the worst-case model. The solution for the optimal robust policies is characterized in the following proposition.

Proposition 1. *The optimal consumption, drift perturbation, and portfolio policy are given, respectively, by*

$$\textit{Consumption}: \quad C_t^* = \beta X_t. \tag{19}$$

$$\textit{Perturbation}: \quad h_t^* = -\epsilon \mathcal{U}_t = -\epsilon \eta_t v_t. \tag{20}$$

$$\textit{Portfolio weight}: \quad \omega_t^* = \underbrace{\frac{\mu_{S,t} - r_{f,t}}{\sigma_S^2 v_t^2}}_{\textit{reference demand}} - \underbrace{\frac{\epsilon}{\sigma_S}\eta_t}_{\textit{uncertainty correction}}. \tag{21}$$

The investor's optimal consumption is a constant fraction of wealth equal to the subjective discount rate β, which is a standard result with logarithmic utility. The optimal drift perturbation h_t^* is negative, and driven by the product of uncertainty aversion and model uncertainty.[21] Because model uncertainty is the product of stochastic disconnect and volatility, for a given level of uncertainty aversion, the drift adjustment can be high even when volatility is low, if there is high disconnect (corresponding to the HL regime). It is negative and increasing in magnitude in the investor's degree of uncertainty aversion and in the level of disconnect. Finally, as Eq. (21) shows, the optimal portfolio holdings decompose into a standard myopic term equal to the Sharpe ratio, plus an uncertainty correction term. The correction term is largest when η_t is largest, that is, in the HL regime where uncertainty is significantly higher than volatility, and lowest when η_t is lowest, i.e., in the LH regime characterized by uncertainty being significantly lower than volatility.[22] Conversely, in the case where η_t is low, which implies high disconnect as volatility is high relative to uncertainty, the uncertainty correction term $\epsilon\eta_t/\sigma_S$ is small, but so is the portfolio holdings of the robust investor because the myopic term $(\mu_{S,t} - r_{f,t})/\sigma_S^2 v_t^2$ shrinks as well.

5. Equilibrium asset prices

Given the optimal demand for the assets expressed by the investor, we now solve for the equilibrium asset prices. An equilibrium is a specification of the dynamics of the risky and risk-less asset prices, including $\mu_{S,t} = \mu_S\left(v_t, \eta_t\right)$ and $r_{f,t} = r_f\left(v_t, \eta_t\right)$, combined with a set of optimal robust consumption and investment policies that support continuous clearing in the markets for the consumption good and the risky asset. The two market clearing conditions are

$$C_t = D_t, \tag{22}$$

$$\omega_t = 1. \tag{23}$$

The next proposition characterizes the equilibrium risk-free rate $r_{f,t}$ and the conditional equity premium under the reference measure, which can be expressed as

$$\frac{1}{dt}\mathbb{E}_t\left[\frac{dS_t + D_t dt}{S_t}\right] - r_{f,t} = \mu_{S,t} - r_{f,t}, \tag{24}$$

given the dynamics for the stock price in Eq. (3).

[20] It can be shown that, under mild regularity assumptions, the solution we provide satisfies the transversality condition: $\lim_{t\to\infty}\mathbb{E}^{\vartheta}\left[V(t, X_t, \eta_t, v_t)\right] = 0$. A formal proof of this sufficient condition is available from the authors upon request.

[21] While in principle there are two roots for the candidate solution of $h_t^* = \pm\epsilon\mathcal{U}_t$, only the negative solution is consistent with the minimization in Eq. (15). Intuitively, the minimization means that the investor evaluates his or her policies under the worst-case alternative in the set of models under consideration.

[22] Disconnect has two effects in the demand function for the risky asset in Eq. (21). First, in the partial equilibrium setting considered here, disconnect directly affects the optimal portfolio holdings of the robust investor. As Eq. (21) shows, the robust investor optimally reduces his or her exposure in the risky asset. Second, in general equilibrium, as we will later show, disconnect increases the risk premium, i.e. the equity premium appearing in the first term in the demand function will be itself a function of volatility and disconnect.

Proposition 2. *In equilibrium, the price–dividend ratio is $S_t/D_t = 1/\beta$. Using the optimal perturbation function h_t^* in Eq. (20), the equilibrium equity premium under the reference measure $\mathbb{P}$ and the risk-free rate are given by*

$$\text{Equity premium}: \quad \mu_{S,t} - r_{f,t} = \sigma_D^2 v_t^2 + \epsilon \sigma_D \eta_t v_t^2. \tag{25}$$

$$\text{Risk-free rate}: \quad r_{f,t} = \beta + \mu_D v_t - \sigma_D^2 v_t^2 - \epsilon \sigma_D \eta_t v_t^2. \tag{26}$$

As the results in Proposition 2 show, the equity premium and risk-free rate are time-varying and non-linear functions of volatility and disconnect. In particular, the uncertainty term embedded in disconnect generates a flight-to-quality-like correlation among asset returns. In periods of increasing uncertainty (increasing disconnect, for a given level of volatility) the demand for the risky asset decreases and the demand for the safe asset increases. Accordingly, in periods of increasing uncertainty the investor requires a higher equity premium to hold the risky asset, and is willing to accept a lower risk-free rate to hold the risk-less asset in equilibrium. The presence of stochastic volatility may amplify or diminish these effects depending on whether it is connected or disconnected from uncertainty, respectively. In the following sections we show that, because both volatility and disconnect jointly drive asset prices, our model is more consistent with the observed dynamics of asset prices, including during high-disconnect episodes.

The structure of the equilibrium equity premium and risk-free rate can be also understood in the context of the optimal portfolio policy obtained in Proposition 1. The negative uncertainty correction implies that the representative investor is more conservative and prefers to hold less of the risky asset and more of the risk-free asset compared to a reference investor. In general equilibrium, the investor must allocate all his or her wealth to the risky asset (see Eq. (23)). Therefore, the last term in the equity premium boosts up the reward from holding the risky asset just enough so that the investor optimally chooses to allocate all his or her wealth to the risky asset. By the same token, the risk-free rate has to decrease in general equilibrium, so the investor optimally chooses not to hold the risk-free asset at all. Finally, it is likely that alternative approaches to model robustness (as discussed in footnote) would also lead to an agent's optimal portfolio holdings that are more conservative and command a higher risk premium (an additional positive ambiguity or uncertainty adjustment) and a lower equilibrium risk free rate.

6. Empirical analysis

6.1. Data

The S&P 500 (logarithmic) returns including dividends obtained from the CRSP database serve as a proxy for the risky-asset return.[23] The risk-free rate is the three-month (constant maturity) Treasury bill rate, which we obtain from the Fred St. Louis Database (Ticker "DGS3MO"). Both series are available at the daily frequency from January 1, 1986 until December 31, 2020; we aggregate them to the weekly frequency.

We construct our weekly measure of uncertainty $\mathcal{U}_t$ using the daily news-based economic policy uncertainty (EPU) index developed by Baker et al. (2016).[24] The daily EPU index is constructed by counting in the archives of well over one thousand US newspapers the number of articles that contain at least one term related to economic policy uncertainty from the following list:"uncertain", "uncertainty", "economic", "economy", "Congress", "deficit", "Federal Reserve", "legislation", "regulation", and "White House". The daily EPU time series is volatile, and we smooth it by computing its moving average over the past week (5 trading days) to construct the uncertainty measure we use at the weekly frequency.

The diffusive term of the stock price process is $\sigma_{S,t} := \sigma_S v_t$ where $\sigma_S = 1$, since we match v_t to the square root of realized variance, which is computed daily from aggregated intraday returns sampled at five minutes frequency (78 observations per trading day). To obtain a weekly measure of realized variance, we sum the daily realized variance estimates. Based on the definition of disconnect in Eq. (12), we construct an empirical proxy for η_t at the weekly frequency by computing the ratio of uncertainty as measured by EPU to realized volatility, and we then normalize this ratio to have an average value over the full sample equal to one.

6.2. Time-series properties of volatility and uncertainty

In Fig. 2, we plot the time series of volatility and uncertainty, as well as the resulting disconnect process. From Panel A, we observe that uncertainty is substantially more volatile than volatility: the standard deviation of the EPU index is 50.1% compared to only 7.3% for realized volatility. While less volatile, realized volatility is more persistent than uncertainty: the first order auto-correlation of volatility is 0.87, compared to 0.74 for the EPU index. In Panel B, we plot the associated normalized disconnect time series. Since it is constructed as a normalized ratio of uncertainty and volatility, it is also volatile and persistent. The shaded areas in the charts mark the three high-disconnect periods described above: the financial crisis, the US 2016 election, and the Covid-19 pandemic. High disconnect occurs whenever η_t is far away from 1.

Consistent with Fig. 1, the correlation between volatility and uncertainty is only 0.44: while the two series co-move on average, they are often disconnected. Regressing uncertainty on volatility and a constant yields an adjusted R^2 of 25%. Therefore, 75% of the time-series variation in uncertainty cannot be explained by volatility—the gap which we attribute to disconnect in the model.

[23] Center for Research in Security Prices, US Stock Database, obtained from Wharton Research Data Services.

[24] Section 6.5 below shows that our results continue to hold when considering alternative empirical measurements of uncertainty.

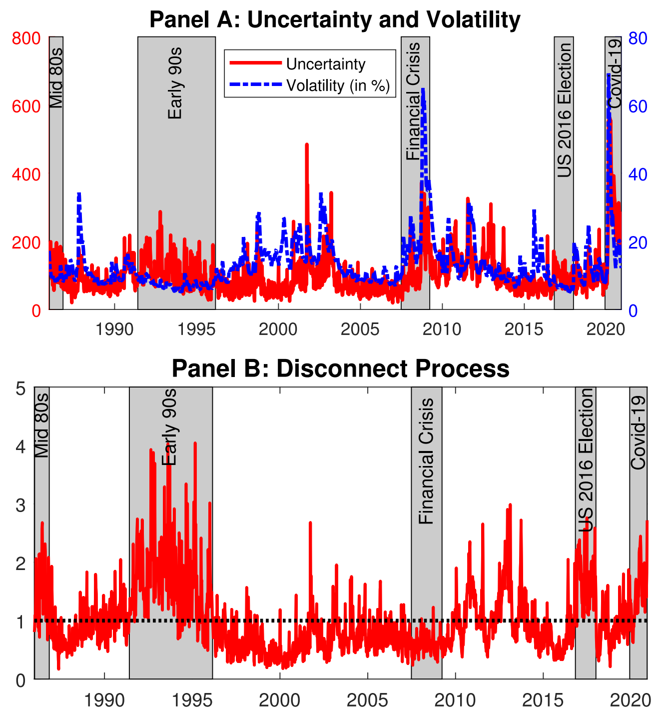

Fig. 2. Time series of volatility, uncertainty and disconnect.
Notes: Panel A shows uncertainty (proxied by the economic policy uncertainty index EPU_t) and volatility (proxied by realized volatility). Panel B shows the disconnect process (η_t) defined as the ratio of the EPU index divided by realized volatility, normalized to have a mean of 1. The sample period is from January 1986 until December 2020, and the data is sampled at the weekly frequency. The shaded areas correspond to five sub-periods associated with high disconnect: the mid 80's (from January 1986 until November 1986), the early 90's (from June 1991 until February 1996), the financial crisis (from July 2007 until March 2009), the US 2016 election (from July 2016 until January 2018), and the Covid-19 pandemic (from January 2020 until December 2020).

Table 1
Average stock excess return and risk-free rate in selected high-disconnect regimes.

	Full Sample	Mid 80's	Early 90's	Financial Crisis	US 2016 election	Covid-19
Time Period Begin	Jan. 1986	Jan. 1986	Jun. 1991	Jul. 2007	Jul. 2016	Jan. 2020
Time Period End	Dec. 2020	Nov. 1986	Feb. 1996	Mar. 2009	Jan. 2018	Dec. 2020
Stock excess return (%)	7.70	18.8	7.77	−33.40	17.29	16.97
Risk-free rate (%)	3.20	6.2	4.22	2.02	0.67	0.38
Uncertainty $\mathcal{U}_t$	101.21	126.79	111.68	130.02	85.00	284.30
Stock return volatility υ_t (%)	17.70	9.78	7.32	28.16	10.51	28.94
Disconnect $\eta_t = \mathcal{U}_t/\upsilon_t$ (scaled)	1.00	1.52	1.77	0.59	1.29	1.49

Notes: This table presents the average annualized stock excess return, risk-free rate, uncertainty ($\mathcal{U}_t$), annualized volatility (υ_t), and disconnect (η_t). These averages are computed over the full sample (from January 1986 until December 2020) and during five sub-periods associated with high disconnect: the mid 80's (from January 1986 until November 1986), the early 90's (from June 1991 until February 1996), the financial crisis (from July 2007 until March 2009), the US 2016 election (from July 2016 until January 2018), and the Covid-19 pandemic (from January 2020 until December 2020). All the data is nominal and sampled at the weekly frequency.

6.3. Average excess returns in high-disconnect periods

Table 1 reports the average stock excess return and risk-free rate, along with the average uncertainty, volatility, and disconnect for the full sample and for the following high-disconnect sub-periods: the mid 80's (from January 1986 until November 1986), the early 90's (from June 1991 until February 1996), the financial crisis (from July 2007 until March 2009), the US 2016 election (from July 2016 until January 2018), and the Covid-19 pandemic (from January 2020 until December 2020). Over the full sample, the average stock excess return is 7.7%, with a relatively high level of the average risk-free rate compared to recent values. Given the paths of uncertainty $\mathcal{U}_t$ and volatility υ_t, the normalized measure of disconnect η_t has an average value of 1 over the full sample by construction.

Before analyzing the most recent disconnect sub-periods, over which we will examine in more detail the performance of the model, we briefly describe two older high-disconnect periods: the mid 80's and the early 90's. Compared to the full sample results, both of these periods are characterized by a very high risk-free rate (about double its level in the full sample for the mid 80's and

30% higher in the early 90's). In addition, in these periods there was a relatively high level of uncertainty (about 30% higher than in the full sample for the mid 80's and 10% higher for the early 90's) and a very low level of volatility (about half is level in the full sample for both mid 80's and early 90's). Accordingly, these are categorized as high-disconnect periods, with average level of disconnect 50% and 70% higher than in the full sample, respectively. Despite these similarities, average stock excess returns were very different in these periods: they were very high in the mid 80's (about double its level in the full sample) but they were about the same as in the full sample in the early 90's. The significantly higher average stock excess returns in a period of high average disconnect experienced in the mid 80's (and in the more recent high-disconnect sub-periods which we analyze next) is consistent with our model. By contrast, the early 90's provide an example in which despite a high average disconnect, stock excess returns were on average similar to their level in the full sample. This may be due to the very high length of the early 90's period (5 years), in which disconnect increased from 1991 until 1993, but then subsequently declined. While in principle we could try to refine this sub-period to further investigate this issue, in this section we only illustrate average stock excess returns in selected periods of high average disconnect. A more formal regression-based analysis of the relation between disconnect and stock excess returns is conducted in Section 6.4.

We now analyze three more recent high-disconnect periods, all of which are characterized by progressively lower average risk-free rates, suggesting monetary policy (among many potential factors other than disconnect) may have played a role in characterizing asset prices. Compared to the full sample results, during the financial crisis period the average stock excess return was extremely negative (−33.4%) and both volatility and uncertainty were very high. However, despite a very high level of average uncertainty (about 30% higher than in the full sample), the extremely high level of average volatility (about 60% higher than in the full sample) implies this is a high-disconnect period, with average level of disconnect about 40% lower than in the full sample.

The period surrounding the US 2016 election was characterized by a higher average stock excess return (more than double its level in the full sample) and significantly lower average volatility, with slightly lower average uncertainty. Even though average uncertainty in this period was relatively close to its full-sample average (only about 15% lower than in the full sample), because average volatility was extremely low during this period (about 40% below its average level in the full sample) this is a high-disconnect period, with average level of disconnect about 30% higher than in the full sample.

Finally, the Covid-19 period was characterized by high average stock excess returns, similar to those around the US 2016 election, but with considerably higher average volatility and uncertainty. Despite the extremely high level of average volatility, comparable to the one observed during the financial crisis, the even more extreme level of average uncertainty (close to 3 times its level in the full sample) implies that this is a high-disconnect period, with average level of disconnect about 50% higher than in the full sample. While disconnect was on average high during the Covid-19 sub-period, it fluctuated over time, mainly driven by volatility. Initially, disconnect declined sharply due to a high spike in volatility in March 2020, while stock prices dropped. Subsequently, it recovered and reached very high levels, as volatility declined in the following months, while financial markets recovered. Meanwhile, uncertainty remained high during most of the Covid-19 period.

Summing up, in three out of four periods in which the average ratio of uncertainty to volatility was relatively high, average stock excess returns were relatively high. Similarly, in a period in which the average ratio of uncertainty to volatility was relatively low, average stock excess returns were extremely low. These illustrative results are consistent with our model. We now turn to formally verify the empirical relation between disconnect and stock excess returns.

6.4. Conditional equity premium and risk-free rate

Proposition 2 implies that the equity premium and the risk-free rate are specific functions of v_t and η_t. While these variables are continuous in the model, and we will analyze them as such, it is useful for interpretation purposes to start by thinking in terms of discrete regimes corresponding to ranges of values: High uncertainty with high volatility (HH), high uncertainty with low volatility (HL), low uncertainty with high volatility (LH), and low uncertainty with low volatility (LL), as described in Fig. 1. For the purpose of this discussion, "high" and "low" are defined based on threshold levels for v_t and η_t, given by their respective mean plus one half of their standard deviation[25]: observations above (respectively,below) the threshold level are "high" (respectively, "low"). Table 2 reports the average stock market excess return and risk-free rate at the two-week horizon in each regime. It also shows the estimated unconditional probability of being in each regime.

The most frequent regime is LL (both volatility and uncertainty are low, η_t close to one), and is empirically characterized by a high average risk-free rate and a low average stock excess return. Regime HH (both volatility and uncertainty are high, η_t close to one) exhibits the lowest average risk-free rate, and considerably higher average stock excess return than in the LL regime. Moving on to the disconnected regimes, we find that the average stock excess return is highest in the HL regime (high uncertainty with low volatility, high η_t) while the average stock excess return in the LH regime (low uncertainty with high volatility, low η_t) dominates only the one in the LL regime. The empirical results are precisely what Proposition 2 implies: the stock market provides a sizable compensation for bearing uncertainty, especially in regimes in which uncertainty is high and disconnected from lower volatility (HL regime, high η_t), compared to the premium earned solely for bearing volatility.

The results in Table 2 correspond to a fixed investment horizon of two weeks. Fig. 3 shows the results as the investment horizon τ ranges from 1 to 12 weeks.

[25] The analysis presented here is robust to alternative thresholds definitions, we omit the complete results for brevity.

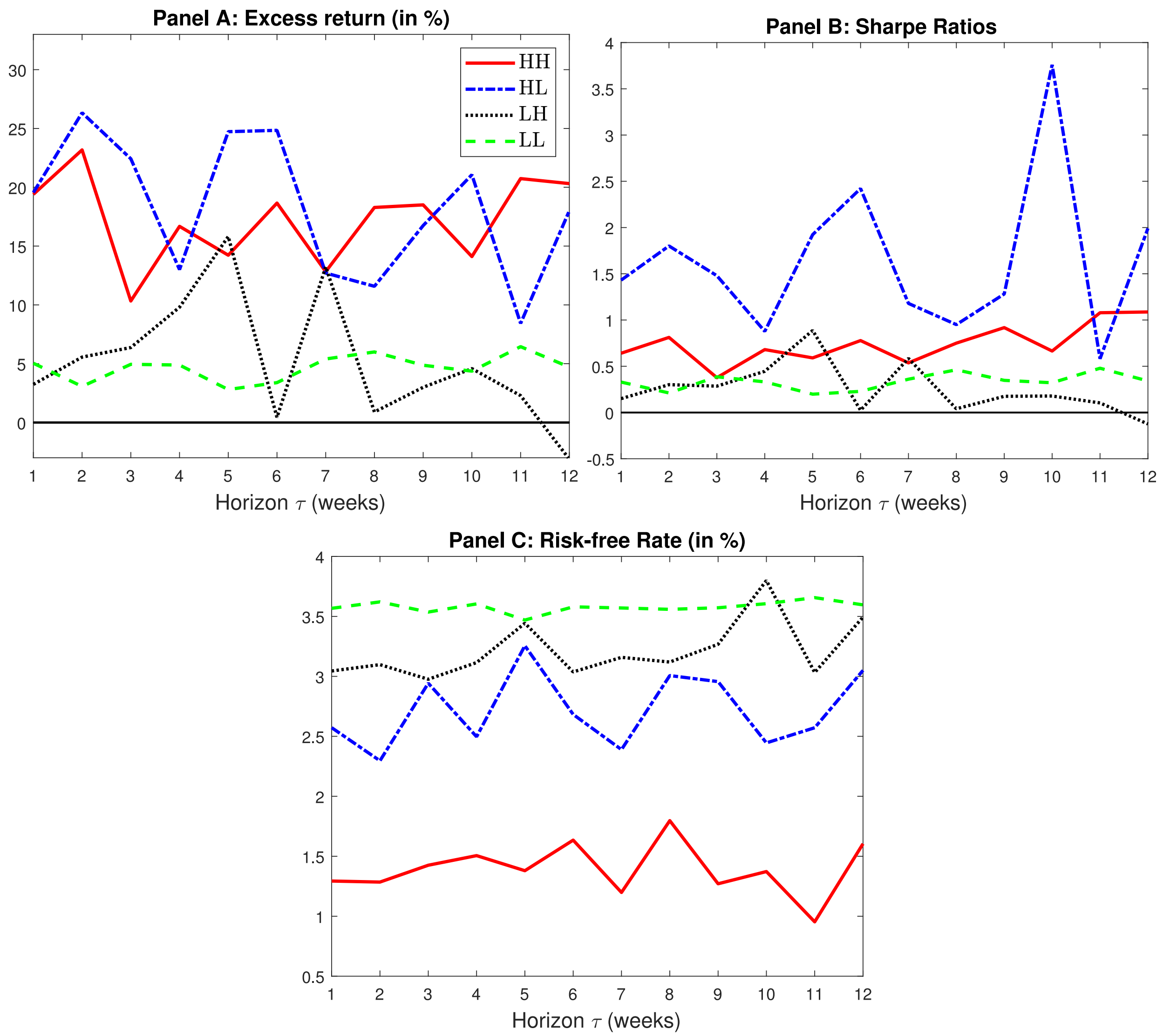

Fig. 3. Stock excess return, Sharpe ratio, and risk-free rate conditional on regimes of uncertainty and volatility.
Notes: This figure shows the annualized average stock excess return (in %), Sharpe ratio, and risk-free rate (in %) over a given horizon $\tau = 1, \ldots, 12$ weeks, conditional on being in one of the four market regimes: (i) both volatility and uncertainty are high (HH), (ii) high uncertainty and low volatility (HL), (iii) low uncertainty and high volatility (LH), or (iv) both low uncertainty and low volatility (LL). The threshold values for volatility and uncertainty are given by their mean plus one half of their standard deviation. We say that volatility and uncertainty are high (respectively, low), when they are above (respectively, below) their threshold values. The sample period is from January 1986 until December 2020, and the data is sampled at the weekly frequency.

Panel A in Fig. 3 plots the stock excess return for each regime as a function of the investment horizon τ. Consistent with our results in Table 2, high-uncertainty regimes (HH and HL) deliver high subsequent stock excess returns for all the horizons considered. The stock excess return in low-uncertainty regimes (LL and LH) are also positive for most horizons (with the only exception for $\tau = 12$ in the LH regime), but they are significantly lower in magnitude compared to those achieved in high-uncertainty regimes. To incorporate the volatility dimension in the comparison, Panel B plots the Sharpe ratios, obtained by scaling the average realized stock excess returns by their respective standard deviation. By this metric, the HL regime achieves the highest performance for most horizons (with the only exception being for $\tau = 11$ weeks), outperforming the results achieved in regime HH. It then follows that, while high-uncertainty regimes offer a high subsequent stock excess return in general (Panel A), regimes with high uncertainty and simultaneous high volatility deliver subsequent more volatile excess returns, leading to investments in regime HL outperforming those in regime HH for any risk-averse investor (Panel B). This is a consequence of volatility persistence (as discussed above, volatility is highly persistent, and in particular more persistent than uncertainty). Finally, Panel C plots the risk-free rate across regimes; the results are also consistent with the evidence contained in Table 2 and with the prediction of Proposition 2. For instance, the risk-free

Table 2
Average stock excess return and risk-free rate in different uncertainty-volatility regimes.

		Volatility Level	
		High	Low
Uncertainty Level	High	$\bar{r}_e = 23.17\%$, $\bar{r}_f = 1.29\%$ $\mathbb{P}[\text{HH}] = 9.23\%$	$\bar{r}_e = 26.33\%$, $\bar{r}_f = 2.39\%$ $\mathbb{P}[\text{HL}] = 10.82\%$
	Low	$\bar{r}_e = 5.56\%$, $\bar{r}_f = 3.1\%$ $\mathbb{P}[\text{LH}] = 10.36\%$	$\bar{r}_e = 3.07\%$, $\bar{r}_f = 3.62\%$ $\mathbb{P}[\text{LL}] = 69.59\%$

Notes: This table reports the estimated unconditional probability of each market regime (HH, HL, LH, and LL). It also shows the average (annualized) risk-free rate and stock excess return at a 2-weeks horizon in each regime. The threshold values for volatility and uncertainty are given by their mean plus one half of their standard deviation. We say that volatility and uncertainty are high (respectively, low), when they are above (respectively, below) their threshold values. Both volatility and uncertainty are sampled at the weekly frequency from January 1986 until December 2020.

rate is lower during high-uncertainty regimes (HH and HL), and is highest in the most likely regime characterized by low uncertainty and low volatility (LL).

Next, we examine the predictions of the model contained in Proposition 2 by regressing both the conditional equity premium and risk-free rate on the variance (υ_t^2) and uncertainty term ($\upsilon_t^2\eta_t$), using the continuum of their values and without distinguishing discrete regimes. Fig. 4 shows the results of those regressions.

We start describing the impact of volatility and uncertainty on the stock excess return, shown in the left panels of Fig. 4. Panel A reports the estimated coefficient from regressing the stock excess return on realized variance (υ_t^2) only, with the shaded area representing a 95% confidence interval. It shows that there is a positive albeit insignificant relationship between the stock excess return and its volatility. In Panel C we do a similar analysis, but adding the uncertainty term ($\upsilon_t^2\eta_t$) to the regression, as required by Proposition 2. We obtain that both volatility and the uncertainty term are statistically significant at the 5% level (except for short horizons up to two weeks). The estimated regression coefficient for the volatility term is negative, and less significant, consistent with the absence of a clearcut volatility-return trade-off in the data. As already discussed (see footnotes), the absence of a return-volatility trade-off is not unexpected given the mixed evidence in the literature; yet any model that assumes volatility aversion is bound to predict a positive trade-off. For the uncertainty term, however, the sensitivity is positive, as predicted by the model. Indeed, it appears that the risk premium is compensation for facing uncertainty rather than volatility. Panel E shows that our model including both volatility and uncertainty delivers a substantially higher (in-sample) adjusted R^2.

Moving on to the impact of volatility and uncertainty on the risk-free rate, we report the results in the panels on the right side of Fig. 4. Panel B reports the estimated coefficient from regressing the risk-free rate on realized variance (υ_t^2) only, with the shaded area representing a 95% confidence interval. It shows that there is a significantly negative relationship between the risk-free rate and the stock return volatility, which is consistent with our result in Proposition 2. In Panel D we do a similar analysis, but adding the uncertainty term ($\upsilon_t^2\eta_t$) to the regression, as required by Proposition 2. We obtain a significantly negative relationship between the risk-free rate and uncertainty, while volatility becomes insignificant. This shows that uncertainty is the main driver of the risk-free rate, and it has a negative sign consistent with our result in Proposition 2. Finally, Panel F shows that our model including both volatility and uncertainty delivers a substantially higher (in-sample) adjusted R^2 for the risk-free rate.

It is clear from our results that uncertainty plays a prominent role in characterizing both the equity premium and the risk-free rate. One possible interpretation is through the lens of a flight-to-quality-like effect. In periods of high uncertainty the demand for risky assets decreases and investors require a higher equity premium to hold the risky asset. Similarly, in periods of high uncertainty the demand for safe assets increases, and their required return declines accordingly. This is consistent with uncertainty been positively related to the equity premium and negatively related to the risk-free rate as indicated by Proposition 2, a relation for which we find empirical validation in the data. By contrast, our empirical analysis shows that volatility negatively affects stock excess returns, and has a negligible impact on the risk-free rate.

Our results on the impact of volatility and uncertainty on the equity premium help explain the challenges faced by previous empirical studies trying to establish a risk-return trade-off using volatility alone as a measure of risk. Comparing panels A and C, it appears that volatility as the sole state variable struggles to generate the variation in the equity premium that is present in the data; the effect of volatility on excess returns in the data is not as clear cut as standard financial theory suggests. By averaging periods when the trade-off between volatility and return works in the expected direction with periods when it does not, the regression in Panel A ends up with a still positive, but barely significant effect. Once we include both volatility and uncertainty in Panel C, volatility is no longer tasked with playing this dual role and we obtain a clean separation of the effect of volatility (negative) and uncertainty (positive) on the equity premium.

Summing up, we find fairly strong support in the data for the model's predictions as given in Proposition 2, most notably the important impact of uncertainty, with the provision that volatility has a negative effect on the equity premium and a negligible effect in the risk-free rate.

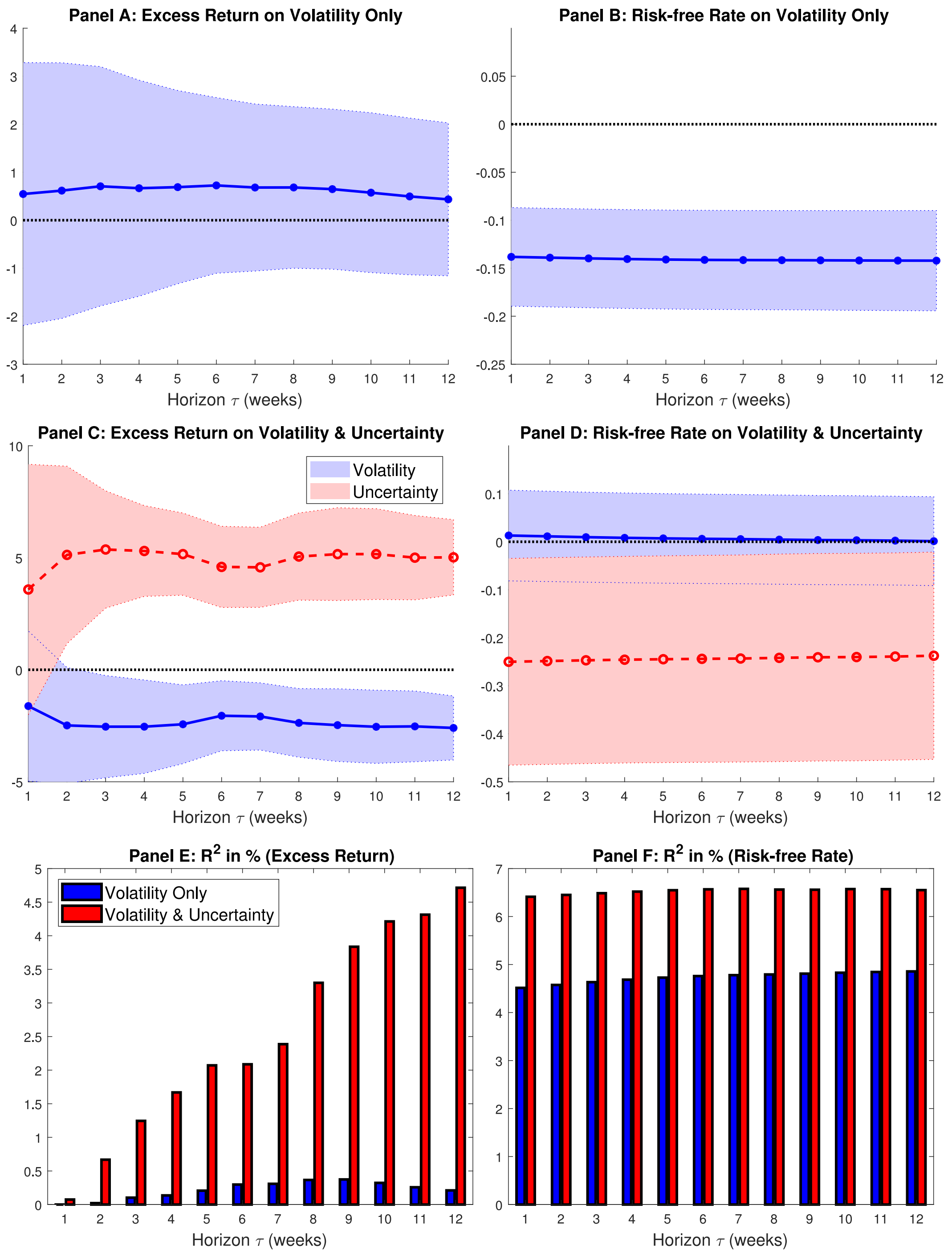

Fig. 4. Conditional stock excess return and risk-free rate, main sample (1986–2020, weekly).
Notes: The left charts correspond to regressions of stock excess return on (squared) volatility (v_t^2) and uncertainty ($v_t^2\eta_2$), and the charts on the right show regressions of the risk-free rate on the same regressors. All regressions are run over horizons of up to 12 weeks. Panels A and B show estimated coefficients from univariate regressions on a variance term only. Panels B and C show estimated coefficients from bivariate regressions on variance and an uncertainty term (based on Proposition 2). The shaded areas represent 95% confidence intervals constructed using auto-correlation and heteroskedasticity corrected standard errors (HAC). Panels D and E show the adjusted R^2 (in %) from the uni- and bi-variate regressions in panels A, B, C and D. The sample period is from January 1986 until December 2020, and the data is sampled at the weekly frequency.

6.5. Robustness to alternative measurements of uncertainty

As explained in Section 6.1, we construct our weekly measure of uncertainty $\mathcal{U}_t$ using the daily economic policy uncertainty (EPU) index of Baker et al. (2016). In this section, we summarize the results obtained using alternative possible measures of uncertainty. Further detailed results are contained in the Online Supplement to the paper.

Different approaches to measure uncertainty have been employed in the literature (see Cascaldi-Garcia et al., 2020 for a survey). Several papers consider the newspaper-based methodology of Baker et al. (2016), but with a different selection of words in order to target different types of uncertainty. For example, Husted et al. (2020) select words related to measure more specifically monetary policy uncertainty, while Caldara and Iacoviello (2019) consider a list of words tailored to identify geopolitical uncertainty. Moreover, besides the news-based component used in its daily version, the monthly frequency EPU index from Baker et al. (2016) contains two additional components added to gauge uncertainty regarding the federal tax code (by counting the number of federal tax code provisions set to expire in future years) and to quantify the disagreement among economic forecasters.[26] The monthly EPU has been further disentangled into several sub-indexes of specific categories, with words lists associated to each of them.[27] The categories are: economic policy, monetary policy, fiscal policy, taxes, government spending, health care, national security, entitlement programs, regulation, trade policy, and sovereign debt and currency crises. Finally, Baker et al. (2021) construct Twitter economic uncertainty indicators, and show they behave similarly to the newspaper-based EPU index.

There are several alternative approaches which measure uncertainty without relying on text analysis from newspapers. Scotti (2016) uses macroeconomic news and survey forecasts to construct an ex-post realized measure of uncertainty about the state of the economy. In contrast to sentiment-based uncertainty measures that rely on analysts' forecasts, Jurado et al. (2015) construct a monthly index of macroeconomic uncertainty as an aggregate volatility of statistical forecasts for hundreds of economic series. In turn, Izhakian (2020) defines ambiguity as probability perturbations (uncertain probabilities) and aversion to ambiguity as aversion to the mean-preserving spreads in these probabilities. He then constructs an uncertainty measure based on the expected volatility of probabilities across the relevant events.

To make sure our results are not driven by the specific uncertainty measure we use, we reproduced the same analysis that led to the construction of Fig. 4 for all the uncertainty measures described in this subsection. Despite these measures being constructed very differently, and being available at varying frequencies ranging from daily to monthly frequency, we find the results highlighted in Fig. 4 are broadly consistent for all the measures considered, with the main caveat that some monthly measures tend to lead to less statistically significant results, as expected given the lower number of observations available.[28] The robustness of our results to several empirical measurements of uncertainty is not surprising given that several uncertainty measures are positively correlated and share an element of counter-cyclicality, as reported by Kozeniauskas et al. (2018).

All the alternative uncertainty measures we considered are not constructed as measures of risk premia, as otherwise the uncertainty measures used would most likely be correlated by construction with the equity premium due to the strong co-movement of all risk premia. For example, given that we use realized volatility as our volatility measure, the VIX would not be a desirable uncertainty measure. This is because the marginal contribution of VIX to explain the equity premium would be its difference with realized volatility, which is the square root of the variance risk premium.[29]

We conclude that our results are robust to several alternative measurements of uncertainty, and hence are most likely not driven by our specific choice of the daily EPU index to construct our weekly uncertainty measure.

6.6. Robustness to a longer sample

In our main empirical analysis our sample is from January 1986 until December 2020, because we have available daily data for this period for both the EPU and realized volatility. As such, even after we aggregate our data to the weekly frequency, we end up with a reasonably large sample.

To consider a longer sample, we repeat the predictive regression analysis conducted in Section 6.4 using data from July 1926 until December 2020. For this longer sample, the data is available at the monthly frequency only. On the one hand, the EPU index is not available at the daily frequency before 1986, so we use instead monthly EPU. On the other hand, intraday returns are not available before 1986, so we construct realized volatility at the monthly frequency based on the aggregation of daily squared returns.[30]

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[26] In addition, the news-based component is different from the one used in the daily series. For the monthly frequency, the focus is only on ten major US newspapers, and the word search is more specific.

[27] For example, articles that fulfill the requirements to be coded as EPU and also contain the term "Federal Reserve" would be included in the monetary policy uncertainty sub-index.

[28] The empirical results from replicating the analysis above with all these alternative uncertainty measures are contained in the online supplement to the paper.

[29] Bollerslev et al. (2009) construct a model in which both the volatility and the volatility-of-volatility of consumption growth rate drive the equity premium. They proxy the latter using the variance risk premium and show empirically that it is able to predict the stock excess return. In contrast to their paper, our theory considers model uncertainty rather than volatility-of-volatility of consumption growth rate (which is constant in our model). Empirically, we refrain from using risk premia to predict the equity premium, and consider instead uncertainty measures, consistent with our modeling framework.

[30] The 1-month maturity risk-free rate data is obtained from Kenneth French's homepage, as the 3-month risk-free rate data obtained from the Fred Database of the St. Louis Fed does not go back that far.

Despite having available only about two thirds of the observations in our main sample,[31] our results for this alternative sample, shown in Fig. 5, are very similar to those obtained in our main sample, shown in Fig. 4. Interestingly, uncertainty is significant in explaining stock excess returns only at short horizons (up to three months ahead), and at long horizons (eleven and twelve months ahead), while it is insignificant for horizons between four and ten months ahead. This is consistent with our results in the main sample: there we found uncertainty was significant in explaining stock excess returns up to 12 weeks ahead, which is equivalent to our results in this subsection, in which we find a significant relation up to three months ahead. Similarly, uncertainty is significant in explaining the risk-free rate up to six months ahead, consistent with our results in the main sample. Finally, using the longer sample yields even higher adjusted R^2 for both the stock excess returns and risk-free rate regressions.

We conclude that our results are robust to alternative samples with longer history, and hence are most likely not driven by our specific sample, which allows us to test our results at the highest frequency available.

6.7. Robustness to the exclusion of high-disconnect periods

To illustrate the importance of the high-disconnect periods in driving the predictability results, we repeat the analysis by purposefully excluding all of these sub-periods from the analysis. More specifically, we exclude the mid 80's (from January 1986 until November 1986), the early 90's (from June 1991 until February 1996), the financial crisis (from July 2007 until March 2009), the US 2016 election (from July 2016 until January 2018), and the Covid-19 pandemic (from January 2020 until December 2020).

While the estimated coefficients of the regression of stock excess return on volatility only remain insignificant, adding uncertainty as an explanatory variable no longer improves the prediction of the stock excess return: both regression coefficients for volatility and uncertainty turn statistically insignificant.[32] This is consistent with our model key prediction: stock excess returns appears to be earned mainly for facing high uncertainty that is disconnected from lower volatility. In regular periods in which volatility and uncertainty move in tandem, both of them have limited ability to predict stock excess returns.

7. Disconnect-managed portfolios

In this section, we examine whether the predictability of the equity risk premium when using uncertainty and volatility translates into superior portfolio performance. As shown in Eq. (23), the equity market clearing condition implies that in equilibrium a robust representative investor would follow an unconditional or passive strategy by investing in equilibrium all his or her wealth in the risky asset. A reference investor who does not worry about model uncertainty but assumes that the equity premium is a function of volatility and uncertainty as suggested by our model in Proposition 2 would follow an active strategy conditioning in both volatility and uncertainty. An investor who disregards the role of uncertainty and posits that the equity premium is a function of volatility only would follow an alternative active strategy conditioning on volatility only.[33] We characterize each of these portfolio strategies through a regression-based approach and compare their relative performance. Recall that the equity premium implied by the model given in Proposition 2 is:

$$\mathrm{ERP}_t = \mu_{S,t} - r_{f,t} = v_t^2 + \epsilon \eta_t v_t^2. \tag{27}$$

So a forecast for the equity premium can be obtained from the regression

$$r_{e,t+\tau} := r_{t,t+\tau} - r_{f,t,t+\tau} = \beta_0 + \beta_v v_t^2 + \beta_\eta \eta_t v_t^2 + \epsilon_{t+\tau}, \tag{28}$$

for any horizon τ. Estimating this equation via OLS yields

$$\hat{r}_{e,t+\tau} = \hat{\beta}_0 + \hat{\beta}_v v_t^2 + \hat{\beta}_\eta \eta_t v_t^2, \tag{29}$$

where $\hat{r}_{e,t+\tau}$ represents the estimated equity premium at horizon τ. Plugging this expression into the optimal policy of a standard logarithmic investor who does not face model uncertainty, gives an optimal active portfolio strategy that conditions on both volatility and uncertainty:

$$\hat{\omega}_t^* = \frac{\mu_{S,t} - r_{f,t}}{v_t^2} = \frac{\hat{\beta}_0}{v_t^2} + \hat{\beta}_v + \hat{\beta}_\eta \eta_t. \tag{30}$$

To consider an alternative active strategy conditioning on stochastic volatility exclusively, we re-estimate Eq. (28) but regressing on the volatility term only. For the portfolio performance analysis, we multiply the estimated portfolio weights in Eq. (30) by a constant $L = 0.15$, which is set to make both dynamic portfolio strategies on average about 100% invested in the index. This is for ease of comparison with the benchmark representative passive investor robust to model uncertainty, who allocates all his or her wealth in equilibrium to the risky asset.

We compare the three portfolio strategies over the full sample, and over the three most recent sub-periods in which volatility and uncertainty were highly disconnected. We highlight that all the portfolio comparisons in this section correspond to a full in-sample analysis with the advantage of hindsight. In other words, we back-test all the portfolio strategies considered. In Section 8, we show that adding uncertainty as a predictor of future stock excess return improves forecast accuracy out-of-sample, over and above the forecast accuracy that can be obtained by using only volatility as a predictor.

[31] This follows because its frequency is monthly, compared to our main sample which is available at the weekly frequency.

[32] The empirical results excluding high-disconnect periods are not shown for brevity, but they are available from the authors upon request.

[33] This special case arises when the representative investor is not averse to uncertainty ($\epsilon = 0$).

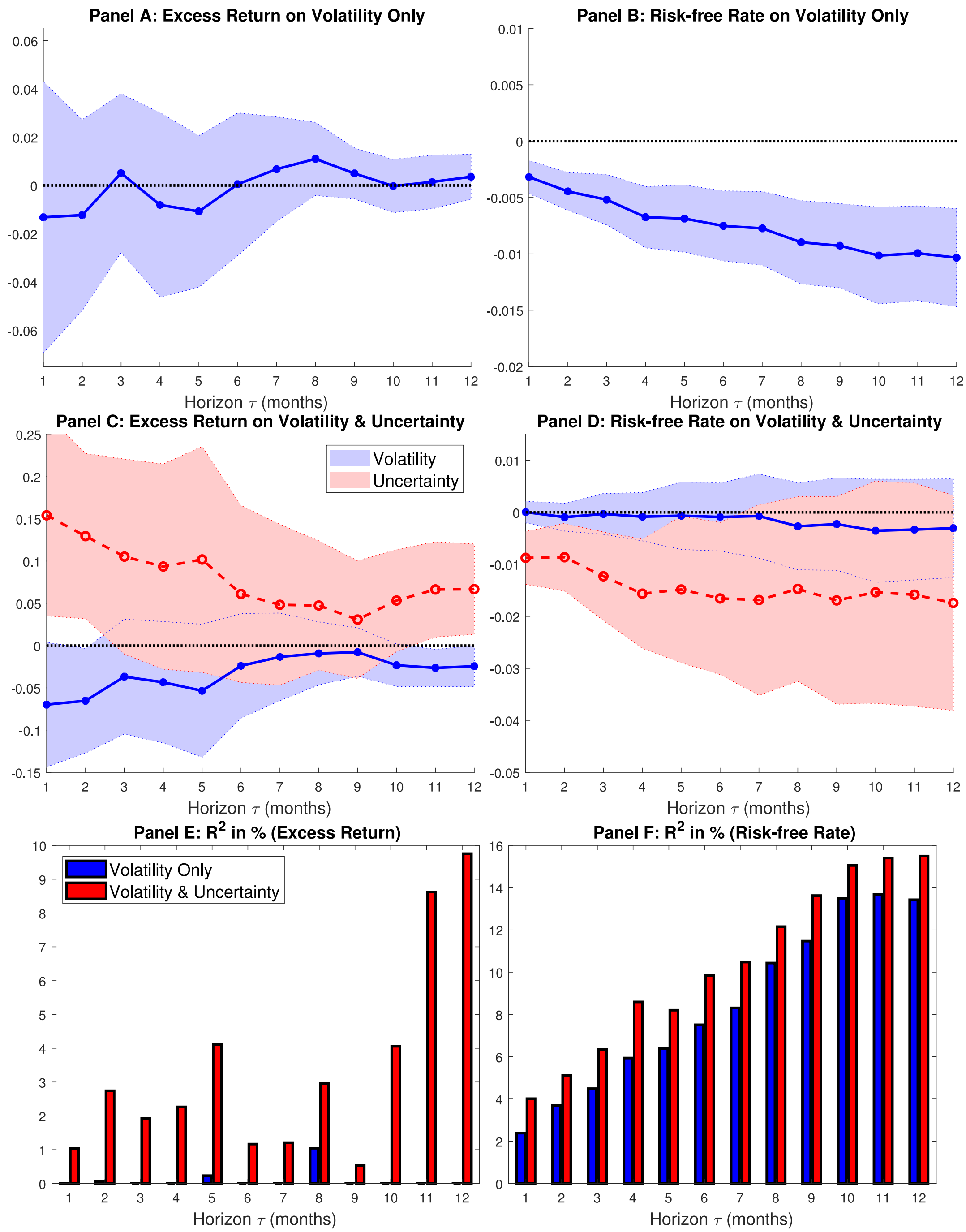

Fig. 5. Conditional stock excess return and risk-free rate, extended sample (1926–2020, monthly).
Notes: This figure is identical to Fig. 4 except that the sample period is from July 1926 until December 2020, and the data is sampled at the monthly frequency.

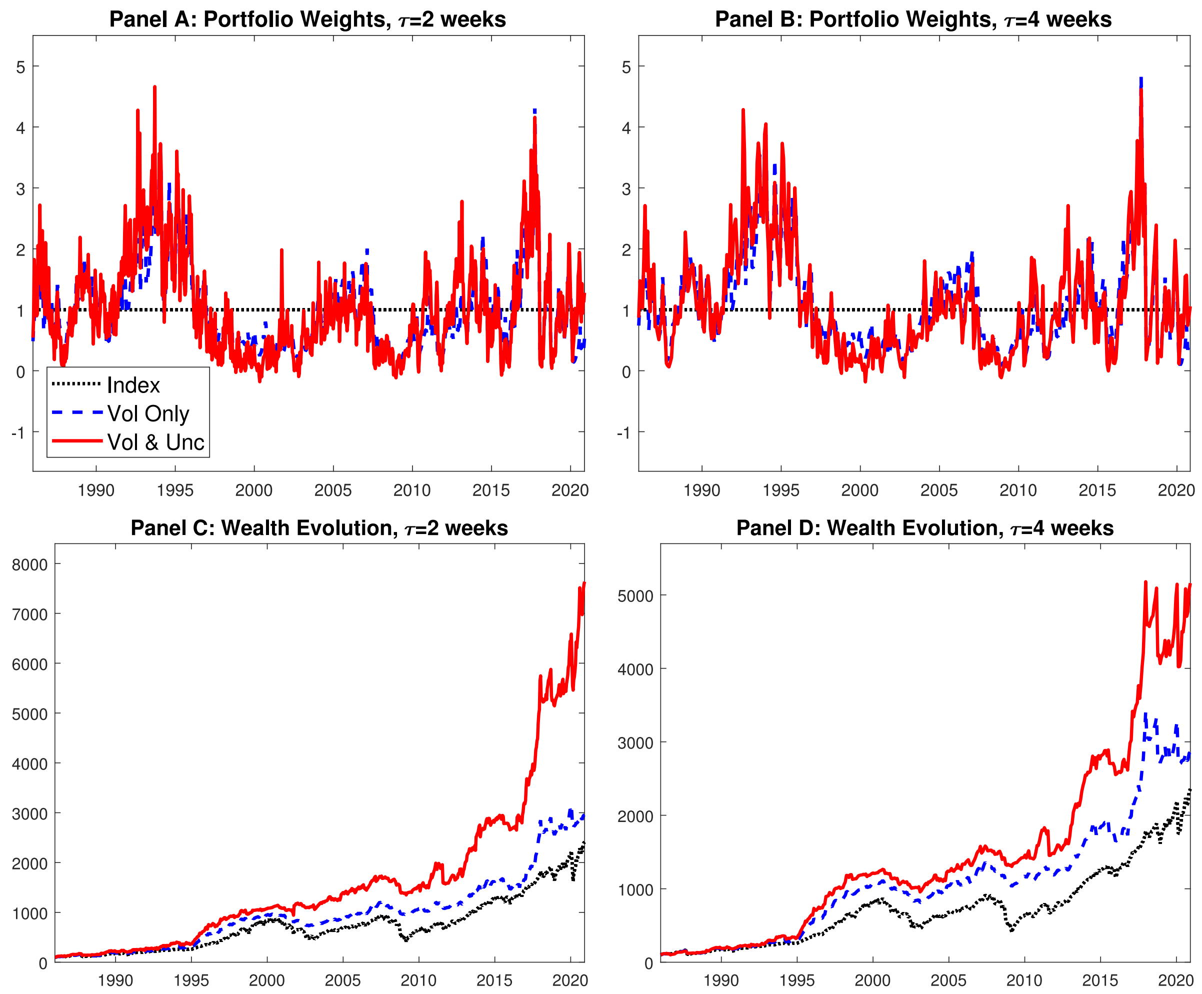

Fig. 6. Portfolios and wealth evolution: full sample.
Notes: Panels A and B show for two- and four-weeks horizons, respectively, the equity portfolio weights according to three strategies: "Index" consists on investing only in the stock ($\omega_t = 1$), "Vol Only" forecasts the equity premium conditioning on volatility only, and "Vol & Unc" forecasts the equity premium conditioning on both volatility and uncertainty. Panels C and D show for two- and four-weeks horizons, respectively, the cumulative wealth evolution associated with each portfolio strategy in Panels A and B. The initial wealth in panels C and D is fixed to 100. The investment and model estimation periods are both from January 1986 until December 2020. The shaded areas correspond to three sub-periods associated with high disconnect: the financial crisis (from July 2007 until March 2009), the US 2016 election (from July 2016 until January 2018), and the Covid-19 pandemic (from January 2020 until December 2020).

7.1. Full sample

We start the portfolio analysis by comparing the above mentioned three different asset allocation strategies (unconditional, conditional on volatility only, conditional on volatility and uncertainty) over the full sample January 1986–December 2020, at horizons of two and four weeks.[34] Panels A and B in Fig. 6 plot the time series of optimal portfolio weights, while panels C and D report the resulting cumulative wealth processes.

The dotted black line represents the unconditional passive buy-and-hold indexing strategy, the dashed blue line represents the active portfolio strategy conditioning on volatility only, and the solid red line represents the active portfolio strategy conditioning on both volatility and uncertainty. Panels A and B show that both active strategies use leverage extensively, as the portfolio holdings ω_t are very often above one. The two active portfolio strategies are correlated, consistent with the idea that on average volatility and uncertainty are connected and hence co-move. They diverge mainly in periods of high disconnect. For example, in the period around the financial crisis characterized by high disconnect driven by a very high level of uncertainty and an extremely high level of volatility, the active strategy conditioning on both volatility and uncertainty was more conservative than the one conditioning on volatility only, thereby avoiding the significant stock market losses in this period. By contrast, in the period around the US

[34] Different horizons of up to 12 weeks deliver similar results, which we omit for brevity.

2016 election which was characterized by levels of uncertainty close to average and simultaneously very low volatility (hence high disconnect), the active strategy conditioning on both volatility and uncertainty was more aggressive than the one conditioning on volatility only, taking advantage of the significant stock market gains during this period. Panels C and D show that the portfolio strategy based on the model (conditioning on both volatility and uncertainty) achieves the highest associated wealth compared to both a strategy conditioning on volatility only and an unconditional passive indexing strategy.

We then evaluate the risk and performance of the wealth excess return associated with each portfolio strategy. We consider the following performance statistics: (i) average excess return, (ii) standard deviation of the excess return, (iii) maximum draw-down ($\mathcal{MDD}\%$),[35] (iv) 5% Value-at-Risk (VaR),[36] (v) Sharpe ratio, and (vi) Treynor ratio. Fig. 7 summarizes key return statistics for the portfolios.

Panel A of Fig. 7 shows that the active portfolio strategy conditioning on both volatility and uncertainty achieves higher excess return on average than the portfolio strategy conditioning on volatility only and the unconditional passive investment strategy for all horizons. Panel B shows that this is not due to excessive risk taking as the standard deviation of the three strategies considered are close to one another, and none of them exhibits the lowest standard deviation across all horizons. Panels C and D show that the portfolio strategy conditioning on both volatility and uncertainty preserves the investor's wealth very effectively, as it achieves the lowest maximum drawdown over most horizons and the lowest value-at-risk in all horizons considered. Finally, panels E and F show that the portfolio strategy conditioning on both volatility and uncertainty delivers the highest risk-adjusted return, as it achieves the highest Sharpe and Treynor ratios for all horizons considered.

We now evaluate the alpha and beta components of the returns for each portfolio strategy. Fig. 8 summarizes the results.

Panel A in Fig. 8 shows that the active portfolio strategy conditioning on both volatility and uncertainty achieves the highest (CAPM) alpha for all horizons considered. Panel B shows that the overall systematic risk beta is similar for both active strategies, with none of them achieving a lower beta for all horizons considered. Interestingly, all the estimated beta coefficients are lower than one, which indicates that the large excess returns are not driven by excessive systematic risk taking.[37] Finally, panels C and D show that the active portfolio strategy conditioning on both volatility and uncertainty achieves the highest alpha for all horizons considered, whether we use CAPM (Panel A), Fama–French three-factor (Panel C), or five-factor (Panel D) models as benchmark.

7.2. Recent high-disconnect regimes

Fig. 2 clearly established that there are several prominent disconnect periods in recent financial markets history. When describing Table 1 we further illustrated the relation between average stock excess return and average disconnect. In addition, Table 2 and Fig. 3 suggest that the impact of disconnect on stock excess return can be very different depending on the relative levels of volatility and uncertainty. In this subsection, we analyze portfolio performance from following the same three strategies implemented in Section 7.1, but focusing on three recent sub-periods characterized by high disconnect: the financial crisis, the US 2016 election, and the Covid-19 crisis. Rather than conditioning on regimes as was done for illustrative purposes in Table 2 and Fig. 3, here we condition on the continuum of volatility and uncertainty values, as in Section 7.1.

7.2.1. Financial crisis

We start by analyzing the 2008 financial crisis, characterized by high uncertainty and extremely high volatility. Table 1 shows that over the two-year time period from July 2007 until March 2009 the average stock market excess return was −33.40%. Accordingly, the excess return of wealth associated with all portfolio strategies considered were negative around this period. For that reason, in this subsection we refrain from using the Sharpe and Treynor ratios, as they are not straightforward to interpret for ranking purposes when average excess returns are negative. Instead, we analyze the average and standard deviation of wealth excess returns.

Panels A and B on Fig. 9 show that the active portfolio strategies achieve a higher terminal wealth than the index during this sub-period. At the two-weeks horizon, the wealth evolution associated to conditioning on volatility only is very similar to the one associated to conditioning on both volatility and uncertainty (Panel A). However, when considering a one-month horizon, the portfolio conditioning on volatility and uncertainty generates the highest associated level of wealth (Panel B).

Panel C shows that the active portfolio strategies achieve higher excess return on average than the passive index strategy for all horizons. However, there is no clear dominant active strategy based on this metric, as none of them achieves the highest excess return consistently across all horizons considered. Panel D shows that the higher excess return achieved by the active strategies is not due to excessive risk taking, as the standard deviation of their excess return is lower than the one associated with the passive index strategy. The standard deviation of excess return associated with the active strategies are very similar to one another for all

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[35] Let W_t denote the current level of wealth process and let $\overline{W}_t := \max_{s\in[0,T]} W_s$ be its running maximum. The drawdown at time t measures the drop of wealth from its running maximum, and is defined as $\mathcal{D}_t := \overline{W}_t - W_t$, $\forall t \geq 0$. The maximum draw-down at time t denoted by $\mathcal{MDD}_t$ records the worst performance that could have happened in the past. It is defined as the running maximum of the drawdown process, given by $\mathcal{MDD}_t := \max_{\tau\in(0,t)} \mathcal{D}_\tau$. Finally, to express the maximum drawdown as the highest proportional wealth loss in the past, we compute $\mathcal{MDD}^{\%} = \mathcal{MDD}_t/\overline{W}_t \in (0,1)$.

[36] The 5% Value-at-Risk is defined as the maximum possible loss (expressed in percentage terms) during the investment period, after excluding all worse outcomes whose combined probability is at most 5%.

[37] The adjusted R^2 obtained from a regression of the excess return for each active portfolio strategy onto the stock market return are 0.6 (conditioning on volatility only) and 0.5 (conditioning on volatility and uncertainty), respectively. This implies that about half of the excess return variance of the active portfolio strategies is not driven by exposure to the aggregate stock market.

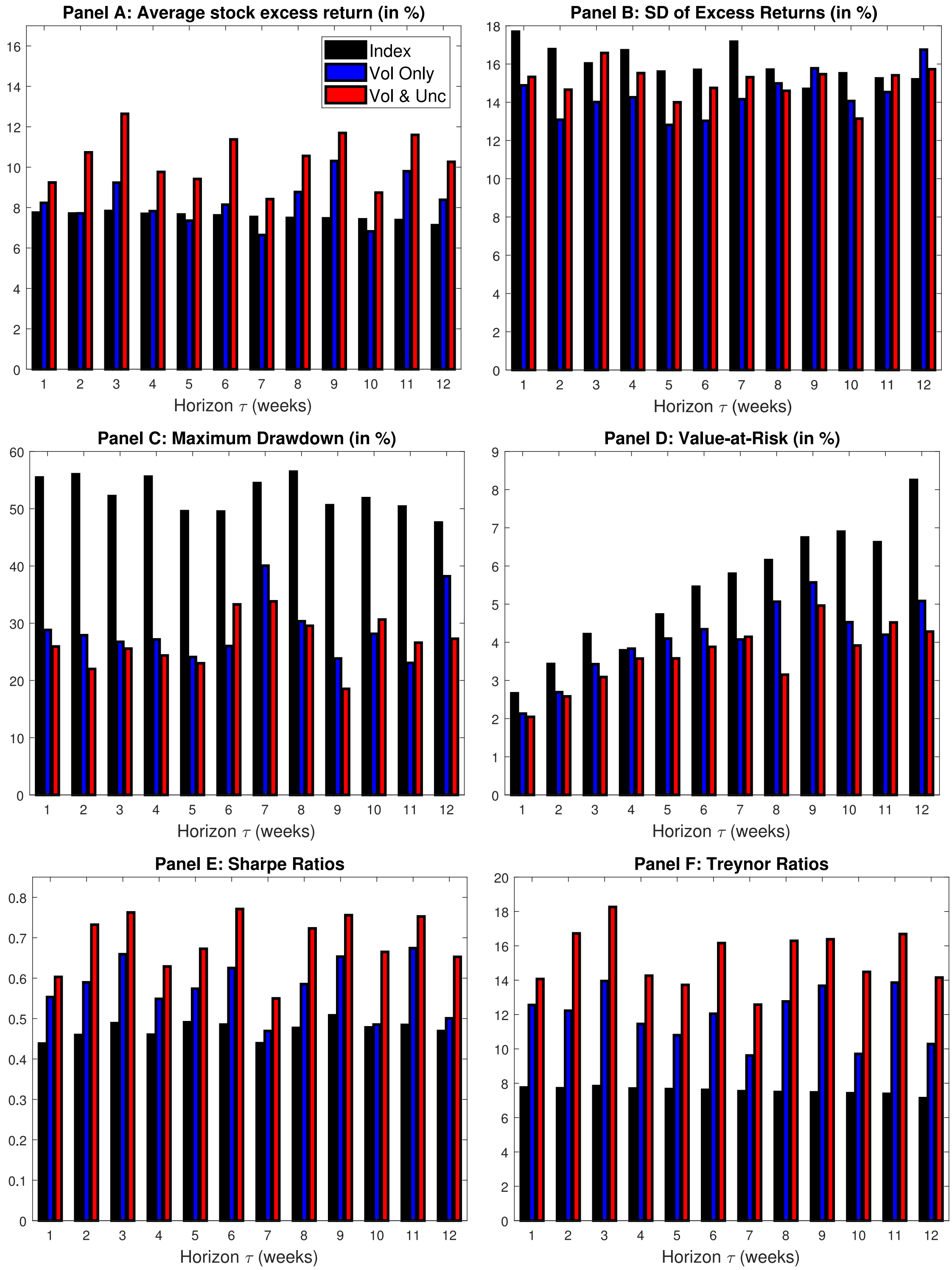

Fig. 7. Portfolio performance measures: full sample.

Notes: This figure reports measures of risk-adjusted wealth excess returns for: index investing, portfolio conditioning on volatility only, and portfolio conditioning on volatility and uncertainty. The statistics shown are: average and standard deviation of excess return (in %), maximum drawdown (in %), 5% Value-at-Risk (in %), Sharpe ratio, and Treynor ratio. The investment and model estimation periods are both from January 1986 until December 2020.

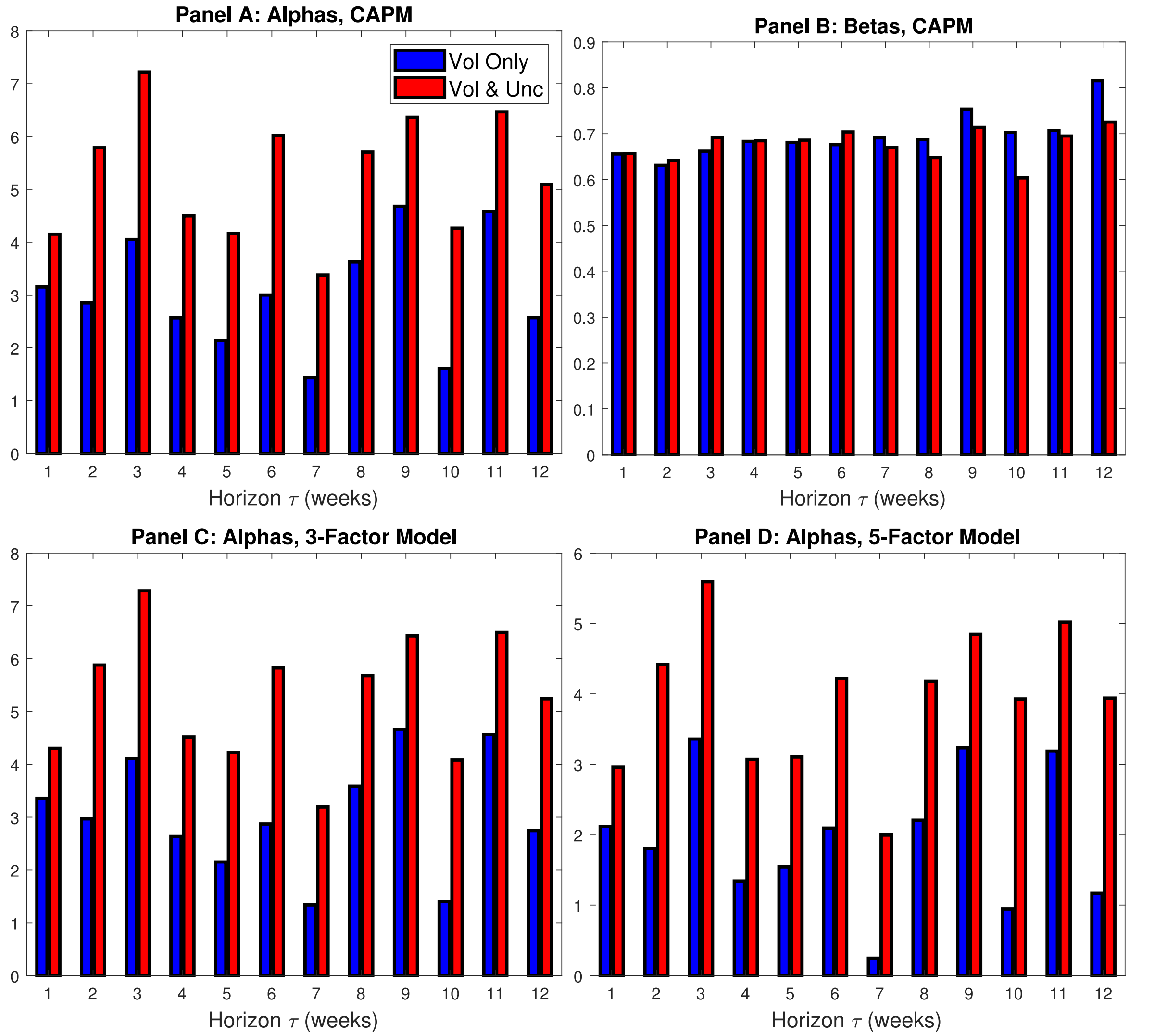

Fig. 8. Alphas and betas: full sample.

Notes: Panels A and B show the estimated alpha and beta from a CAPM model, respectively. These are obtained by regressing the excess return of wealth associated with each active portfolio strategy onto the excess return of the equity market. Panels C and D show estimated alphas from alternative models. The three-factor alpha is obtained from performing the regressions onto the excess return of the market, the SMB ("Small Minus Big") factor, and the HML ("High Minus Low") factor. The five factor alpha is obtained from regressing onto the excess return of the market, the SMB factor, the HML factor, the RMW ("Robust Minus Weak") factor, and the CMA ("Conservative Minus Aggressive") factor. The investment and model estimation periods are both from January 1986 until December 2020.

horizons. Panels E and F show that the active portfolio strategies do equally well in wealth preservation, as they obtain similar levels of maximum drawdown and value-at-risk for most horizons. By contrast, the unconditional passive index strategy is less effective at wealth preservation, as its levels of maximum drawdown and value-at-risk are about double their corresponding levels for its active portfolio strategy counterparts at all horizons.

7.2.2. US 2016 election

We now consider the period surrounding the US 2016 election, from July 2016 until January 2018. During this period, uncertainty remained relatively close to its average value while realized volatility reached record lows. In Fig. 10 we plot the wealth evolution along with risk-adjusted returns for each portfolio strategy around the US 2016election.

Panels A and B show that the conditional active portfolio strategies achieve a higher terminal wealth than the unconditional indexing strategy during this sub-period. At the two-weeks horizon the portfolio conditioning on volatility and uncertainty generates the highest associated level of wealth (Panel A). When considering a one-month horizon, the wealth evolution associated to conditioning on volatility only is similar to the one associated to conditioning on both volatility and uncertainty. Panels C and D

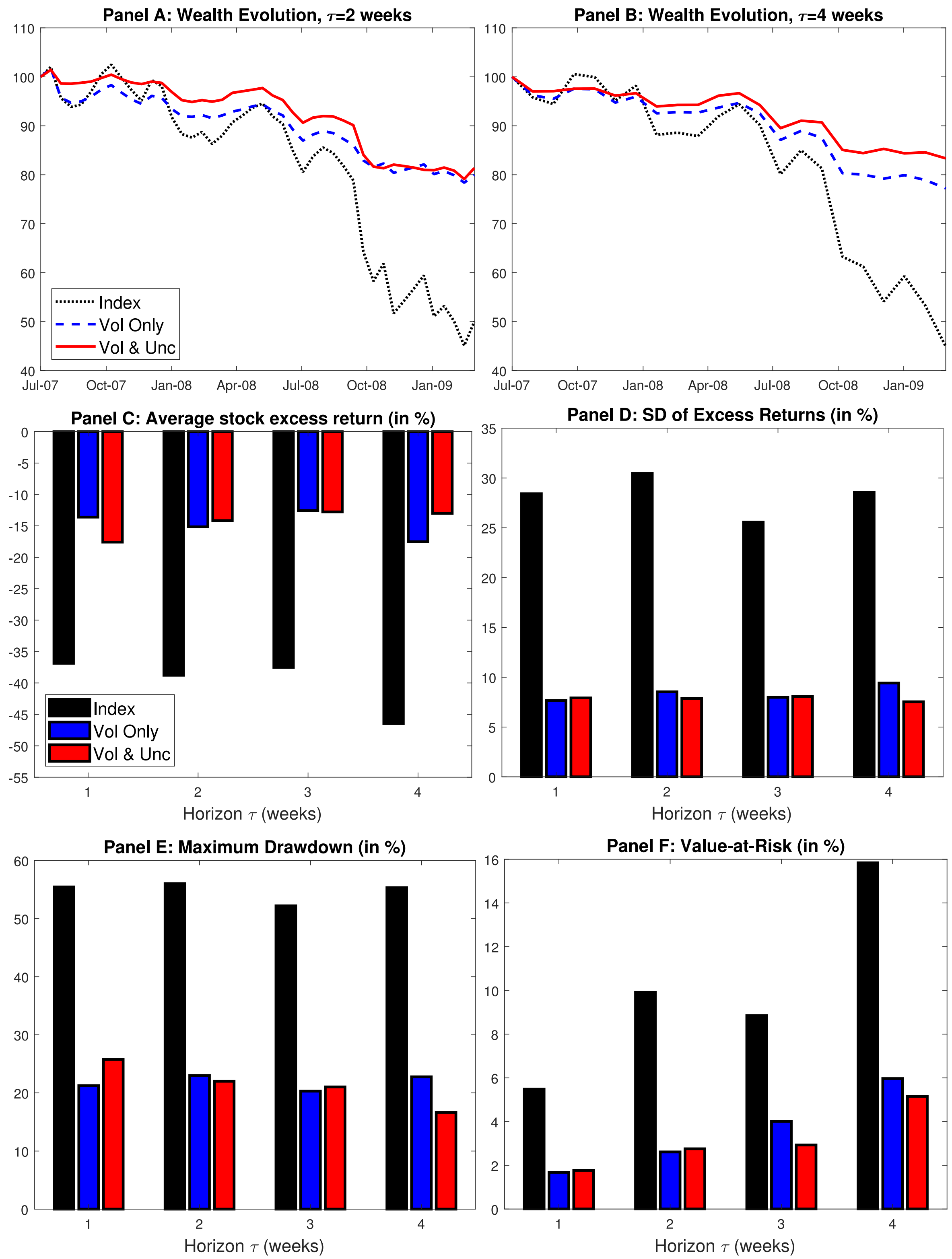

Fig. 9. Wealth evolution and portfolio performance measures around the 2008 Financial Crisis.
Notes: The elements in this figure match closely those from Figs. 6 and 7, except that in this figure the investment period is restricted to the financial crisis period (from July 2007 until March 2009). The estimation period is from January 1986 until December 2020. The initial wealth in panels A and B is fixed to 100.

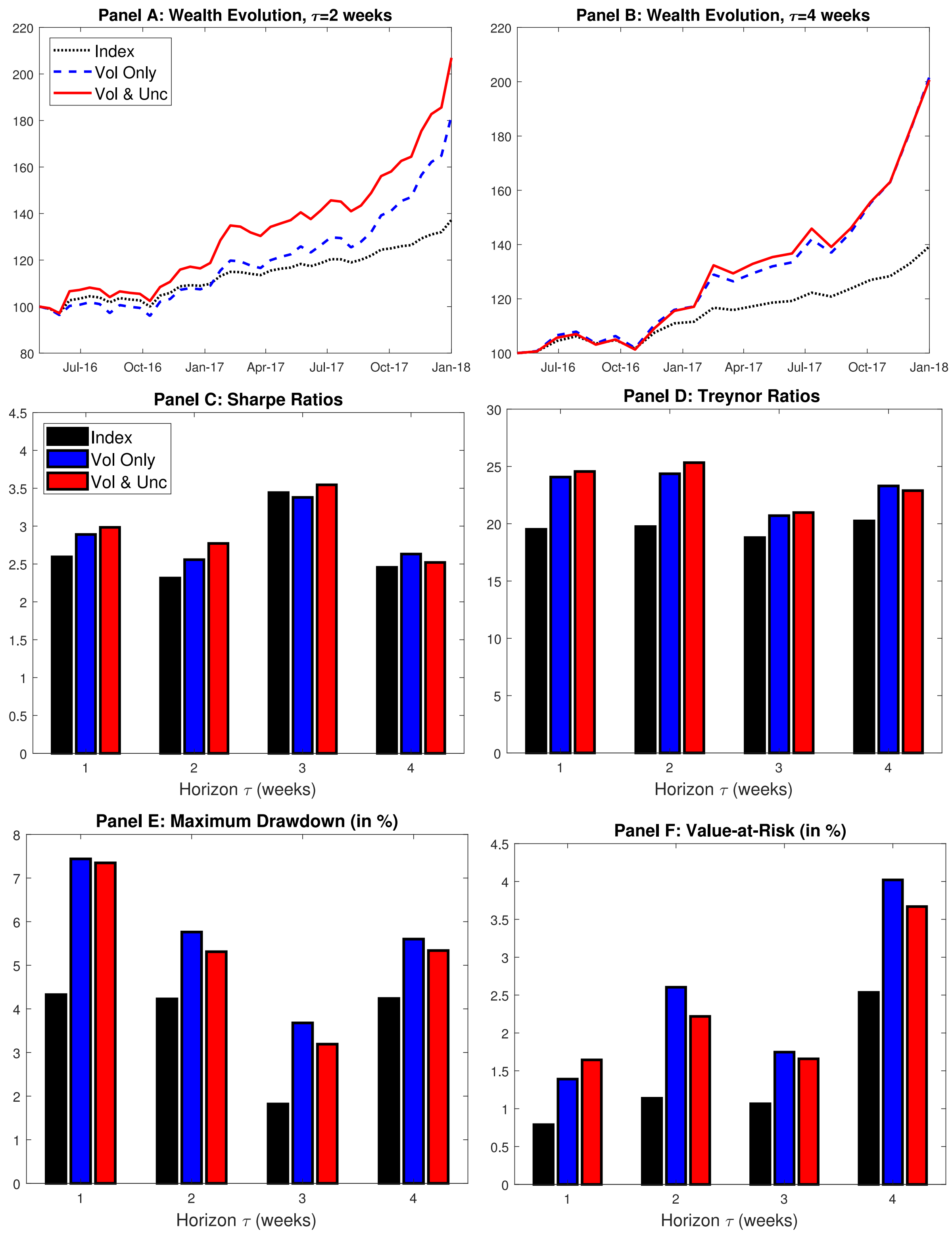

Fig. 10. Wealth evolution and portfolio performance measures around the 2016 US Election.
Notes: The elements in this figure match closely those from Figs. 6 and 7, except that in this figure the investment period is restricted to the 2016 US election period (from July 2016 until January 2018). The estimation period is from January 1986 until December 2020. The initial wealth in panels A and B is fixed to 100.

show that the portfolio conditioning on both volatility and uncertainty achieves the highest risk-adjusted returns in most situations (with the only exception of the one-month horizon, in which conditioning only on volatility slightly dominates), as measured by the Sharpe and Treynor ratios. In addition, it is worth noting that while both active strategies outperform the index, this is partly due to the use of high leverage (the portfolio weights for the active strategies tend to be high, sometimes above 2). Finally, panels E and F show that the active portfolio strategies would not preserve the investor's wealth effectively over this period, as they result in higher maximum drawdowns and value-at-risk than the passive index across all horizons. Comparing the two active strategies only, conditioning on both volatility and uncertainty does a better job at wealth preservation,achieving lower maximum drawdown and value-at-risk across all horizons.

7.2.3. Covid-19

Finally, we consider the recent Covid-19 pandemic crisis, characterized by extremely high volatility (the average volatility in this period is about double the average volatility in the full sample) and comparatively even more extremely high uncertainty (the average uncertainty in this period was close to 3 times the average uncertainty in the full sample).

The Covid-19 pandemic triggered strong market reactions. At the peak of the crisis in mid March 2020, stock market volatility exceeded the highest levels previously seen during the financial crisis and caused equity markets to rapidly lose value, and then subsequently recover. This period is particularly interesting to study within our framework, because of the different timing in the dynamics of volatility and uncertainty. Uncertainty increased substantially in late February and remained at very elevated levels throughout most of the sample period. By contrast, volatility initially spiked sharply and then quickly reverted to levels closer to its full-sample mean. In terms of our measure of disconnect, this implies η_t fell substantially in late February and early March but then increased and stayed high for the remainder of the sample period.[38] In short, around this period the market experienced a regime transition from a more connected regime characterized by extremely high uncertainty and extremely high volatility to a more disconnected regime characterized by extremely high uncertainty and moderately high (but no longer extreme) volatility. In Fig. 10 we collect the wealth evolution for all portfolio strategies together with a set of portfolio performance measures. The period considered in this analysis is from January 2020 until December 2020.

The top two panels of Fig. 11 show that the portfolio conditioning on both volatility and uncertainty performs well throughout the entire Covid-19 period. Panel A shows that, at the two-week horizon, besides protecting the investor's wealth at the onset of the crisis in March and April 2020, the strategy also took advantage of the subsequent recovery of equity markets by rapidly increasing its equity portfolio share. Accordingly, at this horizon the wealth level achieved by the portfolio conditioning on both volatility and uncertainty is the highest. Panel B shows that at the monthly horizon the wealth evolution of the passive index and the portfolio conditioning on both volatility and uncertainty are very similar. By contrast, for the same horizon, the portfolio conditioning on volatility only achieved the lowest wealth level, incurring cumulative losses by the end of 2020. Notably, the portfolio conditioning on both volatility and uncertainty was almost fully invested in the equity market, compared to only half of wealth invested in equities for the portfolio conditioning on volatility only. The high uncertainty around this period led to higher subsequent stock excess return, and only the portfolio conditioning on both volatility and uncertainty took advantage of this compensation for uncertainty, which the portfolio conditioned on volatility alone could not exploit.

Panels C and D show that the portfolio strategy conditioning on both volatility and uncertainty delivers the highest risk-adjusted return, as it achieves the highest Sharpe and Treynor ratios for most horizons considered (all except for the monthly horizon, in which conditioning on volatility and uncertainty delivers a risk-adjusted return similar to the one obtained by the passive index strategy). By contrast, the portfolio strategy conditioning on volatility only obtains the worst performance in most horizons, including negative Sharpe and Treynor ratios in some of them. Finally, panels E and F show that the active portfolio strategies preserve wealth more effectively than the passive index strategy, as they obtain lower maximum drawdowns and value-at-risk for most horizons (again all except for the monthly horizon, in which the three strategies achieve very similar wealth preservation). Meanwhile, both active portfolio strategies preserve wealth equally well; neither results in a lower maximum drawdown or value-at-risk for all horizons.

8. Out-of-sample analysis

In Section 7, all the portfolio comparisons were back-tested in-sample, with the advantage of hindsight. In this section, we examine the same three prediction models for the equity premium, but out-of-sample. We show that adding uncertainty as a predictor of future stock excess return improves forecast accuracy out-of-sample, compared to forecasts conditioning on volatility only, or based on the unconditional average stock excess return. Because in the context of a regression-based portfolio analysis higher stock excess return forecast accuracy drives higher portfolio performance, the results we provide here support the notion that the higher portfolio performance from conditioning on both volatility and uncertainty shown in Section 7 holds also out of sample.

We start by partitioning the full sample in half. The estimation sample consists on the first half of observations (from January 1986 until June 2003), leaving the rest of the sample for implementing the forecasts out-of-sample.[39] We run similar regressions to the one in Eq. (28)), with the stock excess return regressed on: (i) only an intercept (unconditional model), (ii) an intercept and

[38] In the context of our model, such disconnected periods arise exogenously, as would be the case with the occurrence of the Covid-19 pandemic. It is of course plausible that monetary and fiscal policy also played a role in the rapid adjustment of volatility, leading to the high volatility of disconnect around this period.

[39] We replicated the analysis using for the estimation sample other fractions of the full sample, including 40%, 60%, 70%, and 80%. In all cases, the results (omitted for brevity) were qualitatively the same.

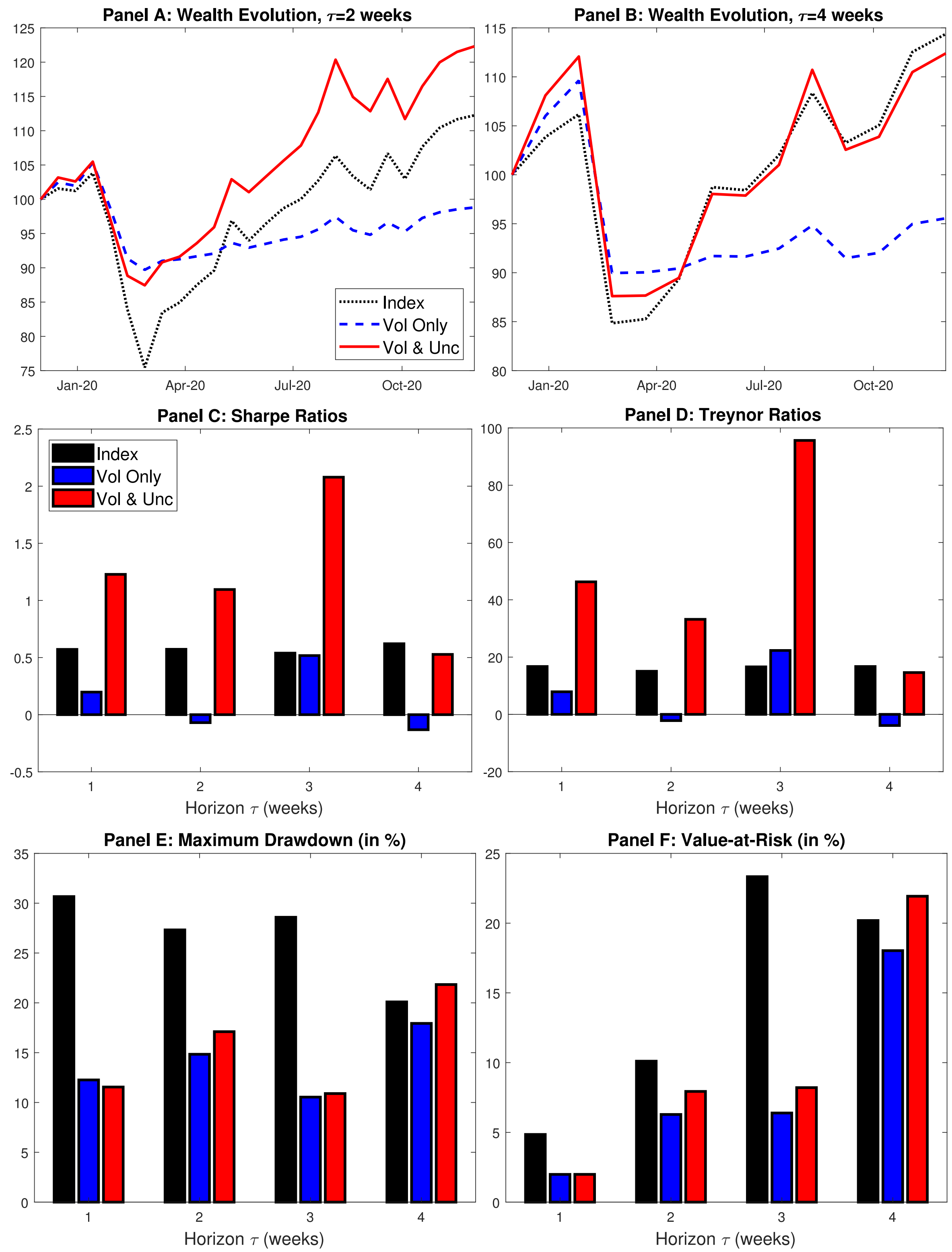

Fig. 11. Wealth evolution and portfolio performance measures around the COVID-19 pandemic.
Notes: The elements in this figure match closely those from Figs. 6 and 7, except that in this figure the investment period is restricted to the COVID-19 pandemic period (from January 2020 until December 2020). The estimation period is from January 1986 until December 2020. The initial wealth in panels A and B is fixed to 100.

Table 3
Root mean squared error (×100) for different out-of-sample stock excess return forecasts.

τ	Unconditional	Volatility	Volatility and Uncertainty
1	1.558	1.552	1.557
2	1.989	1.986	1.978
3	2.277	2.274	2.258
4	2.527	2.522	2.503
5	2.754	2.750	2.725
6	2.950	2.946	2.915
7	3.130	3.126	3.088
8	3.288	3.289	3.241
9	3.438	3.444	3.389
10	3.590	3.595	3.538
11	3.743	3.762	3.703
12	3.888	3.932	3.870

Notes: This table reports the root mean squared error (RMSE), multiplied by 100 for ease of illustration purposes, from out-of-sample forecasts of stock excess return using horizons 1 through 12 weeks, based on three different models: (i) $\widehat{r}_{e,t+\tau} = \widehat{\beta}_0$ (unconditional mean equal to 7.70%), (ii) $\widehat{r}_{e,t+\tau} = \widehat{\beta}_0 + \widehat{\beta}_v v_t^2$ (conditioning on volatility), and (iii) $\widehat{r}_{e,t+\tau} = \widehat{\beta}_0 + \widehat{\beta}_v v_t^2 + \widehat{\beta}_\eta \eta_t v_t^2$ (conditioning on volatility and uncertainty). A lower RMSE indicates a better forecast performance. The estimation period starts by using the first half of the full sample: from January 1986 until June 2003. We then update our estimation one week at a time by using a rolling expanding window. The out-of-sample period corresponds to the remaining observations in our sample, starting with the period from July 2003 until December 2020.

volatility, and (iii) an intercept, volatility, and uncertainty. We run the regressions and make forecasts over horizons ranging from 1 to 12 weeks. Starting from using the first half of observations in the full sample, we update our estimation one week at a time by using a rolling expanding window.

Using the estimated regression coefficients, we go on to predict out of sample stock excess returns over horizons $\tau = 1, \ldots, 12$ weeks in the remaining observations available (starting from the July 2003–December 2020 period). To evaluate the forecast performance we compare the ex-ante forecast of stock excess return according to each of the three regression based models considered above with the corresponding ex-post realized values, by means of the root mean squared error (RMSE). Table 3 shows the results.

Each column in Table 3 reports the RMSE of a forecast of stock excess return for each regression model considered. Remarkably, the results show that the richer model (conditioning on volatility and uncertainty) is more accurate (lower RMSE) than conditioning only on volatility, or not conditioning at all, for all horizons considered except the one-week horizon. This establishes that the in-sample higher performance of the conditional model based on volatility and uncertainty was not due to over-fitting in-sample.

Conditioning only on volatility does not achieve similar success. For horizons from 1 to 7 weeks, forecasting the stock excess return conditioning on volatility is more accurate than not conditioning at all. However, for longer horizons from 8 to 12 weeks, not conditioning provides a better forecast than conditioning on volatility. By contrast, for all horizons considered, conditioning on volatility and uncertainty leads to more accurate forecasts of the stock excess return than not conditioning.

Overall, our results clearly show that a forecast of stock excess return conditioning on both volatility and uncertainty is superior to both a forecast conditioning on volatility only and a forecast based on the unconditional mean. Since in our regression-based portfolio analysis any increase in portfolio performance must be driven by higher accuracy in the stock excess return forecast, our results suggest that the high portfolio performance achieved by conditioning on volatility and uncertainty shown in Section 7 holds also out of sample.

9. Conclusions

In this paper, we develop a novel and tractable framework in which an investor is averse to both volatility and model uncertainty, which are both stochastic and possibly disconnected. We show that in partial equilibrium both volatility and uncertainty affect the optimal consumption and portfolio policies of the investor, and in general equilibrium both drive the equity premium and risk-free rate. The presence of uncertainty as an additional factor in these equilibrium quantities is not exogenously assumed; it is derived in the context of a model where the representative agent is averse to both volatility and uncertainty. We find that allowing uncertainty and volatility to be disconnected results in a better in-sample description and better out-of-sample forecasts.

Empirically, we find that equity returns respond negatively to contemporaneous volatility but respond positively, and consistently across horizons, to the uncertainty component in the model. Our results therefore show that the equity premium appears to be earned for facing uncertainty, especially high uncertainty that is disconnected from lower volatility, rather than being earned for facing volatility as traditionally assumed. In addition, our disconnect measure appears be an informative signal about future realized excess returns. As our empirical analysis has shown, a low (high) level of disconnect is indicative of the likelihood of negative (positive) events materializing in the near future. For example, the model predicts that periods of high uncertainty are followed by subsequently higher excess returns, which are even higher in disconnected regimes characterized by uncertainty being significantly higher than volatility. These results have important implications for optimal portfolio allocation. We show that an investor forecasting the stock excess return conditioning on both volatility and uncertainty as implied by our model achieves superior portfolio performance, over and above the one achieved by conditioning on volatility only, or by not conditioning at all. Therefore, our results suggests that for the successful implementation of a dynamic portfolio policy it is not merely the level of volatility nor uncertainty that matters, but its their relative level to one another, which is precisely what our disconnect measure aims to capture.

Admittedly, our single representative agent setup may be extended in various promising directions. For instance, a model with heterogeneous agents, either optimistic and pessimistic, might be developed to understand how disconnect emerges endogenously. The introduction of rare and potentially disastrous events could be incorporated by adding jumps to the dividend process and add both jump intensity uncertainty (the likelihood of a rare event is not known) and jump size uncertainty (when a rare event materializes, how large its impact is going to be is unknown).

Appendix A

A.1. Derivation of the relative entropy growth

Given the definitions of relative entropy growth and the instantaneous growth rate of relative entropy in Eqs. (8) and (9), we can write:

$$\mathcal{R}(\vartheta_t) := \left.\frac{d}{ds}\right|_{s=0} \mathbb{E}_t^{\vartheta}\left[\log \vartheta_{t+s}\right]. \tag{A.1}$$

We then compute $\mathcal{R}(\vartheta_t)$ as follows:

$$\begin{aligned}\mathcal{R}(\vartheta_t) &= \left.\frac{d}{ds}\right|_{s=0} \mathbb{E}_t^{\vartheta}\left[\int_t^{t+s} h_u dW_u^S - \int_t^{t+s}\frac{h_u^2}{2}du\right]\\ &= \left.\frac{d}{ds}\right|_{s=0} \mathbb{E}_t^{\vartheta}\left[\int_t^{t+s} h_u dW_u^{S,\vartheta} + \int_0^{t}\frac{h_u^2}{2}du\right]\\ &= \left.\frac{d}{ds}\right|_{s=0} \int_t^{t+s}\frac{h_u^2}{2}du = \frac{h_t^2}{2},\end{aligned} \tag{A.2}$$

where the first equality follows from the fact that the solution to Eq. (7) for the change of measure ϑ_t is given by $\vartheta_t = \exp\left\{\int_0^t h_u dW_u^S - \int_0^t \frac{h_u^2}{2}du\right\}$, the second one uses the relation between the Brownian motions under the reference and robust measures, i.e. $dW_t^S = dW_t^{S,\vartheta} + h_t dt$, and the last one is an application of the Leibnitz rule of differentiation of integrals.[40]

A.2. Proof of Proposition 1

We look for a solution of the form $V(t, X_t, v_t, \eta_t) = e^{-\beta t}L(X_t, v_t, \eta_t)$ to the problem in Eq. (17) subject to the entropy growth constraint in Eq. (18). We formulate the Lagrangian for the constrained optimization problem as follows:

$$\begin{aligned}\sup_{\{C_t,\omega_t\}}\inf_{\{h_t\}}\Bigg\{&\log(C_t) - \beta L + \frac{\partial L}{\partial X}\left(X_t\left[r_{f,t} + \omega_t\left(\mu_{S,t}^h - r_{f,t}\right)\right] - C_t\right)\\ &+ \frac{1}{2}\frac{\partial^2 L}{\partial X^2}\omega_t^2 X_t^2\sigma_S^2 v_t^2 + \frac{\partial L}{\partial v}\mu_{v,t}^h + \frac{1}{2}\frac{\partial^2 L}{\partial v^2}\sigma_{v,t}^2 + \frac{\partial L}{\partial \eta}\mu_{\eta,t}^h + \frac{1}{2}\frac{\partial^2 L}{\partial \eta^2}\sigma_{\eta,t}^2\\ &+ \frac{\partial^2 L}{\partial X\partial v}\omega_t X_t\sigma_S v_t\rho_{S,v}\sigma_{v,t} + \frac{\partial^2 L}{\partial X\partial \eta}\omega_t X_t\sigma_S v_t\rho_{S,\eta}\sigma_{\eta,t} + \frac{\partial^2 L}{\partial\eta\partial v}\rho_{S,\eta}\rho_{S,v}\sigma_{v,t}\sigma_{\eta,t}\\ &+\theta\left(\frac{h_t^2}{2} - \frac{\epsilon^2}{2}\mathcal{U}_t^2\right)\Bigg\},\end{aligned} \tag{A.3}$$

subject to $C_t \geq 0$ and $\theta \geq 0$, where θ is the multiplier for the relative entropy constraint in Eq. (18). Solving for the Lagrangian in Eq. (A.3) leads to the following optimal policies

$$C_t^* = \frac{1}{\frac{\partial L}{\partial X}}, \tag{A.4}$$

$$h_t^* = -\epsilon\mathcal{U}_t = -\epsilon\eta_t v_t, \tag{A.5}$$

$$\omega_t^* = \left(-\frac{\frac{\partial L}{\partial X}}{\frac{\partial^2 L}{\partial X^2}X_t}\right)\frac{\mu_{S,t}^h - r_{f,t}}{\sigma_S^2 v_t^2} + \left(-\frac{\frac{\partial^2 L}{\partial X\partial v_t}}{\frac{\partial^2 L}{\partial X^2}X_t}\right)\frac{\rho_{S,v}\sigma_{v,t}}{\sigma_S v_t} + \left(-\frac{\frac{\partial^2 L}{\partial X\partial \eta_t}}{\frac{\partial^2 L}{\partial X^2}X_t}\right)\frac{\rho_{S,\eta}\sigma_{\eta,t}}{\sigma_S v_t}. \tag{A.6}$$

First, the optimal consumption follows from the standard envelope condition. Second, the optimal perturbation policy h_t^*, follows from the minimization of an affine function of h_t subject to a quadratic inequality constraint. The optimum is the negative root

[40] The Leibniz rule states that for a continuous function f, $\frac{d}{du}\left(\int_{a(u)}^{b(u)} f(x,u), dx\right) = \int_{a(u)}^{b(u)}\frac{\partial f}{\partial u}, dx, + f\left(b(u), u\right)\cdot b'(u) - f\left(a(u), u\right)\cdot a'(u)$, where the functions $a(u)$ and $b(u)$ are both have continuous derivatives in u.

$-\epsilon \mathcal{V}_t$, which is the only candidate solution consistent with the minimization. Intuitively, the minimization means that the investor evaluates his or her policies under the worst-case alternative in the set of models under consideration. Third, the optimal portfolio policy in Eq. (A.6) decomposes into a myopic term adjusted by the drift perturbation h_t (first term), and two inter-temporal hedging terms which arise due to the correlation between the stock price process and both, stochastic volatility and disconnect.

Finally, given logarithmic preferences, we conjecture that the value function is of the form[41]

$$L(X, \eta, v) = \frac{\log(X)}{\beta} + \phi(\eta, v). \tag{A.7}$$

Plugging this conjecture of the value function into the optimal policies in Eqs. (A.4), (A.5), and (A.6), respectively, leads to the expressions in the proposition.

A.3. Proof of Proposition 2

The equity premium in Eq. (25) follows from combining the optimal portfolio policy in Eq. (21) with the market clearing condition for the stock in Eq. (23). The price–dividend ratio given in the proposition follows from combining the optimal consumption policy in Eq. (19) with the market clearing condition for consumption in Eq. (22). In addition, one needs to take into account that because $C_t = D_t$ for all t, the price of a claim on the future stream of consumption must be equal to the price of a claim on the future stream of dividends, which implies that $X_t = S_t$. Because the price–dividend ratio is constant, the dynamics of the stock price and dividends are related by $\frac{dS_t}{S_t} = \frac{dD_t}{D_t}$, which implies both have the same drift and diffusion (and $W_t^D = W_t^S,\ t \geq 0$). Identifying terms using Eqs. (1) and (3), and using the equilibrium price–dividend ratio, we get

$$\mu_{S,t} = \mu_S = \beta + \mu_D, \tag{A.8}$$

$$\sigma_S = \sigma_D. \tag{A.9}$$

The risk-free rate in Eq. (26) follows from combining the equity premium in Eq. (25) with the stock return drift and volatility in Eqs. (A.8) and (A.9).

A.4. Equilibrium with CRRA preferences

In this appendix we show how to obtain the solution using the martingale approach for an investor with CRRA preferences. Assuming a complete financial market, the problem of the robust investor is to maximize the expected utility from lifetime consumption:

$$\sup_C \inf_h \int_0^\infty \mathbb{E}_t^\vartheta \left[e^{-\beta t} \frac{C_t^{1-\gamma}}{1-\gamma} \right] dt, \tag{A.10}$$

subject to the budget constraint

$$\int_0^\infty \mathbb{E}_t^\vartheta \left[\xi_t^\vartheta C_t dt \right] \leq \int_0^\infty \mathbb{E}_t^\vartheta \left[\xi_t^\vartheta D_t dt \right], \tag{A.11}$$

and the relative entropy constraint:

$$\frac{h_t^2}{2} \leq \frac{\epsilon^2}{2} \mathcal{V}_t^2, \tag{A.12}$$

where ξ_t^ϑ is the change of measure from the robust probability measure to the risk neutralized measure. The problem of the robust investor can be restated under the reference probability measure as follows

$$\sup_C \inf_h \int_0^\infty \mathbb{E}_t \left[\vartheta_t e^{-\beta t} \frac{C_t^{1-\gamma}}{1-\gamma} \right] dt, \tag{A.13}$$

subject to the budget constraint

$$\mathbb{E}_t \left[\xi_t C_t dt \right] \leq \int_0^\infty \mathbb{E}_t \left[\xi_t D_t dt \right], \tag{A.14}$$

and the relative entropy constraint

$$\frac{h_t^2}{2} \leq \frac{\epsilon^2}{2} \mathcal{V}_t^2, \tag{A.15}$$

where ξ_t is the change of measure from the reference probability measure to the risk neutralized measure. The optimal consumption and optimal perturbation drift for the robust investor are, respectively

$$C_t^* = \left(\frac{\vartheta_t e^{-\beta t}}{\lambda \xi_t} \right)^{\frac{1}{\gamma}}, \quad h_t^* = -\epsilon \mathcal{V}_t, \tag{A.16}$$

[41] It can be verified that this conjecture provides a solution to the problem, with $\phi(\eta, v)$ satisfying an ordinary differential equation, which may be solved in closed form depending on the assumed functional form of $\mu_{S,t}$ and $r_{f,t}$.

where λ is the Lagrange multiplier of the budget constraint. The aggregate resource constraint implies that in equilibrium

$$C_t^* = D_t. \tag{A.17}$$

Substituting the optimal robust consumption policy in Eq. (A.16), we obtain:

$$\xi_t = \frac{e^{-\beta t}}{D_t^{\gamma}} \frac{\vartheta_t}{\lambda}, \tag{A.18}$$

which follows dynamics given by

$$\frac{d\xi_t}{\xi_t} = -r_{f,t}dt - \kappa_t^D dW_t^D - \kappa_t^{\upsilon} dW_t^{\upsilon} - \kappa_t^{\eta} dW_t^{\eta}, \tag{A.19}$$

where $\begin{bmatrix} \kappa_t^D & \kappa_t^{\upsilon} & \kappa_t^{\eta} \end{bmatrix}^{\intercal}$ is the vector of market prices of risk perceived by the reference investor. Because the diffusion of the stock market for general preferences is $\begin{bmatrix} \sigma_{S,t}^D & \sigma_{S,t}^{\upsilon} & \sigma_{S,t}^{\eta} \end{bmatrix}$, the equity premium is given by

$$\mu_{S,t} - r_{f,t} = \begin{bmatrix} \sigma_{S,t}^D & \sigma_{S,t}^{\upsilon} & \sigma_{S,t}^{\eta} \end{bmatrix} \times \begin{bmatrix} \kappa_t^D \\ \kappa_t^{\upsilon} \\ \kappa_t^{\eta} \end{bmatrix} \tag{A.20}$$

$$= \sigma_{S,t}^D \kappa_t^D + \sigma_{S,t}^{\upsilon} \kappa_t^{\upsilon} + \sigma_{S,t}^{\eta} \kappa_t^{\eta}. \tag{A.21}$$

Applying Ito's lemma to Eq. (A.18), and identifying terms with Eq. (A.19) we obtain the equilibrium expressions for the interest rate and market prices of risk, as detailed below. In equilibrium, the interest rate is

$$r_{f,t} = \beta + \gamma \mu_D + \gamma \sigma_D \upsilon_t h_t - \frac{1}{2}\gamma (1+\gamma) \sigma_D^2 \upsilon_t^2, \tag{A.22}$$

and the market price of risk perceived by the reference investor are

$$\kappa_t^D = \gamma \sigma_D \upsilon_t + \epsilon \eta_t \upsilon_t, \quad \kappa_t^{\upsilon} = \kappa_t^{\eta} = 0. \tag{A.23}$$

Accordingly, from Eq. (A.20) the equity premium is given by

$$\mu_{S,t} - r_{f,t} = \sigma_{S,t}^D \left(\gamma \sigma_D \upsilon_t + \epsilon \eta_t \upsilon_t\right) \tag{A.24}$$

$$= \gamma \sigma_D \sigma_{S,t}^D \upsilon_t + \epsilon \sigma_{S,t}^D \eta_t \upsilon_t. \tag{A.25}$$

In the special case in which $\gamma = 1$ (logarithmic preferences), all these results coincide with those in Proposition 2. Moreover, in that special case, the stock price is proportional to dividends, so that

$$S_t = \frac{D_t}{\beta}, \quad \sigma_S^D = \sigma_D \upsilon_t. \tag{A.26}$$

In the more general case of CRRA preferences, however, the solution for the stock price and corresponding stock price volatility is slightly more involved, as we outline below. The stock is a claim to an infinite stream of future dividends, so its price S_t is given by

$$S_t = \int_t^{\infty} \mathbb{E}_t \left[\frac{\xi_u}{\xi_t} D_u\right] du = D_t \int_t^{\infty} e^{-\beta(u-t)} H\left(D_t, \upsilon_t, \vartheta_t, \eta_t, t, u\right) du,$$

where

$$H\left(D_t, \upsilon_t, \vartheta_t, \eta_t, t, u\right) = \mathbb{E}_t \left[\left(\frac{D_u}{D_t}\right)^{1-\gamma} \left(\frac{\vartheta_u}{\vartheta_t}\right)\right] \tag{A.27}$$

The law of iterated expectations implies $H\left(D_t, \upsilon_t, \vartheta_t, \eta_t, t, u\right)$ is a martingale and hence its drift must vanish. Accordingly, $H\left(D_t, \upsilon_t, \vartheta_t, \eta_t, t, u\right)$ is the solution to the following partial differential equation:

$$\begin{aligned} 0 = {} & \frac{\partial H}{\partial D_t} D_t \mu_D + \frac{\partial H}{\partial \upsilon_t} \mu_{\upsilon,t} + \frac{\partial H}{\partial \eta_t} \mu_{\eta,t} + \frac{1}{2}\frac{\partial^2 H}{\partial D_t^2} \left(D_t \sigma_D \upsilon_t\right)^2 + \frac{1}{2}\frac{\partial^2 H}{\partial \upsilon_t^2} \sigma_{\upsilon,t}^2 + \frac{1}{2}\frac{\partial^2 H}{\partial \vartheta_t^2} \left(\epsilon \vartheta_t \eta_t \upsilon_t\right)^2 \\ & + \frac{1}{2}\frac{\partial^2 H}{\partial \eta_t^2} \sigma_{\eta,t}^2 + \frac{\partial^2 H}{\partial D_t \partial \upsilon_t} \sigma_D D_t \upsilon_t \sigma_{\upsilon,t} \rho_{S,\upsilon} - \frac{\partial^2 H}{\partial D_t \partial \vartheta_t} \epsilon \sigma_D D_t \vartheta_t \eta_t \upsilon_t^2 \\ & + \frac{\partial^2 H}{\partial D_t \partial \eta_t} \sigma_D D_t \upsilon_t \sigma_{\eta,t} \rho_{S,\eta} + \frac{\partial^2 H}{\partial \upsilon_t \partial \eta_t} \sigma_{\upsilon,t} \sigma_{\eta,t} \rho_{S,\eta} \rho_{S,\upsilon} + \frac{\partial H}{\partial t}, \end{aligned} \tag{A.28}$$

with the initial condition:

$$H\left(D_t, \upsilon_t, \vartheta_t, \eta_t, t, t\right) = 1. \tag{A.29}$$

In the special case in which $\mu_{\upsilon,t}\left(D_t, \upsilon_t, \vartheta_t, \eta_t\right)$, $\mu_{\eta,t}\left(D_t, \upsilon_t, \vartheta_t, \eta_t\right)$, $\sigma_{\upsilon,t}^D\left(D_t, \upsilon_t, \vartheta_t, \eta_t\right)$, and $\sigma_{\eta,t}^D\left(D_t, \upsilon_t, \vartheta_t, \eta_t\right)$ are specified such that the overall system of state variables is in the linear–quadratic jump diffusion framework, the solution for H is available in closed form.

Given the solution for the stock price S_t, its volatility follows from a straightforward application of Ito's lemma. Meanwhile, the optimal portfolio with CRRA preferences in partial equilibrium can be also obtained in closed form subject to a careful choice of the dynamics of the state variables, as shown by Liu (2007). With CRRA preferences, the optimal portfolio in partial equilibrium includes a hedging demand in addition to the myopic demand obtained in Proposition 1, which is typical of logarithmic preferences.

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