American Options under Stochastic Volatility: Control Variates, Maturity Randomization & Multiscale Asymptotics

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Abstract

American options are actively traded worldwide on exchanges, thus making their accurate and efficient pricing an important problem. As most financial markets exhibit randomly varying volatility, in this paper we introduce an approximation of American option price under stochastic volatility models. We achieve this by using the maturity randomization method known as Canadization. The volatility process is characterized by fast and slow scale fluctuating factors. In particular, we study the case of an American put with a single underlying asset and use perturbative expansion techniques to approximate its price as well as the optimal exercise boundary up to the first order. We then use the approximate optimal exercise boundary formula to price American put via Monte Carlo. We also develop efficient control variates for our simulation method using martingales resulting from the approximate price formula. A numerical study is conducted to demonstrate that the proposed method performs better than the least squares regression method popular in the financial industry, in typical settings where values of the scaling parameters are small. Further, it is empirically observed that in the regimes where scaling parameter value is equal to unity, fast and slow scale approximations are equally accurate.

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1 Introduction

In this paper, we develop approximations for the optimal exercise boundary and price of American options under stochastic volatility, where the volatility process is modulated by fluctuations occurring on fast or slow time scales. We particularly consider the example of an American put with a single underlying asset. It is also made clear that the case of an American call option written on a dividend-paying underlying asset can be handled similarly. In order to derive these approximations, we replace the fixed maturity of the option with an exponentially distributed random variable to introduce an American put with random maturity. We then use singular and regular perturbation techniques to approximately solve the pricing problem associated with the

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random maturity American put. We demonstrate, with the help of numerical experiments, that the resulting approximation of the optimal exercise boundary for this new put can be used to estimate the price of original option with fixed maturity via Monte Carlo. Moreover, we use the approximation of the random maturity put price to form efficient martingale control variates for our simulation method.

In the classical theory of risk-neutral pricing, the price of an American option corresponds to the solution of an optimal stopping problem which, in Markovian models, can also be expressed as a free boundary problem. Even in the simple case of constant volatility, the American option price is not available in closed form. Over the years, many numerical and simulation techniques have been developed to approximately solve either of the two formulations for American option pricing problem. The prominent numerical methods in the constant volatility case are the binomial lattice method of Cox et al. [10], and the approximation method proposed by Brennan and Schwartz [6] where the associated free boundary problem is solved numerically such that the boundary conditions are not violated. Carr [9] introduced the maturity randomization technique where fixed maturity of the American put is successively replaced by random variables corresponding to the arrival times of an independent Poisson process. The partial differential equation (PDE) satisfied by the price of this transformed option becomes similar to the pricing PDE of a perpetual American put, which can be solved explicitly for every instance of maturity randomization. The series of these solutions is then used to approximate the price of original American put. Bouchard et al. [5] proved that the approximation of American put value obtained via maturity randomization converges to the true value with successive randomization iteration of the algorithm. This so-called Canadization method is also extended recently for optimal multiple stopping problems in Lévy models by Leung et al. [22]. Other examples of application of the Canalization method to price American and Russian options are [21], [11], [20].

In order to estimate the true American option value, simulation methods typically solve the discretized version of the option pricing problem, where exercise can happen only at a finite number of fixed times. As the number of exercise opportunities increases to infinity, the price of this discrete exercise American option converges to the true value. In this setting, simulation methods approximately solve the dynamic program associated with the pricing problem. To facilitate this, first, the so-called continuation value function is estimated at each exercise opportunity using simulated underlying sample paths via Monte Carlo, and then, based on these estimates, an approximately optimal exercise policy is defined. This policy is used to exercise on the simulated sample paths, and the average payoff on these exercised sample paths is used as an estimator for the true American option price. The random tree method of Broadie and Glasserman [7] used nested paths simulation to estimate the continuation value function. In the stochastic mesh method, Broadie and Glasserman [8] used a likelihood ratio weighted average on simulated sample paths to estimate the continuation value recursively. These methods were limited in their success due to their slow convergence rate for a given computational budget. In 2001, Longstaff and Schwartz [23] proposed the least squares regression method where the continuation value function is modeled as a linear combination of pre-specified basis functions in the L^2–space. This method has proven to be very popular in practice due to its fast rate of convergence, and it is considered as a benchmark in simulation methods for American option pricing. Recently, an improvement of Longstaff-Schwartz method has been proposed by Gramacy and Ludkovski [15] where the authors adaptively learn the classifiers which are then used in dynamic regression algorithms to estimate the option value function.

Empirical evidence in many studies (see Rubinstein [25]) shows that the implied volatility of options exhibit a smile curve (or skew). In order to capture this phenomenon, stochastic volatility models were proposed (see e.g., Hull and White [17] and Heston [16]). A common theme in all the proposed stochastic volatility models is mean-reversion of a stochastic factor driving the volatility of the underlying process. A detailed discussion of the mean-reverting nature of these models is provided by Fouque et al. [13] where empirical references are given for fast and slow factors in volatility fluctuations. Hence, in [13], the authors proposed multiscale stochastic volatility models which capture both the separation of scales and mean-reverting characteristics of the volatility process.

The problem of pricing European options in the stochastic volatility setting has been well studied. In the Heston model, the European option price appears as a Fourier inversion integral which can be efficiently calculated using numerical methods. Fouque et al. [13] provide an approximation for European option prices for calibration in the multiscale model which is reasonably accurate. Relatively, little has been done to address the problem of American option pricing in stochastic volatility setting. The few existing methods in the literature can be broadly categorized as PDE and non-PDE based methods. The PDE based methods provide an approximation for the price of American option by numerically solving the (at least three-dimensional) free boundary problem. A popular approach is to solve the problem by reformulating it as a linear complementarity problem. In order to solve each discrete complementarity problem, Ikonen and Toivanen use operator splitting methods in [18], and use component-wise splitting methods in [19]. These methods are time-consuming and are typically dependent on sophisticated solver packages. In non-PDE methods, a variant of the Longstaff-Schwartz method with cross-sectional basis functions can be used for pricing as demonstrated by Rambharat and Brockwell [24].

As mentioned earlier, in our work, we develop an approximation for the finite time-horizon American put price and the associated optimal exercise boundary under stochastic volatility. We use the idea of maturity randomization proposed by Carr [9] to reduce the original pricing PDE to a pricing PDE problem corresponding to a perpetual American put. We use perturbation theory and variation of parameters techniques to approximate the put price and the optimal exercise boundary. The optimal boundary approximation is used to exercise simulated underlying paths and estimate the true American option price. We compare the performance of our method with an implementation of the Longstaff-Schwartz method as suggested in [24] and demonstrate better numerical accuracy under typical parameter settings and small computational budget. Fouque and Han [12] provided centered martingales which can be possibly used as control variates for estimating the true option price. However, the true option price function appears in these centered martingales. We replace the true option price function with our approximate price function and use the resulting martingales to form control variates for our simulation method. The proposed control variates lead to a considerable variance reduction in the case of fast mean-reverting stochastic volatility as shown in the numerical examples. In our numerical experiments, we also observe that in the regime where value of scaling parameter is equal to one, fast and slow scale approximations are equally accurate.

For simplicity of presentation, we analyze the two stochastic volatility factors separately. In Section 2 we consider the case of fast mean-reverting volatility process which leads to a singular perturbation problem for the associated pricing PDE. It is evident from the calculations that one does not require to track the fast mean-reverting volatility level to calculate the put price approximation. Next, in Section 3, we derive the approximation for American put price and optimal exercise boundary in the case of slowly fluctuating volatility. In Sections 2.2 and 3.2, we show how to use the price and boundary approximations to form effective control variates using the fast and slow factor approximations respectively. A detailed numerical study comparing our method with the Longstaff-Schwartz method is presented in Section 4. We conclude with few comments on our work and some suggestions for further research in Section 5. The proofs are relegated to Appendix A. We explain the numerical implementation of the respective fast and slow scale control variates in Appendix B.

2 Approximation under fast mean-reverting stochastic volatility

We first consider a fast mean-reverting stochastic volatility model. Here, let X denote the price of a non-dividend paying asset whose dynamics under the risk-neutral probability measure $\mathbb{P}$ is given by the following system of SDEs:

$$dX_t = rX_t\,dt + f(Y_t)X_t\,dW_t^{(1)}, \qquad X_0 = x > 0, \tag{1}$$

$$dY_t = \frac{1}{\varepsilon}(m_1 - Y_t)\,dt + \nu_1\sqrt{\frac{2}{\varepsilon}}\,dW_t^{(2)}, \qquad Y_0 = y, \tag{2}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are one-dimensional Brownian motions with correlation $d\langle W^{(1)}, W^{(2)}\rangle_t = \rho_1 dt$, $\rho_1^2 < 1$ and $\varepsilon > 0$ is the intrinsic time-scale of Y. We choose specific form of the drift function: $(m_1 - Y_t)$ and volatility function: $\sqrt{2}\nu_1$ such that Y is an ergodic process (the Ornstein-Uhlenbeck process) with unique invariant distribution denoted by Φ which is normal $\mathcal{N}(m_1, \nu_1^2)$ and does not depend on ε. Further, we require $f : \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$ to be a continuously differentiable function such that $\int f^2(y)\Phi(y)dy < \infty$. We assume $\varepsilon << 1$ so that the intrinsic time-scale of Y is small and hence it represents a fast mean-reverting stochastic factor of underlying volatility.

Under the risk-neutral probability measure, the price at time $t < T$ of an American put option with maturity $T < \infty$ is

$$P^{\varepsilon}(t, \tilde{x}, \tilde{y}) := \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{t,\tilde{x},\tilde{y}} \left[e^{-r(\tau - t)}(K - X_\tau)^+ \right], \tag{3}$$

where K is the strike price, and $\mathcal{T}_{[t,T]}$ is the set of stopping times τ taking values in $[t, T]$. It can be shown, using the dynamic programming principle, that the American put value is the solution of a free boundary problem. The standard approach to solve this problem is to separate the space of state variables into two regions, the hold- and exercise- region where the boundary of the region is regarded as the optimal exercise boundary. However, it is not possible to calculate an explicit solution for the free boundary problem.

In order to solve this problem approximately, we randomize the maturity of the American put and replace it with an exponentially distributed independent random variable τ_λ with mean $\frac{1}{\lambda} = T$. Following the arguments in Carr [9], we can write the price of American put with random maturity as follows

$$P^{(1)}(x, y) := \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{x,y} \left[e^{-r(\tau \wedge \tau_\lambda)}(K - X_{\tau \wedge \tau_\lambda})^+ \right].$$

As the option maturity is an exponentially distributed random variable, by the memoryless property, the option gets no closer to its random maturity as the time elapses and thus its value suffers no time decay. As the exercise value is also time stationary, the exercise boundary becomes time-independent as well and we need to search for a single critical stock price which depends on the level of stochastic volatility factor. Thus, we look for the optimal exercise boundary parametrized by y.

For the proposed multiscale stochastic volatility model, it is clear that the smooth pasting condition for $P^{(1)}(x, y)$ holds w.r.t. (x, y) from Lemma 2.1 in [28]. Thus, we look for a solution that satisfies the following PDE in the hold region with the smooth pasting conditions:

$$\mathcal{L}^{\varepsilon} P^{(1)}(x, y) + \lambda(K - x)^+ = 0, \quad \text{for } x > x_b(y), \tag{4}$$

$$P^{(1)}(x_b(y), y) = K - x_b(y), \qquad \frac{\partial P^{(1)}}{\partial x}(x_b(y), y) = -1, \qquad \frac{\partial P^{(1)}}{\partial y}(x_b(y), y) = 0,$$

where $x_b(y)$ is the optimal exercise boundary. Here, the operator $\mathcal{L}^\varepsilon$ is given by:

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

where we define

$$\mathcal{L}_0 := \nu_1^2\frac{\partial^2}{\partial y^2}+(m_1-y)\frac{\partial}{\partial y}, \quad \mathcal{L}_1 := \sqrt{2}\rho_1\nu_1 x f(y)\frac{\partial^2}{\partial x\partial y}, \quad \mathcal{L}_2 := \frac{1}{2}f^2(y)x^2\frac{\partial^2}{\partial x^2}+rx\frac{\partial}{\partial x}-(r+\lambda)\cdot.$$

Note that $\mathcal{L}_0$ is the infinitesimal generator of an OU process with unit rate of mean reversion.

In the exercise region, we have

$$P^{(1)}(x,y) = K - x, \ \text{ for } x < x_b(y). \tag{5}$$

For a general stochastic volatility function $f(y)$, there is no analytic solution to the free-boundary problem (4)-(5). In the limit $\varepsilon \searrow 0$, this is a singular perturbation problem and thus our approach is to construct an asymptotic approximation.

2.1 Asymptotic analysis

Specifically, we perform a singular perturbation with respect to the small parameter ε, expanding the solution and exercise boundary in powers of $\sqrt{\varepsilon}$

$$P^{(1)}(x,y) = P_0(x,y) + \sqrt{\varepsilon}P_1(x,y) + \varepsilon P_2(x,y) + \cdots, \tag{6}$$

$$x_b(y) = x_0(y) + \sqrt{\varepsilon}x_1(y) + \cdots. \tag{7}$$

This particular choice of expansion in powers of $\sqrt{\varepsilon}$ allows us to analytically solve for terms in the expansion, independent of ε. We now plug (6) and (7) into (4) - (5), and collect terms of equal powers of $\sqrt{\varepsilon}$. For $x > x_b(y)$, we get

$$\frac{1}{\varepsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0P_1 + \mathcal{L}_1P_0\right) + \left(\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0 + \lambda(K-x)^+\right)$$
$$+ \sqrt{\varepsilon}\left(\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1\right) + \cdots = 0, \tag{8}$$

where we have suppressed the dependence on (x,y) for notational convenience. For $x < x_b(y)$, in (5), we use the asymptotic expansion of (7) in the right hand side and use (6) in the left hand side to perform a Taylor series expansion around $x_0(y)$ to write

$$P_0(x_0(y),y) + \sqrt{\varepsilon}\left(x_1(y)\frac{\partial P_0}{\partial x}\Big|_{x_0(y)} + P_1(x_0(y),y)\right) + \cdots = K - x_0(y) - \sqrt{\varepsilon}x_1(y) + \cdots. \tag{9}$$

Similarly, the boundary conditions can be expanded as

$$\frac{\partial P_0}{\partial x}\Big|_{x_0(y)} + \sqrt{\varepsilon}\left(x_1(y)\frac{\partial^2 P_0}{\partial x^2}\Big|_{x_0(y)} + \frac{\partial P_1}{\partial x}\Big|_{x_0(y)}\right) + \cdots = -1 \tag{10}$$

$$\frac{\partial P_0}{\partial y}\Big|_{x_0(y)} + \sqrt{\varepsilon}\left(x_1(y)\frac{\partial^2 P_0}{\partial y\partial x}\Big|_{x_0(y)} + \frac{\partial P_1}{\partial y}\Big|_{x_0(y)}\right) + \cdots = 0.$$

Zeroth order terms. Collecting terms of order $1/\varepsilon$ in (8) and order 1 in (9) and (10), we have the following PDE and boundary condition:

$$\mathcal{L}_0P_0(x,y) = 0, \quad \text{for } x > x_0(y),$$
$$P_0(x,y) = K - x, \quad \text{for } x \le x_0(y),$$
$$\frac{\partial P_0}{\partial x}\Big|_{x_0(y)} = -1.$$

As $\mathcal{L}_0$ is the infinitesimal generator of Y, zero is an eigenvalue with constant as its eigenfunction. Thus, the first equation implies that we may seek P_0 of the form $P_0 = P_0(x)$. Along with the second equation above, it is easy to see that P_0 is independent of y everywhere. As P_0 is independent of y on either side of the exercise boundary $x_0(y)$, we can conclude that x_0 is also independent of y. We make use of this observation going forward.

First order terms. Collecting terms of order $1/\sqrt{\varepsilon}$ in (8) and order $\sqrt{\varepsilon}$ in (9) and (10) leads to the following PDE and boundary condition:

$$\begin{aligned} \mathcal{L}_0 P_1(x,y) &= 0, \quad \text{for } x > x_0, \\ P_1(x,y) &= 0, \quad \text{for } x \le x_0, \\ x_1(y)\frac{\partial^2 P_0}{\partial x^2}\Big|_{x_0} + \frac{\partial P_1}{\partial x}\Big|_{x_0} &= 0, \end{aligned} \tag{11}$$

where we have used the result that P_0 is independent of y. As with P_0, we can also take P_1 to be independent of y. Thus, we can see that x_1 is also independent of y.

Second order terms. Matching terms of order 1 in (8) and order ε in (9) leads to

$$\begin{aligned} \mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 + \lambda(K-x)^+ &= 0, \quad \text{for } x > x_0, \\ P_2(x,y) &= 0, \quad \text{for } x \le x_0, \end{aligned} \tag{12}$$

where we have used that P_1 does not depend on y. The first equation is a Poisson equation for P_2 with respect to the infinitesimal generator $\mathcal{L}_0$ with the source term $\mathcal{L}_2 P_0 + \lambda(K-x)^+$. A well-behaved solution (polynomial growth at infinity) for P_2 exists if and only if $\mathcal{L}_2 P_0 + \lambda(K-x)^+$ is centered with respect to the invariant distribution of the diffusion whose infinitesimal generator is $\mathcal{L}_0$. Thus, the centering condition becomes

$$\langle \mathcal{L}_2 P_0 \rangle + \lambda(K-x)^+ = 0, \tag{13}$$

where the angled brackets indicate taking the average of the argument with respect to Φ, the invariant distribution of Y. Since P_0 does not depend on y, the centering condition becomes $\langle \mathcal{L}_2 \rangle P_0 + \lambda(K-x)^+ = 0$ which is equivalent to the ODE

$$\frac{1}{2}\bar{\sigma}^2 x^2 \frac{d^2 P_0}{dx^2} + rx\frac{dP_0}{dx} - (r+\lambda)P_0 + \lambda(K-x)^+ = 0$$

where $\bar{\sigma}^2 := \langle f^2 \rangle = \int f^2(y)\Phi(dy)$. The solution to the above ODE can be obtained from the calculations performed in [9].

Proposition 1. *In the stochastic volatility model* (1)–(2)*, the zeroth order approximation of randomized maturity American put price is given by*

$$P_0(x) = \begin{cases} a_{01}\left(\frac{x}{K}\right)^{\beta_1} + a_{02}\left(\frac{x}{x_0}\right)^{\beta_1} & \text{if} \quad x > K, \\ b_{01}\left(\frac{x}{K}\right)^{\beta_2} + a_{02}\left(\frac{x}{x_0}\right)^{\beta_1} + KR - x & \text{if} \quad x_0 < x \le K, \\ K - x & \text{if} \quad x \le x_0, \end{cases} \tag{14}$$

where the constants are

$$\begin{aligned} &\gamma = \frac{1}{2} - \frac{r}{\bar{\sigma}^2}, && R = \frac{\lambda}{\lambda + r}, && \Delta = \sqrt{\gamma^2 + \frac{2\lambda}{R\bar{\sigma}^2}}, \\ &p = \frac{\Delta - \gamma}{2\Delta}, && q = 1 - p, && \hat{p} = \frac{\Delta - \gamma + 1}{2\Delta}, \\ &\hat{q} = 1 - \hat{p}, && a_{01} = \left(qKR - \hat{q}K\right), && a_{02} = qKR\frac{r}{\lambda}, \\ &b_{01} = \left(\hat{p}K - pKR\right), && \beta_1 = \gamma - \Delta, && \beta_2 = \gamma + \Delta. \end{aligned}$$

The zeroth order approximation to the exercise boundary is given by

$$x_0 = K\left(\frac{pRr}{\lambda(\hat{p}-Rp)}\right)^{\frac{1}{\beta_2}}.$$

By plugging the expressions back in the above mentioned formulas, it can be easily verified that $x_0 \le K$ and $P_0(x)$ is continuous at K and x_0. From (12), we have

$$\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0 - \lambda(K-x)^+ = -\left(\mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0\rangle\right),$$

which follows from (13). Therefore,

$$P_2 = -\frac{1}{2}\left(\phi(y) + \tilde{c}(x)\right) x^2 \frac{\partial^2 P_0}{\partial x^2}, \tag{15}$$

where function ϕ solves $\mathcal{L}_0 \phi = f^2(y) - \bar{\sigma}^2$ and $\tilde{c}(x)$ is an arbitrary finite-valued function independent of y.

Third order terms. Collecting terms of order $\sqrt{\varepsilon}$ in (8) and order $\varepsilon^{3/2}$ in (9), we obtain the following PDE and boundary condition:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0 \quad \text{for } x > x_0.$$

This is a Poisson equation in y for $P_3(x,y)$, which imposes a solvability condition on the source term $\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1$, leading to

$$\langle \mathcal{L}_2\rangle P_1 = -\langle \mathcal{L}_1 P_2\rangle. \tag{16}$$

We introduce a new convenient notation for the correction term $\tilde{P}_1 := \sqrt{\varepsilon} P_1$. Thus, plugging P_2, given by (15), into (16) gives:

$$\langle \mathcal{L}_2\rangle \tilde{P}_1 = V_3\left(2x^2 \frac{\partial^2 P_0}{\partial x^2} + x^3 \frac{\partial^3 P_0}{\partial x^3}\right), \tag{17}$$

where $V_3 := \sqrt{\frac{\varepsilon}{2}}\rho\nu\langle f(y)\phi'(y)\rangle$. The above ordinary differential equation can be solved to obtain the following result:

Proposition 2. *The correction term $\tilde{P}_1$ in the price expansion is given by*

$$\tilde{P}_1(x) = \begin{cases} \hat{a}_1\left(\frac{x}{x_0}\right)^{\beta_1} - \frac{V_3\beta_1^2(\beta_1-1)}{\bar{\sigma}^2\Delta}\log(x)\left(a_{01}\left(\frac{x}{K}\right)^{\beta_1} + a_{02}\left(\frac{x}{x_0}\right)^{\beta_1}\right) & \text{if } x > K, \\ \hat{a}_2\left(\frac{x}{x_0}\right)^{\beta_1} + \hat{a}_3\left(\frac{x}{K}\right)^{\beta_2} & \\ +\frac{V_3\log(x)}{\bar{\sigma}^2\Delta}\left[b_{01}\beta_2^2(\beta_2-1)\left(\frac{x}{K}\right)^{\beta_2} - a_{02}\beta_1^2(\beta_1-1)\left(\frac{x}{x_0}\right)^{\beta_1}\right] & \text{if } x_0 < x \le K, \\ 0 & \text{if } x \le x_0, \end{cases}$$

where

$$\begin{aligned} \hat{a}_1 &= \frac{V_3}{2\Delta^2\bar{\sigma}^2}\Big(a_{01}\beta_1^2(\beta_1-1) + b_{01}\beta_2^2(\beta_2-1)\Big)\Big(\big(\frac{x_0}{K}\big)^{\beta_2} - \big(\frac{x_0}{K}\big)^{\beta_1}\Big) \\ &\quad + \frac{V_3}{\Delta\bar{\sigma}^2}\Big(\beta_1^2(\beta_1-1)\big(a_{01}\big(\frac{x_0}{K}\big)^{\beta_1}\log(K) + a_{02}\log(x_0)\big) - b_{01}\big(\frac{x_0}{K}\big)^{\beta_2}\beta_2^2(\beta_2-1)\log(x_0/K)\Big) \\ \hat{a}_2 &= \frac{V_3}{2\Delta^2\bar{\sigma}^2}\left(\frac{x_0}{K}\right)^{\beta_2}\Big(a_{01}\beta_1^2(\beta_1-1) + b_{01}\beta_2^2(\beta_2-1)(1+2\Delta\log(K))\Big), \\ &\quad + \frac{V_3\log(x_0)}{\Delta\bar{\sigma}^2}\Big(a_{02}\beta_1^2(\beta_1-1) - b_{01}\beta_2^2(\beta_2-1)\left(\frac{x_0}{K}\right)^{\beta_2}\Big), \\ \hat{a}_3 &= -\frac{V_3}{2\Delta^2\bar{\sigma}^2}\Big(a_{01}\beta_1^2(\beta_1-1) + b_{01}\beta_2^2(\beta_2-1)(1+2\Delta\log(K))\Big). \end{aligned}$$

Let $\tilde{x}_1 := \sqrt{\varepsilon}x_1$. Then, the correction term in the optimal exercise boundary expansion is given by

$$\tilde{x}_1 = -\frac{d\tilde{P}_1}{dx}\Big|_{x_0} \Big/ \frac{d^2P_0}{dx^2}\Big|_{x_0}$$

where

$$\begin{aligned}\frac{d\tilde{P}_1}{dx}\Big|_{x_0} &= \frac{\hat{a}_2\beta_1}{x_0} + \frac{\hat{a}_3\beta_2}{K}\Big(\frac{x_0}{K}\Big)^{\beta_2-1} \\ &\quad + \frac{V_3}{\bar{\sigma}^2\Delta}\Big[\frac{b_{01}\beta_2^2(\beta_2-1)}{K}\Big(\frac{x}{K}\Big)^{\beta_2-1}(1+\beta_2\log x_0) + \frac{a_{02}\beta_1^2(\beta_1-1)}{x_0}(1+\beta_1\log x_0)\Big], \\ \frac{d^2P_0}{dx^2}\Big|_{x_0} &= \frac{b_{01}\beta_2(\beta_2-1)}{K^2}\Big(\frac{x_0}{K}\Big)^{\beta_2-2} + \frac{a_{02}\beta_1(\beta_1-1)}{x_0^2}.\end{aligned}$$

Remark 1. Ting et al. [27] performed a similar analysis as in Proposition 2, but for the case of a perpetual American option i.e. $\lambda = 0$, under fast-mean reverting stochastic volatility.

In the constant volatility case, Carr [9] used successive jump times of an independent Poisson process to randomize the maturity of American put in order to closely approximate the true American put price. In this work, we use an independent exponential random variable, which corresponds to the first jump time of an independent Poisson process, to randomize the maturity of the American put. Therefore, the obtained price approximation $P_0+\sqrt{\varepsilon}P_1$ may not be accurate enough to approximate the true American put price P^ε. The approximation can be improved by replacing the fixed maturity with successively increasing arrival times of an independent Poisson process as in Carr [9]. In this approach, the number of different coefficient calculations in the approximation terms increases with the number of boundary conditions and poses a significant computational challenge. This direction has been reserved for future research.

However, it has been empirically observed and theoretically proved (cf. Andersen [2], Belomestny [4], Agarwal and Juneja [1]) that simulation methods based on even a crude approximation of the optimal exercise boundary produce good estimates for the true American option price. Thus, in Section 2.2, we use the exercise boundary approximation $x_0 + \sqrt{\varepsilon}x_1$ in our setting to price the *finite maturity* American put via Monte Carlo.

Remark 2. American call options on a dividend paying underlying can also be priced using the proposed methodology under the multiscale stochastic volatility model. In the case of an underlying with continuous dividend yield, we can use the American option put-call parity from Corollary 1 [26] to transform the original American call option pricing problem to an American put option pricing problem. It can be further checked that the free boundary problem for the new American put problem can be solved using the maturity randomization technique as in Section 2. Finally, to incorporate the discrete cash dividends, a correction to the exercise boundary condition can be introduced based on Equation (37) of [9].

2.2 Control variate for finite maturity American option pricing

Here we discuss how to form control variates for our simulation method to price an American put with *finite maturity* T under fast mean-reverting stochastic volatility. Let τ^* denote the optimal stopping time in (3). As in [12], by applying Itô's Lemma, we can write

$$P^\varepsilon(0,x,y) := e^{-r\tau^*}(K - X_{\tau^*})^+ - \mathcal{U}_0(P^\varepsilon) - \frac{1}{\sqrt{\varepsilon}}\mathcal{U}_1(P^\varepsilon),$$

where $\mathcal{U}_0$ and $\mathcal{U}_1$ are defined as

$$\mathcal{U}_0(P^\varepsilon) := \int_0^{\tau^*} e^{-rs}\frac{\partial P^\varepsilon}{\partial x}(s, X_s, Y_s)f(Y_s)X_s\, dW_s^{(1)},$$
$$\mathcal{U}_1(P^\varepsilon) := \int_0^{\tau^*} e^{-rs}\frac{\partial P^\varepsilon}{\partial y}(s, X_s, Y_s)\sqrt{2}\nu_1\, dW_s^{(2)}.$$

As an explicit formula for P^ε is not available, we use the approximation of the price $P^{(1)}$ of the random maturity American put as given in Propositions 1 and 2 to calculate an approximation for the martingales $\mathcal{U}_0$ and $\mathcal{U}_1$. Since, the terms P_0 and P_1 in the approximation are independent of y, we only need to approximate $\mathcal{U}_0(P^\varepsilon)$ which is done by using

$$\hat{\mathcal{U}}_0(P^{(1)}; x_b) := \int_0^{\hat{\tau}} e^{-rs}\frac{\partial}{\partial x}(P_0 + \sqrt{\varepsilon}P_1)(X_s)f(Y_s)X_s\, dW_s^{(1)},$$

where $\hat{\tau}$ is an approximation to optimal stopping time τ^* defined using the optimal exercise boundary approximation $x_0 + \sqrt{\varepsilon}x_1$:

$$\hat{\tau} := \inf\{t : X_t \leq x_0 + \sqrt{\varepsilon}x_1\} \wedge T$$

where x_0 and x_1 are constants independent of y. Hence, for a set of N independent sample paths of the underlying process X, a Monte Carlo estimator with the martingale control variate is given by

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-r\hat{\tau}}(K - X_{\hat{\tau}}^{(i)})^+ - \hat{\mathcal{U}}_0^{(i)}(P^{(1)}; x_b)\right] \tag{18}$$

where $X_{\hat{\tau}}^{(i)}$ is the value of the ith underlying process sample path at $\hat{\tau}$. We provide details of the numerical implementation of control variates in Appendix B.

3 Slow scale volatility approximations

We conduct a similar analysis for the case when stochastic volatility is slowly fluctuating. Note that we do not require ergodicity of the slow scale factor Z. However, for simplicity of presentation, we assume a specific form of the drift and volatility functions: $(m_2 - Z_t)$ and $\sqrt{2}\nu_2$ respectively for the volatility driving process Z. In this case, the dynamics of X is given by the following system of SDEs:

$$dX_t = rX_t\, dt + f(Z_t)X_t\, dW_t^{(1)}, \qquad X_0 = x,$$
$$dZ_t = \delta(m_2 - Z_t)\, dt + \nu_2\sqrt{2\delta}\, dW_t^{(3)}, \qquad Z_0 = z,$$

where $W_t^{(1)}$ and $W_t^{(3)}$ are one-dimensional Brownian motions with correlation $d\langle W^{(1)}, W^{(3)}\rangle_t = \rho_2 dt$, $\rho_2^2 < 1$ and $1/\delta > 0$ is the intrinsic time-scale of Z. We assume $\delta << 1$ so that the intrinsic time-scale of Z is large and hence it represents a slowly fluctuating stochastic factor of underlying volatility. Here, we require $f : \mathbb{R} \to \mathbb{R}_+\backslash\{0\}$ to be a continuously differentiable function.

The price at time t of an American put option with maturity $T < \infty$ is

$$P^\delta(t, \tilde{x}, \tilde{z}) := \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{t,\tilde{x},\tilde{z}}\left[e^{-r(\tau-t)}(K - X_\tau)^+\right], \tag{19}$$

where K is the strike price, and $\mathcal{T}_{[t,T]}$ is the set of stopping times τ taking values in $[t,T]$. The price of American put with independent exponentially distributed random maturity τ_λ, with mean $\frac{1}{\lambda}=T$, is

$$P^{(1)}(x,z) := \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_{x,z}\left[e^{-r(\tau\wedge\tau_\lambda)}(K-X_{\tau\wedge\tau_\lambda})^+\right].$$

Let us denote by $x_b(z)$ the optimal exercise boundary for American option with randomized maturity defined above. Again, we look for a solution $P^{(1)}(x,z)$ that satisfies the following PDE in the hold region with the boundary conditions and smooth pasting conditions:

$$\begin{aligned} \mathcal{L}^\delta P^{(1)}(x,z)+\lambda(K-x)^+ = 0, \quad \text{for } x > x_b(z), \\ P^{(1)}(x_b(z),z) = K - x_b(z), \\ \frac{\partial P^{(1)}}{\partial x}(x_b(z),z) = -1, \\ \frac{\partial P^{(1)}}{\partial z}(x_b(z),z) = 0, \end{aligned} \tag{20}$$

where the operator $\mathcal{L}^\delta := \mathcal{L}_2+\sqrt{\delta}\mathcal{M}_1+\delta\mathcal{M}_2$, with

$$\begin{aligned} \mathcal{L}_2 &:= \frac{1}{2}f^2(z)x^2\frac{\partial^2}{\partial x^2}+rx\frac{\partial}{\partial x}-(r+\lambda)\cdot, \\ \mathcal{M}_1 &:= \sqrt{2}\rho_2\nu_2 f(z)x\frac{\partial^2}{\partial x\partial z}, \qquad \mathcal{M}_2 := \nu_2^2\frac{\partial^2}{\partial z^2}+(m_2-z)\frac{\partial}{\partial z}. \end{aligned}$$

In the exercise region, we have

$$P^{(1)}(x,z) = K - x, \quad \text{for } x < x_b(z). \tag{21}$$

In the limit $\delta \searrow 0$, the above problem is a regular perturbation problem and we look for an asymptotic approximation.

3.1 Asymptotic analysis

We expand $P^{(1)}(x,z)$ and the optimal exercise boundary $x_b(z)$ in powers of $\sqrt{\delta}$ as follows

$$\begin{aligned} P^{(1)}(x,z) &= P_0(x,z)+\sqrt{\delta}P_1(x,z)+\delta P_2(x,z)+\cdots, \\ x_b(z) &= x_0(z)+\sqrt{\delta}x_1(z)+\cdots. \end{aligned}$$

We plug the asymptotic expansion formulas and repeat the procedure as followed in Section 2.1. Matching terms of order 1 (with respect to $\sqrt{\delta}$) in (20)-(21) leads to the following PDE and boundary condition:

$$\begin{aligned} \mathcal{L}_2 P_0+\lambda(K-x)^+ = 0, \qquad \text{for } x > x_0(z), \\ P_0 = K - x, \quad \text{for } x \le x_0(z), \\ \left.\frac{\partial P_0}{\partial x}\right|_{x_0(z)} = -1. \end{aligned} \tag{22}$$

We know from our previous calculations, that $P_0(x,z)$ is given as in Proposition 1 where the coefficients are defined as in (14) with $\bar{\sigma}$ replaced by $f(z)$.

Next, by matching order $\sqrt{\delta}$ terms in (20)-(21), we get

$$\begin{aligned} \mathcal{L}_2 P_1 + \mathcal{M}_1 P_0 &= 0, \qquad x > x_0(z) \\ P_1 &= 0, \qquad x \le x_0(z). \end{aligned} \tag{23}$$

The source term in (23), has a term with derivative with respect to the volatility factor z. In order to solve this PDE, we define a new quantity, $\mathcal{V} := \partial P_0/\partial z$. The following result enables us to construct the solution to (23).

Lemma 1. *We assume that $\mathcal{V}$ is continuous at $x = x_0(z)$ and is differentiable at $x = K$. Then, $\mathcal{V}$ is given by*

$$\mathcal{V}(x,z) = \begin{cases} a_1\left(\frac{x}{x_0(z)}\right)^{\beta_1} + \frac{f'(z)\beta_1(\beta_1-1)}{f(z)\Delta}\log x\left(a_{01}\left(\frac{x}{K}\right)^{\beta_1} + a_{02}\left(\frac{x}{x_0(z)}\right)^{\beta_1}\right) & \text{if} \quad x > K, \\ a_2\left(\frac{x}{x_0(z)}\right)^{\beta_1} + a_3\left(\frac{x}{K}\right)^{\beta_2} & \\ -\frac{f'(z)\log(x)}{f(z)\Delta}\left(b_{01}\beta_2(\beta_2-1)\left(\frac{x}{K}\right)^{\beta_2} - a_{02}\beta_1(\beta_1-1)\left(\frac{x}{x_0(z)}\right)^{\beta_1}\right) & \text{if} \quad x_0(z) < x \le K, \\ 0 & \text{if} \quad x \le x_0(z) \end{cases}$$

where a_1, a_2 and a_3 are coefficients given by (26), (27) *and* (28) *in Appendix A.2.*

The proof of the above lemma is given in Appendix A. Once, we have solved for $\mathcal{V}$, we proceed to find the correction term P_1 in the price approximation under slowly fluctuating stochastic volatility.

Proposition 3. *The correction term P_1 in the price approximation, solution of* (23), *is given by*

$$P_1(x,z) = \begin{cases} \eta_1\left(\frac{x}{x_0(z)}\right)^{\beta_1} + V_2(z)a_1\beta_1 f(z)\log(x)\left(\frac{x}{x_0(z)}\right)^{\beta_1} + \frac{V_2(z)f'(z)}{2\Delta^2}\beta_1 & \\ \times\log(x)(\beta_1-1)(\beta_2+\beta_1\Delta\log(x))\left(a_{01}\left(\frac{x}{K}\right)^{\beta_1} + a_{02}\left(\frac{x}{x_0(z)}\right)^{\beta_1}\right) & \text{if} \quad x > K, \\ \eta_2\left(\frac{x}{x_0(z)}\right)^{\beta_1} + \eta_3\left(\frac{x}{K}\right)^{\beta_2} + V_2(z)f(z)\log(x)\left(a_2\beta_1\left(\frac{x}{x_0(z)}\right)^{\beta_1}\right. & \\ \left.-a_3\beta_2\left(\frac{x}{K}\right)^{\beta_2}\right) - \frac{V_2(z)f'(z)}{2\Delta^2}b_{01}\beta_2 & \\ \times\log(x)(\beta_2-1)\left(\beta_1-\beta_2\Delta\log(x)\right)\left(\frac{x}{K}\right)^{\beta_2} & \\ +\frac{V_2(z)f'(z)}{2\Delta^2}a_{02}\beta_1(\beta_1-1)\log(x)\left(\beta_2+\beta_1\Delta\log(x)\right)\left(\frac{x}{x_0(z)}\right)^{\beta_1} & \text{if} \quad x_0(z) < x \le K, \\ 0 & \text{if} \quad x \le x_0(z) \end{cases}$$

where $V_2(z) := \frac{\sqrt{2}\rho_2\nu_2}{\Delta f^2(z)}$ and η_1, η_2 and η_3 are given in (29), (30) *and* (31). *The correction term in optimal exercise boundary expansion is*

$$x_1(z) = -\frac{dP_1}{dx}\Big|_{x_0(z)} \Big/ \frac{d^2P_0}{dx^2}\Big|_{x_0(z)}$$

where

$$\begin{aligned}
\frac{dP_1}{dx}\Big|_{x_0(z)} &= \frac{\eta_2\beta_1}{x_0(z)} + \frac{\eta_3\beta_2}{x_0(z)}\left(\frac{x_0(z)}{K}\right)^{\beta_2} + \frac{V_2(z)f(z)}{x_0}\Big(a_2\beta_1\left(1+\beta_1\log(x_0(z))\right) \\
&\quad - a_3\beta_2\left(1+\beta_2\log(x_0(z))\right)\left(\frac{x_0(z)}{K}\right)^{\beta_2}\Big) - \frac{V_2(z)f'(z)b_{01}\beta_2(\beta_2-1)}{2\Delta^2 x_0(z)}\Big(\big(\beta_1 \\
&\quad - \Delta\beta_2\log(x_0(z))\big)\big(1+\beta_2\log(x_0(z))\big) - \Delta\beta_2\log(x_0(z))\Big)\left(\frac{x_0(z)}{K}\right)^{\beta_2} \\
&\quad + \frac{V_2(z)f'(z)a_{02}\beta_1(\beta_1-1)}{2\Delta^2 x_0(z)}\Big(\left(\beta_2+\Delta\beta_1\log(x_0(z))\right)\left(1+\beta_1\log(x_0(z))\right) \\
&\quad + \Delta\beta_1\log(x_0(z))\Big), \\
\frac{d^2P_0}{dx^2}\Big|_{x_0(z)} &= \frac{b_{01}\beta_2(\beta_2-1)}{K^2}\Big(\frac{x_0(z)}{K}\Big)^{\beta_2-2} + \frac{a_{02}\beta_1(\beta_1-1)}{x_0(z)^2}.
\end{aligned}$$

We will use the derived exercise boundary approximation $x_0(z) + \sqrt{\delta}x_1(z)$ under the slow scale to price the finite maturity American put via Monte Carlo.

3.2 Control variate for finite maturity American option pricing

We discuss how to form control variates for our simulation method to price an American put with *finite maturity* T under slow stochastic volatility. Let τ^* denote the optimal stopping time in (19). Applying Itô's Lemma, we have

$$P^\delta(0,x,z) := e^{-r\tau^*}(K - X_{\tau^*})^+ - \mathcal{U}_0(P^\delta) - \sqrt{\delta}\,\mathcal{U}_2(P^\delta),$$

where $\mathcal{U}_0$ and $\mathcal{U}_2$ are defined as

$$\begin{aligned}
\mathcal{U}_0(P^\delta) &:= \int_0^{\tau^*} e^{-rs}\frac{\partial P^\delta}{\partial x}(s,X_s,Z_s)f(Z_s)X_s\,dW_s^{(1)}, \\
\mathcal{U}_2(P^\delta) &:= \int_0^{\tau^*} e^{-rs}\frac{\partial P^\delta}{\partial z}(s,X_s,Z_s)\sqrt{2}\nu_2\,dW_s^{(3)}.
\end{aligned}$$

We use the approximation of the price $P^{(1)}$ of the random maturity American put as given in Propositions 1 (with $\bar\sigma$ replaced by $f(z)$) and 3 to calculate an approximation for the martingales. We approximate the centered martingales $\mathcal{U}_0(P^\delta)$ and $\mathcal{U}_2(P^\delta)$ as follows:

$$\begin{aligned}
\hat{\mathcal{U}}_0(P^{(1)};x_b) &:= \int_0^{\hat\tau} e^{-rs}\frac{\partial}{\partial x}(P_0 + \sqrt{\delta}P_1)(X_s,Z_s)f(Z_s)X_s\,dW_s^{(1)}, \\
\hat{\mathcal{U}}_2(P^{(1)};x_b) &:= \int_0^{\hat\tau} e^{-rs}\frac{\partial}{\partial z}(P_0 + \sqrt{\delta}P_1)(X_s,Z_s)\sqrt{2}\nu_2\,dW_s^{(3)},
\end{aligned}$$

where $\hat\tau$ is the approximation to the optimal stopping time τ^* defined using the optimal exercise boundary approximation $x_0(z) + \sqrt{\delta}x_1(z)$:

$$\hat\tau := \inf\{t : X_t \le x_0(Z_t) + \sqrt{\delta}x_1(Z_t)\} \wedge T.$$

The Monte Carlo estimator in this case is

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-r\hat\tau}(K - X_{\hat\tau}^{(i)})^+ - \hat{\mathcal{U}}_0^{(i)}(P^{(1)};x_b) - \sqrt{\delta}\hat{\mathcal{U}}_2^{(i)}(P^{(1)};x_b)\right]. \tag{24}$$

We provide details of the numerical implementation of control variates in Appendix B.

4 Numerical examples

In this section, we compare the performance of estimators suggested in the fast mean-reverting and slowly fluctuating volatility cases in (18) and (24) respectively with the popular least squares regression method of Longstaff and Schwartz [23]. The numerical tests are conducted using a workstation with two 64-bit Quad-Core Intel Xeon processors and 16 GB RAM. For the purpose of comparison studies, we choose the following functional form of volatility driving function for both fast and slow scale: $f(y) = e^y$.

Before we proceed, we need to calculate two constants in the fast mean-reverting stochastic volatility model of (1)–(2) which are required to evaluate the terms in approximation formulas. Firstly, the average volatility $\bar{\sigma} = \left(\int f^2(y)\Phi(dy)\right)^{1/2}$ is given by $\bar{\sigma} = e^{(m_1+\nu_1^2)}$, where m_1 and ν_1 are the corresponding model parameters. Next, in order to evaluate the correction term P_1, we need to calculate $V_3 = \sqrt{\frac{\varepsilon}{2}}\rho\nu\langle f(y)\phi'(y)\rangle$ where ϕ solves $\mathcal{L}_0\phi = f^2(y) - \bar{\sigma}^2$. With the specified choice of f, we compute

$$V_3 = -\frac{\rho}{\nu_1^2}\sqrt{\frac{\varepsilon}{2}}e^{3m_1}\left(e^{9\nu_1^2/2} - e^{5\nu_1^2/2}\right).$$

4.1 Performance of estimator under fast mean-reverting volatility

We consider an American put with maturity $T < \infty$ under the fast mean-reverting stochastic volatility model (1)–(2). The model parameters with initial conditions and option parameters are given in Table 4.1. The numerical experiments are performed with various practically relevant time scale parameter ε values. Typically, for the fast-scale approximations to be accurate enough, we need that $\varepsilon << T$ where unit of time to maturity is the number of years. The underlying sample paths are simulated with the Euler discretization scheme (cf. [14, page 81]) with time step size $\Delta t = 10^{-3}$. Each discrete time step corresponds to an exercise opportunity.

K	Y_0	r	m_1	ν_1	ρ_1	T
100	-1.0	0.10	-2.0	1.0	-0.3	1.0

Table 4.1: *Parameters for fast mean-reverting stochastic volatility model and option parameters.*

In the proposed method, we use the optimal exercise boundary approximation $x_0 + \sqrt{\varepsilon}x_1$ derived in Section 2 to exercise the simulated sample paths. As we use an approximation of the optimal exercise boundary to exercise the sample paths, the proposed policy is suboptimal. The resulting option price estimator is thus lower biased. Further, we illustrate the variance reduction achieved by using the option price approximation $P_0 + \sqrt{\varepsilon}P_1$ in the martingale control variate for the proposed estimator. We use the Longstaff and Schwartz (LS) method as implemented in the numerical examples by Rambharat and Brockwell [24] to compare the performance of our method. As chosen in [24], we use the following set of Hermite polynomials as basis functions for LS method:

$$\begin{array}{lllll} L_0(X_t), & L_1(X_t), & L_2(X_t), & L_3(X_t), & L_4(X_t), \\ L_1(Y_t), & L_2(Y_t), & L_3(Y_t), & L_4(Y_t), & L_1(X_t)\times L_1(Y_t), \\ L_2(X_t)\times L_2(Y_t), & L_3(X_t)\times L_3(Y_t), & L_4(X_t)\times L_4(Y_t) & & \end{array}$$

where $L_0(x) = 1$, and for $n \geq 1$, $L_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$. The lower biased estimator in the LS method is obtained by using the standard two phase implementation where in Phase 1, we use M simulated underlying sample paths to estimate the coefficients in continuation value function representation. In Phase 2, we use the continuation value estimates to exercise on a new set of N underlying sample paths. As we compare the lower biased estimators, a higher value indicates a better estimator.

ε^{-1}	Proposed method				VR	Longstaff-Schwartz		Run time	Inter.
	Price	(s.e.)	Price CV	(s.e.)	Ratio	Price	(s.e.)	(secs)	est.
100	15.204	(0.0577)	15.216	(0.0213)	7.3	14.781	(0.1677)	60	15.409
	15.220	(0.0527)	15.222	(0.0209)	6.4	14.860	(0.1505)	90	(0.021)
	15.229	(0.0462)	15.220	(0.0350)	12.0	14.895	(0.1351)	120	
75	14.871	(0.0806)	14.882	(0.0234)	11.9	14.554	(0.1447)	60	15.183
	14.876	(0.0535)	14.878	(0.0208)	6.6	14.659	(0.1254)	90	(0.019)
	14.874	(0.0454)	14.873	(0.0180)	6.4	14.661	(0.1176)	120	
50	14.410	(0.0714)	14.417	(0.0260)	7.5	14.252	(0.1297)	60	14.940
	14.437	(0.0674)	14.420	(0.0180)	14.0	14.321	(0.1153)	90	(0.023)
	14.414	(0.0498)	14.422	(0.0176)	8.0	14.355	(0.0969)	120	
25	13.648	(0.0742)	13.643	(0.0317)	5.5	13.757	(0.1376)	60	14.312
	13.651	(0.0678)	13.648	(0.0250)	7.4	13.770	(0.1336)	90	(0.020)
	13.653	(0.0570)	13.643	(0.0221)	6.7	13.805	(0.1058)	120	

Table 4.2: *Comparison of standard errors (s.e.) of the option price estimate using the proposed method without and with control variate (CV) for in-the-money option with $S_0 = 90$. LS method price is obtained using the first four Hermite polynomials as basis functions. The variance reduction (VR) ratio is calculated as the square of the ratio of standard errors in two different cases. The results are shown for different values of the run time (in seconds) for a single iteration.*

To estimate the true price of the American option V_0 in absence of unbiased estimators, we aim to calculate a lower biased estimator $\hat{v}_n$ and an upper biased estimator $\hat{V}_n$ i.e. $\mathbb{E}[\hat{v}_n] \leq V_0 \leq \mathbb{E}[\hat{V}_n]$ where n denotes the number of independent replications of the respective algorithms. Then, as explained in Glasserman [14, Pg.431], we can form a conservative confidence interval for V_0 as $\left(\hat{v}_n - l_n, \hat{V}_n + H_n\right)$ where l_n and H_n denote the halfwidth of a certain level of confidence interval for lower and upper biased estimator, respectively. The estimator of the true value V_0 can then be taken as the midpoint of the confidence interval. However, due to the high number of exercise opportunities of the American option being considered, even a reasonable implementation of the dual method proposed by Andersen and Broadie [3] is unable to provide a meaningful upper biased estimator due to the exponential increase in computational budget with the number of exercise times. Hence, to get a proxy for the true price, we use *interleaving* estimator of Longstaff and Schwartz [23]. The bias of *interleaving* estimator is unclear as it mixes the high bias from the backward recursion with the low bias resulting from suboptimal exercise. For our purpose, the value of *interleaving* estimator will act as a proxy to the true price and proximity to this value will provide a reasonable accuracy comparison. In our experiments, we use 4×10^5 sample paths and the first seven Hermite polynomials and their cross products as basis functions to calculate the *interleaving* estimator (Inter. est.) in different settings where the running time of a single iteration is 1.5×10^3 seconds.

We illustrate the performance of proposed method when compared to the LS method under the cases of in-the-money, at-the-money and out-of-the-money options in Table 4.2, 4.3 and 4.4 respectively where the reported results are based on 50 independent iterations of the algorithms with varying computational budgets. The computational budget for a single iteration is specified in terms of running time (in seconds) of the algorithm. In each phase of the LS algorithm, we use the same number of sample paths – $20000, 30000$ and 40000 respectively for the correspondingly increasing computational budget. We once again emphasize that in case of

ε^{-1}	Proposed method				VR	Longstaff-Schwartz		Run time	Inter.
	Price	(s.e.)	Price CV	(s.e.)	Ratio	Price	(s.e.)	(secs)	est.
100	11.043	(0.1054)	11.042	(0.0308)	11.7	9.693	(0.3781)	60	11.190
	11.033	(0.0649)	11.040	(0.0268)	5.9	10.181	(0.2638)	90	(0.031)
	11.057	(0.0524)	11.047	(0.0208)	6.4	10.364	(0.2114)	120	
75	10.714	(0.0901)	10.723	(0.0297)	9.2	9.464	(0.4034)	60	10.943
	10.717	(0.0834)	10.717	(0.0293)	8.1	9.835	(0.2877)	90	(0.028)
	10.724	(0.0595)	10.715	(0.0288)	6.8	10.098	(0.2110)	120	
50	10.281	(0.0867)	10.300	(0.0349)	6.2	9.203	(0.4078)	60	10.644
	10.311	(0.0712)	10.302	(0.0242)	8.7	9.592	(0.3186)	90	(0.27)
	10.296	(0.0632)	10.296	(0.0218)	8.4	9.669	(0.2239)	120	
25	9.617	(0.0807)	9.626	(0.0345)	5.5	8.199	(0.4817)	60	10.308
	9.628	(0.0744)	9.616	(0.0280)	7.1	8.847	(0.3625)	90	(0.029)
	9.612	(0.0681)	9.610	(0.0299)	5.2	9.018	(0.2676)	120	

Table 4.3: *At-the-money option with* $S_0 = 100$.

fast mean-reverting stochastic volatility, both the option price and optimal exercise boundary first order approximations are independent of the current level of stochastic volatility factor Y_t. Only the average of the variance $\bar{\sigma}^2$ plays a role.

It can be observed in Table 4.2, that our proposed method performs better than the LS method for small values of the scaling parameter ε. We are also able to achieve considerable variance reduction using the control variates proposed in Section 2.2. As expected, the pricing accuracy reduces with increasing value of the scaling parameter ε. In Table 4.3, we can see that the proposed method provides a more accurate estimator than the lower biased LS estimator, uniformly for all values of the scaling parameter ε. Further, in the case of out-of-the-money options (Table 4.4), we observed that for the given computational budget, the two phase implementation of the LS method provided insignificant estimates of the option price. This observation again emphasizes that the proposed method provides a better approach to estimate the true American option value when the stochastic volatility is fast mean-reverting.

4.2 Performance of estimator under slowly fluctuating volatility

To test the performance of the proposed estimator in the slowly fluctuating volatility setting, we consider an American put with the model parameters, initial conditions and option parameters given in Table 4.5. The numerical experiments are performed with different slow scale parameter δ values where small values are practically relevant. The underlying sample paths are simulated with the Euler discretization scheme with time step size $\Delta t = 10^{-3}$. We use the optimal exercise boundary approximation $x_0 + \sqrt{\delta}x_1$ derived in Section 3 to exercise the simulated sample paths. In the case of slowly fluctuating volatility, the developed control variates do not exhibit significant variance reduction and hence, the values are not reported.

Unlike, the fast-scale stochastic volatility, the current level of volatility factor Z_t is extremely important in the slowly fluctuating stochastic volatility model. It can be observed in Table 4.6 that for $\delta = 0.1$, our proposed method provides a slightly lower estimator than the LS method but with considerably smaller standard error. As expected, the performance of the estimator deteriorates as the value of δ increases. We implemented control variates by using approximate centered martingales $\mathcal{U}_0$ and $\mathcal{U}_2$ but found that no considerable variance reduction is achieved in the case of slowly fluctuating volatility. This particular empirical observation

ε^{-1}	Proposed method				VR	Run time	Inter.
	Price	(s.e.)	Price CV	(s.e.)	Ratio	(secs)	est.
100	8.097	(0.0903)	8.107	(0.0258)	12.3	60	8.228
	8.112	(0.0727)	8.105	(0.0255)	8.1	90	(0.031)
	8.114	(0.0599)	8.103	(0.0207)	8.4	120	
75	7.847	(0.0702)	7.834	(0.0243)	8.3	60	8.015
	7.823	(0.0620)	7.826	(0.0271)	5.2	90	(0.029)
	7.835	(0.0621)	7.829	(0.0210)	8.8	120	
50	7.487	(0.0794)	7.495	(0.0357)	5.0	60	7.881
	7.489	(0.0618)	7.493	(0.0286)	4.7	90	(0.035)
	7.496	(0.0570)	7.492	(0.0265)	4.6	120	
25	6.965	(0.0837)	6.962	(0.0410)	4.2	60	7.507
	6.978	(0.0532)	6.972	(0.0264)	4.1	90	(0.032)
	6.959	(0.0541)	6.970	(0.0241)	5.0	120	

Table 4.4: *Out-of-the-money option with $S_0 = 110$. The two phase Longstaff and Schwartz method produced very small values of option price estimator which are not reported.*

K	Z_0	r	m_2	ν_2	ρ_2	T
100	-1.0	0.10	-2.0	1.0	-0.3	1.0

Table 4.5: *Parameters for slowly fluctuating stochastic volatility model and option parameters.*

motivates replacing the option maturity with the increasing arrival times of an independent Poisson process. However, the optimal exercise boundary approximation remains reasonably accurate and in Table 4.7, we can see that for a small computational budget, the proposed method provides a more accurate estimator than the lower biased LS estimator, uniformly for all values of the scaling parameter δ. Further, in the case of out-of-the-money option (Table 4.8), we observed that the two phase implementation of LS method, like in the case of fast mean-reverting stochastic volatility, provided insignificant estimates of the option price. This observation again emphasizes that the proposed method provides a better approach to estimate the true American option value in the case of both fast mean-reverting and slowly fluctuating stochastic volatility.

Remark 3. In order to compare the accuracy of the option price approximation formula in different stochastic volatility settings – fast and slow scale – we set the value of the scaling parameters $\varepsilon = \delta = 1$. We empirically observed that the boundary approximations in both the methods provide an equally accurate Monte Carlo estimator for the true American option price. Thus, when $\varepsilon = \delta = 1$, any of the boundary approximation formula derived in Section 2 and 3 can be used to form the Monte Carlo estimator.

5 Conclusion

We introduced a new method to approximately solve the important problem of American option pricing under stochastic volatility by combining PDE asymptotic techniques with Monte Carlo simulation. We particularly study the case of an American put option with a single underlying asset. We derived closed-form approximations for the price of put option with random maturity and its optimal exercise boundary up to the first order, when volatility is driven by a fast

δ	Proposed method		Longstaff-Schwartz		Run	Inter.
	Price	(s.e.)	Price	(s.e.)	time (secs)	est.
0.1	13.345	(0.1090)	13.228	(0.9989)	60	15.019
	13.370	(0.0895)	13.343	(0.8989)	90	(0.019)
	13.373	(0.0700)	13.473	(0.6110)	120	
1.0	12.986	(0.1124)	13.592	(0.8595)	60	14.898
	13.027	(0.1049)	13.709	(0.5602)	90	(0.019)
	13.030	(0.0807)	13.789	(0.7763)	120	
10	12.089	(0.1382)	13.071	(0.5722)	60	13.910
	12.114	(0.1080)	13.060	(0.6429)	90	(0.021)
	12.111	(0.0804)	13.119	(0.6349)	120	
25	12.632	(0.1250)	13.720	(0.1318)	60	14.310
	12.620	(0.1031)	13.765	(0.1375)	90	(0.022)
	12.598	(0.0923)	13.803	(0.1287)	120	

Table 4.6: *Comparison of standard errors of the option price estimate using the proposed method without and with control variate (CV). The Longstaff-Schwartz method price is obtained using first four Hermite polynomials as basis functions. The results are shown for different values of the run time for single iteration. In-the-money option with* $S_0 = 90$.

mean-reverting or a slowly fluctuating factor. We then proposed a simulation method which uses the optimal exercise boundary approximation for price estimation and numerically showed that it performs better than a reasonable implementation of the least squares regression method under typical parameter settings and small computational budget. We also achieved significant improvement in pricing accuracy by using the derived asymptotic price approximation to form control variates in the proposed method when the stochastic volatility is fast mean-reverting. It was observed that similar improvement in pricing accuracy is not replicated when the stochastic volatility fluctuates on the slow scale. In the multiscale stochastic volatility model, it can be shown that the approximations derived separately for different scales of fluctuations essentially combine.

In our work, we randomized the maturity of the put with an exponentially distributed random variable and showed that under typical scaling parameter regimes, the exercise boundary approximations provide accurate estimators when used with Monte Carlo simulation. Understandably, the accuracy of these approximations can be iteratively improved by replacing the fixed maturity with successively increasing arrival times of an independent Poisson process. But the number of different coefficient calculations increases with the number of boundary conditions which poses a significant computational challenge. The development of an iterative approach to achieve this task provides a promising direction for future research.

A Proofs

A.1 Proof of Proposition 2

It is evident from the zeroth order term P_0 in (14) that we solve for the correction term $\tilde{P}_1$ in two separate regions. Firstly, for $x > K$, from (17), we get the following ODE

$$\frac{1}{2}\bar{\sigma}^2 x^2 \frac{d^2\tilde{P}_1}{dx^2} + rx\frac{d\tilde{P}_1}{dx} - (r+\lambda)\tilde{P}_1 = c_1 x^{\beta_1}$$

δ	Proposed method		Longstaff-Schwartz		Run	Inter.
	Price	(s.e.)	Price	(s.e.)	time (secs)	est.
0.1	9.883	(0.1116)	5.144	(0.4283)	60	10.709
	9.850	(0.0706)	5.723	(0.5083)	90	(0.026)
	9.887	(0.0718)	5.885	(0.4376)	120	
1.0	9.890	(0.1362)	5.512	(0.5939)	60	10.714
	9.878	(0.0939)	5.908	(0.5967)	90	(0.021)
	9.864	(0.0931)	6.555	(0.5844)	120	
10	8.879	(0.0952)	6.984	(0.5531)	60	9.871
	8.822	(0.0986)	7.439	(0.5832)	90	(0.022)
	8.861	(0.0741)	7.707	(0.4379)	120	
25	9.224	(0.1079)	8.280	(0.5165)	60	10.215
	9.227	(0.0794)	8.665	(0.4431)	90	(0.020)
	9.243	(0.0795)	8.843	(0.2598)	120	

Table 4.7: *At-the-money option with* $S_0 = 100$.

where

$$c_1 = V_3\beta_1^2(\beta_1 - 1)\left(\frac{a_{01}}{K^{\beta_1}} + \frac{a_{02}}{x_0^{\beta_1}}\right).$$

We use a transformation to make the linear operator $\mathcal{L}_2$ which has x–dependent coefficients, into the constant coefficient heat operator. To this end, we define the new variable $y = \log(x)$. Then, the transformation to the heat equation with a source term is given by

$$\frac{d^2\tilde{P}_1}{dy^2} + \left(\frac{2r}{\bar{\sigma}^2} - 1\right)\frac{d\tilde{P}_1}{dy} - \frac{2(r+\lambda)}{\bar{\sigma}^2}\tilde{P}_1 = \frac{2c_1}{\bar{\sigma}^2}\exp(\beta_1 y). \tag{25}$$

We use variation of parameters to solve the above ODE. The two linearly independent solutions to the homogeneous part of (25) are $u_1(y) = \exp(\beta_2 y)$ and $u_2(y) = \exp(\beta_1 y)$. It is easy to see that $\beta_2 > 1 > 0 > \beta_1$. The Wronskian of the two linearly independent solutions is $W(y) = (\beta_1 - \beta_2)\exp((\beta_1 + \beta_2)y) = -2\Delta\exp(2\gamma y)$ and the general solution for the ODE (25) can be calculated as $A(z)u_1(y) + B(z)u_2(y)$, where

$$A(y) = \int -\frac{1}{W(y)}F(y)u_2(y)dy, \qquad B(y) = \int \frac{1}{W(y)}F(y)u_1(y)dy,$$

and $F(y) = 2c_1\exp(\beta_1 y)/\bar{\sigma}^2$ is the source term. The solution for the first order correction term can then be computed via substituting back to the x variable

$$\tilde{P}_1(x) = a_1 x^{\beta_1} + \tilde{a}_1 x^{\beta_2} - \frac{V_3\beta_1^2(\beta_1 - 1)}{\bar{\sigma}^2\Delta}\log(x)\Big(a_{01}\Big(\frac{x}{K}\Big)^{\beta_1} + a_{02}\Big(\frac{x}{x_0}\Big)^{\beta_1}\Big), \text{ for } x > K,$$

for unknown constants a_1 and $\tilde{a}_1$.

Similarly, in the region $x_0 < x \leq K$, from the solution in (14), we get

$$\frac{1}{2}\bar{\sigma}^2 x^2\frac{d^2\tilde{P}_1}{dx^2} + rx\frac{d\tilde{P}_1}{dx} - (r+\lambda)\tilde{P}_1 = c_2 x^{\beta_1} + c_3 x^{\beta_2}$$

where

$$c_2 = V_3\beta_1^2(\beta_1 - 1)\frac{a_{02}}{x_0^{\beta_1}}, \qquad c_3 = V_3\beta_2^2(\beta_2 - 1)\frac{b_{01}}{K^{\beta_2}}.$$

δ	Proposed method		Run	Inter.
	Price	(s.e.)	time (secs)	est.
0.1	7.304	(0.1069)	60	7.623
	7.325	(0.0830)	90	(0.016)
	7.329	(0.0783)	120	
1.0	7.737	(0.1356)	60	8.193
	7.721	(0.1128)	90	(0.024)
	7.727	(0.0905)	120	
10	6.714	(0.0962)	60	7.351
	6.709	(0.0728)	90	(0.022)
	6.709	(0.0847)	120	
25	6.893	(0.1175)	60	7.428
	6.888	(0.0854)	90	(0.022)
	6.912	(0.0834)	120	

Table 4.8: *Out-of-the-money option with $S_0 = 110$. The two phase Longstaff and Schwartz method produced very small values of option price estimator which are not reported.*

The solution in this case is

$$\tilde{P}_1(x) = a_2 x^{\beta_1} + a_3 x^{\beta_2} + \frac{V_3 \log(x)}{\bar{\sigma}^2 \Delta}\Big(b_{01}\beta_2^2(\beta_2 - 1)\Big(\frac{x}{K}\Big)^{\beta_2} - a_{02}\beta_1^2(\beta_1 - 1)\Big(\frac{x}{x_0}\Big)^{\beta_1}\Big), \text{ for } x_0 < x \le K,$$

for unknown constants a_2 and a_3. For $x \le x_0$, we have $\tilde{P}_1(x) = 0$.

To solve for the unknown constants, we first note that as $\beta_2 > 0$, it is easy to see from the condition $\lim_{x \uparrow \infty} \tilde{P}_1(x) = 0$ that $\tilde{a}_1 = 0$. For the remaining constants a_1, a_2, a_3, we use the continuity condition at $x = K$ and $x = x_0$, and differentiability condition at $x = K$ to obtain three linear equations. After a fair bit of algebra we can solve for the constants. Then, $\tilde{P}_1(x)$ is given by the formula stated in Proposition 2.

For the correction term in the expansion of the optimal exercise boundary, we refer to (11) and get

$$\tilde{x}_1 = -\frac{\partial \tilde{P}_1}{\partial x}\Big|_{x_0} \Big/ \frac{\partial^2 P_0}{\partial x^2}\Big|_{x_0}$$

where P_0 is given in (14) and $\tilde{P}_1$ is as derived above. □

A.2 Proof of Lemma 1

In this proof, we do not explicitly mention the dependence of $x_0(\cdot)$ on z for ease of notation. For smooth function P_0, we compute the following by differentiating (22) with respect to z

$$\frac{1}{2} f^2(z) x^2 \frac{\partial^2 \mathcal{V}}{\partial x^2} + r x \frac{\partial \mathcal{V}}{\partial x} - (r + \lambda)\mathcal{V} + f(z) f'(z) x^2 \frac{\partial^2 P_0}{\partial x^2} = 0$$

where $f'(z) = \partial f(z)/\partial z$. In order to solve the above equation, we first consider the homogeneous equation $\mathcal{L}_2 \mathcal{V} = 0$. We use a transformation to make the linear operator $\mathcal{L}_2$ which has (x, z)–dependent coefficients, into the z–dependent coefficient heat operator as shown below. To this end, we define the new variable $y = \log(x)$. Then, the transformation to the heat equation is

$$\frac{1}{2}f^2(z)\frac{\partial^2\mathcal{V}}{\partial y^2} + \Big(r - \frac{1}{2}f^2(z)\Big)\frac{\partial\mathcal{V}}{\partial y} - (r+\lambda)\mathcal{V} = 0.$$

In case of $x > K$, the source term is given by $c_1 x^{\beta_1}$ where

$$c_1 = -f(z)f'(z)\beta_1(\beta_1 - 1)\big(a_{01}/K^{\beta_1} + a_{02}/x_0^{\beta_1}\big).$$

We use variation of parameters technique to conclude that $\mathcal{V}$ is given by

$$\mathcal{V}(x,z) = a_1\big(\frac{x}{x_0}\big)^{\beta_1} + \frac{f'(z)\beta_1(\beta_1-1)}{f(z)\Delta}\log x\Big(a_{01}\big(\frac{x}{K}\big)^{\beta_1} + a_{02}\big(\frac{x}{x_0}\big)^{\beta_1}\Big), \text{ for } x > K.$$

Similarly, in the region $x_0 < x \leq K$, we get

$$\frac{1}{2}f^2(z)\frac{d^2\mathcal{V}}{dy^2} + \Big(r - \frac{1}{2}f^2(z)\Big)\frac{d\mathcal{V}}{dy} - (r+\lambda)\mathcal{V} = c_2 x^{\beta_1} + c_3 x^{\beta_2}$$

where

$$c_2 = -f(z)f'(z)\beta_1(\beta_1-1)\frac{a_{02}}{x_0^{\beta_1}}, \qquad c_3 = -f(z)f'(z)\beta_2(\beta_2-1)\frac{b_{01}}{K^{\beta_2}}.$$

The solution in this case is

$$\begin{aligned}\mathcal{V}(x,z) = a_2\big(\frac{x}{x_0}\big)^{\beta_1} + a_3\big(\frac{x}{K}\big)^{\beta_2} - \frac{f'(z)\log(x)}{f(z)\Delta}\Big(b_{01}\beta_2(\beta_2-1)\Big(\frac{x}{K}\Big)^{\beta_2} \\ - a_{02}\beta_1(\beta_1-1)\Big(\frac{x}{x_0}\Big)^{\beta_1}\Big), \text{ for } x_0 < x \leq K.\end{aligned}$$

From the expression of $\mathcal{V}$, we can see that for fixed z, it remains continuously differentiable with respect to x in the respective regions. Thus, we expect that the function and its first derivative also remain continuous across the boundary. Hence, we can find the unknown coefficients a_1, a_2 and a_3 by using the continuity condition at $x = x_0$ and differentiability condition at $x = K$. This procedure provides us the following:

$$\begin{aligned}a_1 &= -\frac{f'(z)}{2f(z)\Delta^2}\big(a_{01}\beta_1(\beta_1-1) + b_{01}\beta_2(\beta_2-1)\big)\Big(\big(\frac{x_0}{K}\big)^{\beta_2} - \big(\frac{x_0}{K}\big)^{\beta_1}\Big) \\ &\quad - \frac{f'(z)}{f(z)\Delta}\Big(\beta_1(\beta_1-1)\big(a_{01}\big(\frac{x_0}{K}\big)^{\beta_1}\log(K) + a_{02}\log(x_0)\big) - b_{01}\beta_2(\beta_2-1)\big(\frac{x_0}{K}\big)^{\beta_2}\log\big(\frac{x_0}{K}\big)\Big), \quad (26)\\ a_2 &= -\frac{f'(z)}{2f(z)\Delta^2}\big(\frac{x_0}{K}\big)^{\beta_2}\big(a_{01}\beta_1(\beta_1-1) + b_{01}\beta_2(\beta_2-1)(1 + 2\Delta\log(K))\big) \\ &\quad - \frac{f'(z)}{f(z)\Delta}\log(x_0)\big(a_{02}\beta_1(\beta_1-1) - b_{01}\beta_2(\beta_2-1)\big(\frac{x_0}{K}\big)^{\beta_2}\big), \quad (27)\\ a_3 &= \frac{f'(z)}{2f(z)\Delta^2}\big(a_{01}\beta_1(\beta_1-1) + b_{01}\beta_2(\beta_2-1)(1 + 2\Delta\log(K))\big). \quad (28)\end{aligned}$$

□

A.3 Proof of Proposition 3

For the region $x > K$, plugging back $\mathcal{V}$ in (23), we get the following PDE

$$\mathcal{L}_2 P_1 = c_4 x^{\beta_1} + c_5 x^{\beta_1}(1 + \beta_1\log(x))$$

with

$$c_4 = -\sqrt{2}\rho_2\nu_2\beta_1 a_1 f(z)/x_0^{\beta_1}, \qquad c_5 = -\frac{\sqrt{2}\rho_2\nu_2}{\Delta}\beta_1(\beta_1-1)f'(z)\big(a_{01}/K^{\beta_1} + a_{02}/x_0^{\beta_1}\big).$$

By defining a new variable $y = \log(x)$, we can reduce the above equation to a heat equation with only z–dependent coefficients and a source term. The transformation is given by,

$$\frac{1}{2}f^2(z)\frac{\partial^2 P_1}{\partial y^2} + \Big(r - \frac{1}{2}f^2(z)\Big)\frac{\partial P_1}{\partial y} - (r+\lambda)P_1 = c_4\exp(\beta_1 y) + c_5(1+\beta_1 y)\exp(\beta_1 y).$$

In the remaining proof, we do not explicitly mention the dependence of $x_0(\cdot)$ on z for ease of notation.The solution using variation of parameters technique is given by

$$P_1(x,z) = \tilde{\eta}_1 x^{\beta_1} - \frac{c_4}{\Delta f^2(z)}x^{\beta_1}\log(x) - \frac{c_5}{2\Delta^2 f^2(z)}x^{\beta_1}\log(x)(\beta_2 + \beta_1\Delta\log(x)), \text{ for } x > K,$$

where $\tilde{\eta}_1$ is an unknown coefficient dependent on z. For $x_0 < x \le K$, plugging back $\mathcal{V}$ in (23), we get the following PDE

$$\mathcal{L}_2 P_1 = c_6 x^{\beta_1} + c_7 x^{\beta_2} - c_8 x^{\beta_2}(1+\beta_2\log(x)) + c_9 x^{\beta_1}(1+\beta_1\log(x))$$

with

$$c_6 = -\sqrt{2}\rho_2\nu_2 f(z) a_2\beta_1/x_0^{\beta_1}, \qquad c_7 = -\sqrt{2}\rho_2\nu_2 f(z) a_3\beta_2/K^{\beta_2}$$

$$c_8 = \frac{-\sqrt{2}\rho_2\nu_2 f'(z)\beta_2(\beta_2-1)}{\Delta}\frac{b_{01}}{K^{\beta_2}} \qquad c_9 = \frac{-\sqrt{2}\rho_2\nu_2 f'(z)\beta_1(\beta_1-1)}{\Delta}\frac{a_{02}}{x_0^{\beta_1}}.$$

We repeat the procedure discussed above to get, for $x_0 < x \le K$,

$$\begin{aligned}P_1(x,z) &= \tilde{\eta}_2 x^{\beta_1} + \tilde{\eta}_3 x^{\beta_2} - \frac{c_6}{\Delta f^2(z)}x^{\beta_1}\log(x) + \frac{c_7}{\Delta f^2(z)}x^{\beta_2}\log(x)\\ &\quad + \frac{c_8}{2\Delta^2 f^2(z)}x^{\beta_2}\log(x)(\beta_1 - \Delta\beta_2\log(x)) - \frac{c_9}{2\Delta^2 f^2(z)}x^{\beta_1}\log(x)(\beta_2 + \Delta\beta_1\log(x))\end{aligned}$$

where $\tilde{\eta}_2$ and $\tilde{\eta}_3$ are unknown coefficients dependent on z. Let us define $V_2(z) := \frac{\sqrt{2}\rho_2\nu_2}{\Delta f^2(z)}$. Then, we have

$$\begin{aligned}P_1(x,z) &= \eta_1\big(\frac{x}{x_0}\big)^{\beta_1} + V_2(z)a_1 f(z)\beta_1\log(x)\big(\frac{x}{x_0}\big)^{\beta_1} + \frac{V_2(z)f'(z)}{2\Delta^2}\beta_1(\beta_1-1)\log(x)\big(\beta_2\\ &\qquad \times + \beta_1\Delta\log(x)\big)\Big(a_{01}\big(\frac{x}{K}\big)^{\beta_1} + a_{02}\big(\frac{x}{x_0}\big)^{\beta_1}\Big), \qquad \text{for } x > K,\\ P_1(x,z) &= \eta_2\big(\frac{x}{x_0}\big)^{\beta_1} + \eta_3\big(\frac{x}{K}\big)^{\beta_2} + V_2(z)a_2 f(z)\beta_1\log(x)\big(\frac{x}{x_0}\big)^{\beta_1} - V_2(z)a_3 f(z)\beta_2\log(x)\big(\frac{x}{K}\big)^{\beta_2}\\ &\quad - \frac{V_2(z)f'(z)}{2\Delta^2}\beta_2(\beta_2-1)b_{01}\log(x)\big(\beta_1 - \beta_2\Delta\log(x)\big)\big(\frac{x}{K}\big)^{\beta_2}\\ &\quad + \frac{V_2(z)f'(z)}{2\Delta^2}\beta_1(\beta_1-1)a_{02}\log(x)\big(\beta_2 + \beta_1\Delta\log(x)\big)\big(\frac{x}{x_0}\big)^{\beta_1}, \qquad \text{for } x_0 \le x \le K,\end{aligned}$$

for unknown coefficients η_1, η_2 and η_3. Next, we use the continuity condition at $x = x_0$ and $x = K$ and differentiability condition at $x = K$ to calculate these coefficients which are given as follows

$$
\begin{aligned}
\eta_1 = V_2(z)\Big[& a_{01}\beta_1\beta_2(\beta_1-1)f'(z)\Big(1-\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}\Big) - 2a_{01}\Delta^2\beta_1(\beta_1-1)f'(z)\log(K)\big(2+\beta_1\log(K)\big) \\
& - 2a_{01}\Delta\beta_1^2(\beta_1-1)f'(z)\log(K)\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2} \\
& - 2a_{02}\Delta\beta_1(\beta_1-1)f'(z)\log(x_0)\big(\frac{K}{x_0}\big)^{\beta_1}\big(\beta_2+\Delta\beta_1\log(x_0)\big) \\
& - b_{01}\beta_1\beta_2(\beta_2-1)f'(z)\Big(\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}-1\Big) + 2b_{01}\Delta\beta_2^2(\beta_2-1)f'(z)\log(K)\Big(\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}-1\Big) \\
& - 2b_{01}\Delta\beta_1\beta_2(\beta_2-1)f'(z)\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}\big(\log(K)-\log(x_0)\big) \\
& + 2b_{01}\Delta^2\beta_2^2(\beta_2-1)f'(z)\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}\big(\log^2(K)-\log^2(x_0)\big) \\
& - 2(a_1-a_2)\Delta^2\beta_1\beta_2 f(z)\log(K)\big(\frac{K}{x_0}\big)^{\beta_1} - 2(a_1-a_2)\Delta^2\beta_1 f(z)\big(\frac{K}{x_0}\big)^{\beta_1}\Big(\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}-\beta_1\log(K)\Big) \\
& + 2a_2\Delta^2\beta_1 f(z)\big(\frac{K}{x_0}\big)^{\beta_1}\big(\beta_1\log(x_0)-1\big) - 2(a_2\beta_2\log(x_0)-a_1)\Delta^2\beta_1 f(z)\big(\frac{K}{x_0}\big)^{\beta_1} \\
& - 2a_3\Delta^2\beta_2 f(z)\Big(\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}-1\Big) \\
& - 4a_3\Delta^3\beta_2 f(z)\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2}\big(\log(K)-\log(x_0)\big)\Big]\Big/\big(4\Delta^3(K/x_0)^{\beta_1}\big), \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\eta_2 = -V_2(z)\Big[& a_{01}\beta_1\beta_2(\beta_1-1)f'(z)\big(\frac{x_0}{K}\big)^{\beta_2} + 2a_{01}\Delta\beta_1^2(\beta_1-1)f'(z)\log(K)\big(\frac{x_0}{K}\big)^{\beta_2} \\
& - 4a_{02}\Delta^2\beta_1\beta_2 f'(z)\log(x_0) + 2a_{02}\Delta^2\beta_1^2(\beta_1-1)f'(z)\log^2(x_0) \\
& + b_{01}\beta_1\beta_2(\beta_2-1)f'(z)\big(\frac{x_0}{K}\big)^{\beta_2} - 2b_{01}\Delta\beta_2^2(\beta_2-1)f'(z)\log(K)\big(\frac{x_0}{K}\big)^{\beta_2} \\
& + 2b_{01}\Delta\beta_1\beta_2(\beta_2-1)f'(z)\big(\frac{x_0}{K}\big)^{\beta_2}\big(\log(K)-\log(x_0)\big) \\
& - 2b_{01}\Delta^2\beta_2^2(\beta_2-1)f'(z)\big(\frac{x_0}{K}\big)^{\beta_2}\big(\log^2(K)-\log^2(x_0)\big) \\
& + 2(a_1-a_2)\Delta^2\beta_1 f(z)\big(\frac{K}{x_0}\big)^{\beta_1-\beta_2} + 4a_2\Delta^2\beta_1 f(z)\log(x_0) \\
& + 2a_3\Delta^2\beta_2 f(z)\big(\frac{x_0}{K}\big)^{\beta_2} + 2a_3\Delta^3\beta_2 f(z)\big(\frac{x_0}{K}\big)^{\beta_2}\big(\log(K)-\log(x_0)\big)\Big]\Big/4\Delta^3, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\eta_3 = V_2(z)\Big[& a_{01}\beta_1(\beta_1-1)f'(z)\big(\beta_2+2\Delta\beta_1\log(K)\big) \\
& + b_{01}\beta_2(\beta_2-1)f'(z)\Big(\beta_1-2\Delta^2\log(K)\big(2+\beta_2\log(K)\big)\Big) \\
& + 2a_3\Delta^2\beta_2 f(z)\big(1+2\Delta\log(K)\big) + 2(a_1-a_2)\Delta^2\beta_1 f(z)\big(\frac{K}{x_0}\big)^{\beta_1}\Big]\Big/4\Delta^3. \qquad (31)
\end{aligned}
$$

□

B Implementation of control variates

In order to achieve variance reduction for the proposed Monte Carlo price estimator, we use control variates as introduced in Sections 2.2 and 3.2. Here, we discuss their implementation in the numerical examples.

B.1 Fast Factor Control Variate

Let us recall the centered martingale which is used as a control variate under the fast mean-reverting stochastic volatility. It is given by

$$\hat{\mathcal{U}}_0(P^{(1)};x_b) = \int_0^{\hat{\tau}} e^{-rs}\frac{\partial}{\partial x}(P_0 + \sqrt{\varepsilon}P_1)(X_s)f(Y_s)X_s dW_s^{(1)}$$

where $\hat{\tau}$ is the exercise time approximated using the boundary approximation $x_0 + \sqrt{\varepsilon}x_1$.

In the simulation method, we discretize the time scale with a time step $\Delta t = 10^{-3}$. Thus, the total number of time steps required to sample a path of the underlying process X for pricing an American put with maturity T is $L = T/\Delta t$. Next, we generate a set of N independent underlying sample paths for X and Y using the Euler discretization scheme and denote them as $\big(X_0^{(i)}, X_1^{(i)}, \cdots, X_L^{(i)}\big)_{i=1}^N$ and $\big(Y_0^{(i)}, Y_1^{(i)}, \cdots, Y_L^{(i)}\big)_{i=1}^N$ where to simplify the notation we have used $X_n^{(i)} := X_{n\Delta t}^{(i)}$ and $Y_n^{(i)} := Y_{n\Delta t}^{(i)}$ for $n = 0, 1, \cdots, L$. On these sample paths, we use the exercise policy based on the approximate optimal exercise time defined as

$$\hat{\tau} := \min\big\{n \geq 0 : X_n \leq x_0 + \sqrt{\varepsilon}x_1\big\} \wedge L.$$

The control variate $C^{(i)}$ which approximates $\hat{\mathcal{U}}_0(P^{(1)};x_b)$ for the i^{th} underlying path is formed as follows by using the explicit formulas for the derivatives of P_0 and P_1

$$C^{(i)} := \sum_{n=0}^{\hat{\tau}^{(i)}-1} e^{-r(n+1)\Delta t}\Big[DP_0(X_n^{(i)}) + \sqrt{\varepsilon}DP_1(X_n^{(i)})\Big] f(Y_n^{(i)})X_n^{(i)}\sqrt{\Delta t}N_{i,n+1}^{(1)}$$

where

$$DP_0(X_n) := \begin{cases} a_{01}\frac{\beta_1}{K}\big(\frac{X_n}{K}\big)^{\beta_1-1} + a_{02}\frac{\beta_1}{x_0}\big(\frac{X_n}{x_0}\big)^{\beta_1-1} & \text{if} \quad X_n > K, \\ b_{01}\frac{\beta_2}{K}\big(\frac{X_n}{K}\big)^{\beta_2-1} + a_{02}\frac{\beta_1}{x_0}\big(\frac{X_n}{x_0}\big)^{\beta_1-1} - 1 & \text{if} \quad x_0 < X_n \leq K, \\ -1 & \text{if} \quad X_n \leq x_0, \end{cases}$$

$$\sqrt{\varepsilon}DP_1(X_n) := \begin{cases} \hat{a}_1\frac{\beta_1}{x_0}\left(\frac{X_n}{x_0}\right)^{\beta_1-1} - \frac{V_3\beta_1^2(\beta_1-1)}{\bar{\sigma}^2\Delta K}\left(a_{01}\left(\frac{X_n}{K}\right)^{\beta_1-1}\right. & \\ \left.+a_{02}\left(\frac{X_n}{x_0}\right)^{\beta_1-1}\right)(1+\beta_1\log(X_n)) & \text{if} \quad X_n > K, \\ \hat{a}_3\frac{\beta_1}{x_0}\left(\frac{X_n}{x_0}\right)^{\beta_1-1} + \hat{a}_4\frac{\beta_2}{x_0}\left(\frac{X_n}{K}\right)^{\beta_2-1} & \\ +\frac{V_3}{\bar{\sigma}^2\Delta}\left(\frac{b_{01}\beta_2^2(\beta_2-1)}{K}\left(\frac{X_n}{K}\right)^{\beta_2-1}(1+\beta_2\log(X_n))\right. & \\ \left.-\frac{a_{02}\beta_1^2(\beta_1-1)}{x_0}\left(\frac{X_n}{x_0}\right)^{\beta_1-1}(1+\beta_1\log(X_n))\right) & \text{if} \quad x_0 < X_n \leq K, \\ 0 & \text{if} \quad X_n \leq x_0, \end{cases}$$

and $\big(N_{i,1}^{(1)}, \cdots, N_{i,\hat{\tau}^{(i)}}^{(1)}\big)$ is the realization of a sequence of independent standard normal random variables generated to simulate the i^{th} sample path of Brownian motion $W^{(1)}$ in the Euler scheme.

B.2 Slow Factor Control Variate

In the case of slowly fluctuating volatility, the first martingale is

$$\hat{\mathcal{U}}_0(P^{(1)};x_b) = \int_0^{\hat{\tau}} e^{-rs}\frac{\partial}{\partial x}(P_0 + \sqrt{\delta}P_1)(X_s, Z_s)f(Z_s)X_s dW_s^{(1)}.$$

Here, the derivative of P_0 and P_1 with respect to x can be calculated explicitly from the results in Section 3. The other centered martingale is

$$\hat{\mathcal{U}}_2(P^{(1)}; x_b) = \int_0^{\hat{\tau}} e^{-rs} \frac{\partial}{\partial z}(P_0 + \sqrt{\delta} P_1)(X_s, Z_s) h(Z_s) dW_s^{(3)}. \tag{32}$$

In (32), the derivative $\partial P_0/\partial z$ is denoted as $\mathcal{V}$ which is given explicitly in Lemma 1. The main difficulty is to calculate the derivative of P_1 with respect to z as it has an implicit dependence on the variable. To overcome this issue, we use a finite difference estimate of $\partial P_1/\partial z$. We suppress the dependence of $x_0(\cdot)$ on z for notational convenience.

Thus, the control variate $C^{(i)}$ which approximates $\hat{\mathcal{U}}_0(P^{(1)}; x_b) + \sqrt{\delta}\hat{\mathcal{U}}_2(P^{(1)}; x_b)$ for the i^{th} underlying path is $C^{(i)} := C_1^{(i)} + \sqrt{\delta} C_2^{(i)}$, where

$$C_1^{(i)} := \sum_{n=0}^{\hat{\tau}^{(i)}-1} e^{-r(n+1)} \Big[DP_0(X_n^{(i)}, Z_n^{(i)}) + \sqrt{\delta} DP_1(X_n^{(i)}, Z_n^{(i)}) \Big] f(Z_n^{(i)}) X_n^{(i)} \sqrt{\Delta t} N_{i,n+1}^{(1)},$$

$$DP_0(X_n, Z_n) := \begin{cases} a_{01} \frac{\beta_1}{K} \left(\frac{X_n}{K}\right)^{\beta_1 - 1} + a_{02} \frac{\beta_1}{x_0} \left(\frac{X_n}{x_0}\right)^{\beta_1 - 1} & \text{if} \quad X_n > K, \\ b_{01} \frac{\beta_2}{K} \left(\frac{X_n}{K}\right)^{\beta_2 - 1} + a_{02} \frac{\beta_1}{x_0} \left(\frac{X_n}{x_0}\right)^{\beta_1 - 1} - 1 & \text{if} \quad x_0 < X_n \le K, \\ -1 & \text{if} \quad X_n \le x_0, \end{cases}$$

$$DP_1(X_n, Z_n) := \begin{cases} \frac{\eta_1 \beta_1}{X_n} \left(\frac{X_n}{x_0}\right)^{\beta_1} + \frac{V_2(Z_n) a_1 \beta_1 f(Z_n)}{X_n} \left(\frac{X_n}{x_0}\right)^{\beta_1} (1 + \beta_1 \log(X_n)) \\ + \frac{V_2(Z_n) \beta_1 (\beta_1 - 1) f'(Z_n)}{2\Delta^2 X_n} \left(a_{01} \left(\frac{X_n}{K}\right)^{\beta_1} + a_{02} \left(\frac{X_n}{x_0}\right)^{\beta_1} \right) \\ \times \big(\beta_2 (1 + \beta_1 \log(X_n)) + \Delta \beta_1 \log(X_n)(2 + \beta_1 \log(X_n)) \big) & \text{if } X_n > K, \\ \frac{\eta_2 \beta_1}{X_n} \left(\frac{X_n}{x_0}\right)^{\beta_1} + \frac{\eta_3 \beta_2}{X_n} \left(\frac{X_n}{K}\right)^{\beta_2} \\ + \frac{V_2(Z_n) a_2 \beta_1 f(Z_n)}{X_n} \left(\frac{X_n}{x_0}\right)^{\beta_1} \big(1 + \beta_1 \log(X_n)\big) \\ - \frac{V_2(Z_n) a_3 \beta_2 f(Z_n)}{X_n} \left(\frac{X_n}{K}\right)^{\beta_2} \big(1 + \beta_2 \log(X_n)\big) \\ - \frac{V_2(Z_n) b_{01} \beta_2 (\beta_2 - 1) f'(Z_n)}{2\Delta^2 X_n} \left(\frac{X_n}{K}\right)^{\beta_2} \\ \times \Big(\big(\beta_1 - \Delta \beta_2 \log(X_n)\big)\big(1 + \beta_2 \log(X_n)\big) - \Delta \beta_2 \log(X_n) \Big) \\ + \frac{V_2(Z_n) a_{02} \beta_1 (\beta_1 - 1) f'(Z_n)}{2\Delta^2 X_n} \left(\frac{X_n}{x_0}\right)^{\beta_1} \\ \times \Big(\big(\beta_2 + \Delta \beta_1 \log(X_n)\big)\big(1 + \beta_1 \log(X_n)\big) + \Delta \beta_1 \log(X_n) \Big) & \text{if } x_0 < X_n \le K, \\ 0 & \text{if } X_n \le x_0. \end{cases}$$

Further,

$$C_2^{(i)} := \sum_{n=0}^{\hat{\tau}^{(i)}-1} e^{-r(n+1)\Delta t} \Big[\mathcal{V}(X_n^{(i)}, Z_n^{(i)}) + \sqrt{\Delta} P_1^z(X_n^{(i)}, Z_n^{(i)}) \Big] \sqrt{2} \nu_2 \sqrt{\Delta t} \Big(\rho_2 N_{i,n+1}^{(1)} + \sqrt{1 - \rho_2^2} N_{i,n+1}^{(3)} \Big),$$

and for small $h > 0$,

$$P_1^z(X_n, Z_n) := \frac{-P_1(X_n, Z_n + 2h) + 4P_1(X_n, Z_n + h) - 3P_1(X_n, Z_n)}{h}.$$

$\big(N_{i,1}^{(1)}, \cdots, N_{i,\hat{\tau}^{(i)}}^{(1)}\big)$ and $\big(N_{i,1}^{(3)}, \cdots, N_{i,\hat{\tau}^{(i)}}^{(3)}\big)$ are the realizations of sequence of independent standard normal random variables generated to simulate the i^{th} sample path of Brownian motion $W^{(1)}$ and $W^{(3)}$ in the Euler scheme.

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