Optimal Trading with Predictable Return and Stochastic Volatility

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Abstract

We consider a class of dynamic portfolio optimization problems that allow for models of return predictability, transaction costs, and stochastic volatility. Determining the dynamic optimal portfolio in this general setting is almost always intractable. We propose a multiscale asymptotic expansion when the volatility process is characterized by its time scales of fluctuation. The analysis of the nonlinear Hamilton-Jacobi-Bellman PDE is a singular perturbation problem when volatility is fast mean-reverting; and it is a regular perturbation when the volatility is slowly varying. These analyses can be combined for multifactor multiscale stochastic volatility model. We present formal derivations of asymptotic approximations and demonstrate how the proposed algorithms improve our Profit&Loss using Monte Carlo simulations.

1 Introduction

Dynamic portfolio optimization provides institutional investors in active asset management a framework for determining optimal investment strategies. This central and essential problem has a long history dating back to Mossin (1968), Samuelson (1969), and Merton (1969, 1971). In his seminal paper, Merton (1971) derives explicit solutions for the continuous-time portfolio optimization problem. In this classical setting, the stocks are modeled as geometric Brownian motions (with *constant* volatilities), and the objective is to maximize the expected utility of terminal wealth by allocating investment capital between risky stocks and a riskless money-market account. Under the constant relative risk aversion (CRRA) utility, Merton shows that the optimal control is a fixed mix strategy.

While this work has brought forth important structural insights, its restrictive assumptions about investor objectives and market dynamics (necessary for exact analytical solutions) have prevented more widespread applications to practical trading algorithms. Following this seminal paper, there has been a significant literature aiming to relax its assumptions and to incorporate the impact of various frictions, such as transaction costs and stochastic volatility, on the optimal portfolio choice.

A tractable alternative is to formulate the dynamic portfolio optimization problem as a linear-quadratic control. Gârleanu and Pedersen (2013) derive a closed-form optimal dynamic portfolio policy for a model with linear dynamics for return predictors, quadratic transaction costs, and quadratic penalty terms for risk. However, the explicit solution depends sensitively on the quadratic cost structure with linear dynamics. The goal of this article is to study the dynamic portfolio optimization problem allowing for more realistic market dynamics without sacrificing model tractability. Specifically, our model captures a number of common features of practical interest, while maintaining tractability by viewing the more flexible model as a perturbation around the well-understood constant volatility problem.

Return predictability. The usual goals of hedge fund managers and proprietary traders are to predict future security returns and trade to profit from their predictions. Such predictions are not limited to

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simple unconditional bullish or bearish forecasts of future returns, but often involve predictions on short and long-term expected returns using a factor-based approach such as momentum and mean-reversion. Different factors often have different predicting strengths and mean-reversion speeds.

Transaction costs. Dynamic portfolio optimization often involves frequent turnover and hence significant transaction costs. Such trading costs can arise from sources ranging from the bid-offer spread or execution commissions to price impact, where the manager's own trading affects the subsequent evolution of prices. Intuitively, the investor would like to keep his portfolio close to the "optimal" portfolio in the absence of transaction costs; however, due to transaction costs, it may not be optimal to trade all the way to the target all the time.

Stochastic volatility. Stochastic volatility has been recognized as an important factor of asset price modeling because it is seen as an explanation of a number of well-known empirical findings such as volatility smile and volatility clustering. The need for multifactor modeling of the volatility process is alluded to by Chacko and Viceira (2005); they observe that vastly different estimates of the mean-reversion speed of volatility can be obtained by using high and low frequency data.

In this paper, our central innovation is to propose a framework for the dynamic trading problems allowing for many features relevant for practical trading algorithms described above. Our formulation maintains the tractability of the Gârleanu and Pedersen problem by analyzing the dynamic trading problem under stochastic volatility under the lens of multiscale asymptotics. We further demonstrate that our formulation provides explicit correction terms to the constant volatility strategy which can be efficiently computed for a large class of volatility models of practical interest; moreover, through Monte-Carlo simulations, we show how the proposed algorithms improve the trading profit&loss.

Specifically, our dynamic portfolio optimization problem is analytically tractable. In many stochastic volatility models of practical interest (e.g. Heston, exponential Ornstein-Uhlenbeck, and the 3/2-model), the correction terms to the constant volatility strategies can be explicitly given. Moreover, the correction terms give rise to economically sensible trading strategies. We find that under fast-scale stochastic volatility, the investor should optimally deleverage his portfolio when the current volatility level is higher than the long-term average, regardless of the return-volatility correlation. On the other hand, the return-volatility correlation plays a more important role under the slow-scale stochastic volatility. When the correlation between the volatility and return factors is positive, the investor optimally decreases his trading rate as he anticipates a higher return estimate is accompanied by a higher volatility. Moreover, we demonstrate that the effect of slow-scale stochastic volatility is more significant than the fast-scale volatility in our infinite-horizon optimal trading problem. In fact, the leading order correction in the fast-scale volatility expansion vanishes identically and one has compute the second order expansion to consider the principal effect of fast-scale volatility.

1.1 Literature review

Our paper is related to three different strands of literature: the literature of dynamic portfolio choice with return predictability and transaction costs, the modeling of price impact in algorithmic trading, and the use of asymptotic approximation in the presence of multiscale stochastic volatility.

First, we consider the literature on dynamic portfolio choice. The vast body of work begins with the seminal paper of Merton (1971). Following this paper, there has been a significant literature aiming to incorporate various frictions, such as transaction costs, stochastic volatility, and partial information, on the dynamic portfolio optimization problem. Transaction costs were first introduced into the Merton portfolio problem by Magill and Constantinides (1976) and later further investigated by Dumas and Luciano (1991). Liu and Loewenstein (2002) study the optimal trading strategy for a CRRA investor in the presence of transaction costs and obtain closed-form solutions when the finite horizon is uncertain. Bichuch and Sircar (2014) analyze the problem using asymptotic approximations and find approximations to the optimal policy and the optimal long-term growth rate.

There is also significant literature on portfolio optimization that incorporates return predictability (see, e.g., Campbell and Viceira (1996)). Balduzzi and Lynch (1999, 2000) illustrate the impact of return predictability and transaction costs on the utility costs and the optimal rebalancing rule by discretizing the sate space of the dynamic program. Wachter (2002) solves, in closed form, the optimal portfolio choice problem for an investor with utility over consumption under mean-reverting returns without transaction costs.

Several authors have also considered the portfolio problems under more realistic market dynamics such as stochastic interest rates or stochastic volatility. For the case of stochastic interest rates the reader is referred to Korn and Kraft (2002). Kraft (2005), Boguslavskaya and Muravey (2015) consider a variation of the Merton problem within the framework of the Heston model and finite time horizon. Chacko and Viceira (2005) consider a similar problem with a different specification of the market price of risk and a slightly different stochastic volatility model; they also note the need for multifactor volatility model to adequately capture the persistence and variability characteristics of the volatility process that are most relevant to long-term investors. Fouque et al. (2013) build on this empirical observation and study the Merton portfolio optimization problem in the presence of multiscale stochastic volatility using asymptotic approximations.

Moreover, there is also an emerging body of literature on partial information and expert opinions. Fouque et al. (2014) analyze the Merton problem when the growth rate is an unobserved Gaussian process. By applying the Kalman filter on observations of the stock price, they track the level of the growth rate and determine the optimal portfolio maximizing expected terminal utility. Frey et al. (2012) investigate optimal portfolio strategies in a market where the drift is driven by an unobserved Markov chain. Information on the state of this chain is obtained from stock prices and expert opinions in the form of signals at random discrete time points. Using hidden Markov model filtering results and Malliavin calculus, Sass and Haussmann (2004) numerically determine the optimal strategy under a multi-stock market model where prices satisfy a stochastic differential equation with instantaneous rates of return modeled as a continuous time Markov chain with finitely many states.

Gârleanu and Pedersen (2013) achieve a closed-form solution for a model with linear dynamics for return predictors, quadratic functions for transaction costs, and quadratic penalty terms for risk. Glasserman and Xu (2013) develop a linear-quadratic formulation for portfolio optimization that offers robustness to modeling errors or mis-specifications. Moallemi and Saglam (2012) allow for more flexible models with trading constraints and risk considerations, but at the expense of restricting to the class of linear rebalancing policies. In similar spirit, Passerini and Vazquez (2015) extend the model of Gârleanu and Pedersen (2013) to include linear trading costs and using both limit and market orders. They find that the presence of linear costs induces a "no-trading" zone when using market orders, and a corresponding "market-making" zone when using limit orders. The more complex models are not analytically tractable, and Passerini and Vazquez propose a heuristic "recipe" that approximates the value function by dropping certain terms in the Hamilton-Jacobi-Bellman (HJB) equation.

Second, there is a large body of work on the modeling of price impact in algorithmic trading. The typical problem studied in this literature is the so-called "optimal execution problem." This arises when an investor holding a large number of shares wants to liquidate his position over a given horizon. Rapid selling of the stock may depress the stock price, while order slicing may add to the uncertainty in the sale price. This tradeoff between expected execution cost and risk is first formulated by Almgren and Chriss in a couple of seminal papers (Almgren and Chriss, 1999, 2001). Under the assumptions that execution costs are linear in the trading rate and the choice of risk criterion is the quadratic variation, Almgren and Chriss derive a closed-form analytical solution to the optimal execution problem.

The Almgren and Chriss model has then been generalized in various directions. A number of authors have investigated the optimal execution problem with respect to different risk criteria. For example, Schied and Schöneborn (2009) consider the maximization of expected utility of the proceeds of an asset sale; while Gatheral and Schied (2011) quantify the risk associated with a liquidation strategy as the time-averaged value-at-risk (VaR) and provide a closed-form solution to the optimal execution problem. More recently, limit order book dynamics has been incorporated into the optimal execution problem. This leads to the concept of transient price impact, that is, price impact that decays over time. Obizhaeva and Wang (2013) proposed one of the first models for linear transient price impact. This model has been generalized by

Gatheral et al. (2012) and Alfonsi et al. (2012). For a recent survey of the market impact models used in algorithmic order execution, we refer to Gatheral and Schied (2013).

Third, there is also a literature on the use of asymptotic approximation in the presence of multiscale stochastic volatility. This approximation technique has attracted considerable interest recently in derivative pricing and optimal investment problems. As detailed in the recent book of Fouque et al. (2011), singular and regular perturbation methods have been developed over a number of years to provide effective approximations for the *linear* option pricing problems.

More recently, asymptotic analysis has been extended to simplify a number of *nonlinear* problems. Jonsson and Sircar (2002a,b) apply singular perturbation to the partial hedging problem and optimal investment problem, both for fast mean-reverting stochastic volatility. Fouque et al. (2013) extend the results for the nonlinear Merton problem for general utility functions using multiscale stochastic volatility asymptotics. While the basic solution approach is similar, we stress that our work differs from these papers in several critical ways. First and foremost, transaction costs are not taken into account in the aforementioned literature; and we believe that the explicit modeling of transaction costs is crucial for a practical trading algorithm to keep turnover under control. Second, we consider a mean-variance optimization problem with an infinite trading horizon, which is more popular among industry practitioners. Finally, as opposed to the asymptotic expansion in Fouque et al. (2013), we compute explicitly up to the second order correction in the fast-scale stochastic volatility.

Summary In the table below we summarize the models for dynamic trading in the literature. Type refers to continuous or discrete-time model; (g)BM stands for (geometric) Brownian motion; SV stands for stochastic volatility; pred. stands for predictability; proportional refers to proportional transaction costs.

Table 1: Dynamic trading models: problems, models and solution approaches.

	Type	Price dynamics	Trading friction	Objective	Solution
Merton (1971)	Continuous	gBM	None	Utility	Analytic
Liu and Loewenstein (2002)	Continuous	gBM	proportional	Utility	Analytic
Bichuch and Sircar (2014)	Continuous	gBM+SV	proportional	Utility	Asymptotic
Kraft (2005)	Continuous	gBM+Heston	None	Utility	Analytic
Chacko and Viceira (2005)	Continuous	gBM+3/2	None	Utility	Analytic
Fouque et al. (2013)	Continuous	gBM+SV	None	Utility	Asymptotic
Fouque et al. (2014)	Continuous	gBM+pred.	None	Utility	Asymptotic
Gârleanu and Pedersen (2009, 2013)	Continuous/discrete	BM+pred.	Quadratic	Mean-variance	Analytic
Moallemi and Saglam (2012)	Discrete	BM+pred.	Quadratic	Mean-variance	linear rebalancing rules
Passerini and Vazquez (2015)	Continuous	BM+pred.	Linear-quadratic	Mean-variance	Approximate
This paper	**Continuous**	**BM+pred.+SV**	**Quadratic**	**Mean-variance**	**Asymptotic**

Some qualitative effects of stochastic volatility mean-reversion on the optimal trading strateiges are discussed in the recent working paper Gârleanu and Pedersen (2014).

1.2 Organization and Results

In Section 2 we introduce the Gârleanu and Pedersen (2013) model in discrete time. This section serves to provide some structural intuitions of the optimal trading problem. Section 3 introduces the continuous-time model with multiscale stochastic volatility. We derive the HJB equation for the optimal portfolio problem and give the analytical solution in the special case of constant volatility. To keep the presentation manageable, we focus on the analysis of the two factors separately. We begin in Section 4 with the case of fast mean-reverting stochastic volatility, which leads to a singular perturbation problem for the associated HJB PDE. In Section 5, we analyze the case of slowly fluctuating volatility, which leads to a regular perturbation problem. Section 6 discusses how the fast and slow results can be combined for approximations under multiscale stochastic volatility. In Section 7, we illustrate our results with numerical examples. Section 8 concludes and suggests directions of extension.

2 Introduction in discrete time

The goal of this section is to use discrete-time dynamic programming to illustrate how transaction costs influence investment decisions. In their seminal paper, Gârleanu and Pedersen (2013) examine a dynamic, transaction-cost-sensitive version of the Markowitz portfolio optimization problem (1968; 1952) with multiple stocks and multiple return predictors, examining how the portfolio should dynamically be adjusted as new information arrives. For expositional purposes, we will focus on the special case when there is just a single stock and a single return predictor.

Denote by q_t the investor's position in this stock at time t. The excess return is given by $r_{t+1} = P_{t+1} - (1+r^f)P_t$, where P_t is stock price at time t and r^f is the risk-free rate. We suppose that at each time t, the investor has an estimate of the stock's anticipated return x_t, so that

$$r_{t+1} = x_t + \widehat{\varepsilon}_{t+1},$$

where $\widehat{\varepsilon}$ is white noise with mean zero and variance σ^2. We assume mean-reverting dynamics for x_t:

$$x_{t+1} - x_t = -\varphi x_t + \varepsilon_t,$$

where ε is an independent white noise with mean zero and variance Ω.

We assume quadratic transaction costs: a trade of size Δq incurs transaction costs $\frac{1}{2}K(\Delta q)^2$ for some constant $K > 0$. The interpretation is that trades move the market transiently by an amount linear in the trade size Δq.

2.1 Investor's problem and dynamic programming

At time 0, starting with position q_{-1} of stock and return estimate x_0 for the coming period's return, the investor chooses q_0 to maximize the discounted lifetime risk-adjusted return less transaction cost, i.e.

$$V(q_{-1}, x_0) = \max_{q_0, q_1, \ldots} \mathbb{E}\left[\sum_{t=0}^{\infty}(1-\rho)^{t+1}\left(r_{t+1}q_t - \frac{\gamma}{2}\sigma^2 q_t^2\right) - \frac{(1-\rho)^t}{2}\left(\Delta q_t\right)^2 K\right],$$

where the constant γ is a risk aversion parameter and ρ is the discount rate. The principle of dynamic programming states that

$$V(q_{-1}, x_0) = \max_{q_0}\left\{-\frac{1}{2}\left(\Delta q_0\right)^2 K + (1-\rho)\left(q_0 x_0 - \frac{\gamma}{2}\sigma^2 q_0^2\right) + (1-\rho)\mathbb{E}\left[V(q_0, x_1)\right]\right\}. \tag{1}$$

This is a linear-quadratic stochastic control problem, so it is natural to use the linear-quadratic *ansatz*

$$V(q, x) = -\frac{1}{2}A_{qq}q^2 + A_{qx}qx + \frac{1}{2}A_{xx}x^2 + A_0 \tag{2}$$

for some constants $A_{qq}, A_{qx}, A_{xx}, A_0$. To find these constants and the optimal investment policy, we substitute the ansatz (2) into the dynamic programming equation (1). The left hand side reads

$$-\frac{1}{2}A_{qq}q_{-1}^2 + A_{qx}q_{-1}x_0 + \frac{1}{2}A_{xx}x_0^2 + A_0;$$

while the right hand side is a quadratic polynomial in q_0:

$$\begin{aligned}\max_{q_0}\Bigg\{ & -\frac{1}{2}q_0^2\left(K + (1-\rho)\gamma\sigma^2 + (1-\rho)A_{qq}\right) \\ & + q_0\left(x_0(1-\rho)(1-\varphi)A_{qx} + x_0(1-\rho) + Kq_{-1}\right) \\ & + \left(\frac{1}{2}(1-\rho)A_{xx}\left(x_0^2(1-\varphi)^2 + \Omega^2\right) + A_0(1-\rho) - \frac{1}{2}Kq_{-1}^2\right)\Bigg\}.\end{aligned}$$

Writing this as $-\frac{1}{2}q_0^2 P + q_0 Q + R$ we see that $q_0 = P/Q$ and the maximum is

$$\frac{1}{2}\frac{Q^2}{P} + R,$$

and matching coefficients determines the values of A_{qq}, A_{qx}, A_{xx}, and A_0.

2.2 Interpretation of the optimal policy

Differentiating the dynamic programming equation (1) with respect to q_{-1} gives

$$-A_{qq}q_{-1} + A_{qx}x_0 = -(q_{-1} - q_0)K. \tag{3}$$

To interpret this relation, let q_* maximize the value function for given x_0, i.e.

$$q_* = \arg\max_q V(q, x_0) = \frac{A_{qx}x_0}{A_{qq}}.$$

At first one might expect $q_0 = q_*$; but this is not true due to market frictions: if q_{-1} is far from q_*, the investor would incur large transaction costs to do that trade. Instead, by rearranging (3) we obtain

$$q_0 = q_{-1}\left(1 - \frac{A_{qq}}{K}\right) + \frac{A_{qq}}{K}q_*.$$

Although the "target amount" is q_*, due to transaction costs it is not optimal to trade all the way there – instead the investor goes to a choice just part way between q_{-1} and q_*. Note that one can see, using the explicit formula for A_{qq}, that $0 < A_{qq}/K < 1$.

3 Continuous-time model

We now return to the continuous-time model of dynamic portfolio optimization problems with return predictability, transaction costs, and stochastic volatility. For expositional purposes, we will consider a single asset with price P_t and a single return predictor x_t. The dynamics of the price is given by

$$dP_t = \alpha_t\,dt + \sigma(Y_t, Z_t)\,dB_t,$$

where B_t is a standard Brownian motion. Without loss of generality, we will decompose the drift α_t into a constant $\bar{\alpha}$ and intraday component x_t with zero long-term mean: $\alpha_t := \bar{\alpha} + x_t$. We will model the signal x_t with an Ornstein-Uhlenbeck process,

$$dx_t = -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}. \tag{4}$$

3.1 Multiscale stochastic volatility

We work under the multiscale stochastic volatility framework used in Fouque et al. (2003, 2013) for option pricing and portfolio optimization, where there is one fast volatility factor, and one slow. Here, the volatility is a function σ of a fast factor Y and a slow factor Z: $\sigma(Y_t, Z_t)$. The volatility-driving factors (Y, Z) are described by:

$$\begin{aligned} dY_t &= \frac{1}{\varepsilon}b(Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t)\,dW_t^{(1)}, \\ dZ_t &= \delta c(Y_t)\,dt + \sqrt{\delta}g(Y_t)\,dW_t^{(2)}, \end{aligned} \tag{5}$$

where $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathbb{P})$ with instantaneous correlation as follows:

$$d\langle W^{(0)}, W^{(i)}\rangle_t = \rho_i \, dt, \quad i = 1, 2, \qquad d\langle W^{(1)}, W^{(2)}\rangle_t = \rho_{12} \, dt,$$

where $|\rho_1| < 1$, $|\rho_2| < 1$, $|\rho_{12}| < 1$, and $1 + \rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, in order to ensure positive definiteness of the covariance matrix of the three Brownian motions. We also assume that the Brownian motions $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are independent of B_t. The model is described by the coefficients $\bar{\alpha}, \kappa, \eta, \sigma, a, b, c, g$. The parameters ε and δ, when small, characterize the fast and slow variation of Y and Z volatility factors respectively.

We assume that the process $Y_t = Y_{t/\varepsilon}^{(1)}$ in distribution, where $Y^{(1)}$ is an ergodic process with unique invariant distribution Φ, independent of ε. Moreover, we assume that $Z_t = Z_{\delta t}^{(1)}$ in distribution, where $Z^{(1)}$ is a diffusion process with drift and diffusion coefficients c and g respectively. We do not need any ergodicity assumptions on $Z^{(1)}$ for the slow scale asymptotics in the limit $\delta \downarrow 0$.

3.2 Market friction

Trading is costly in our model, and the transaction cost (TC) associated with trading dq_t shares within a time interval dt is

$$TC(u_t) = Ku_t^2, \quad dq_t = u_t \, dt.$$

where q_t is his position at time t, so that u_t is the rate of trading. The level of transaction cost is parameterized by some positive constant $K > 0$. The interpretation is that the transaction price of the asset is above the unaffected price process when $u_t > 0$; and the difference is proportional to the rate of trading.

Remark 3.1. *Gârleanu and Pedersen (2013) further take K to be proportional to the risk σ^2 which simplifies some of the formula in the case of multiple assets, but it is not necessary for their analysis, and consequently nor for our asymptotic analysis here.*

Remark 3.2. *Gârleanu and Pedersen (2013) also consider persistent transaction costs and obtain explicit solutions. We do not incorporate this for simplicity but the anslysis could be extended to that case by increasing the dimension of the problem to include the persistent factor.*

3.3 Hamilton-Jacobi-Bellman equation

The investor's objective is to choose the dynamic trading strategy $(u_t)_t$ to maximize the present value of all future expected excess returns, penalized for risks and trading costs,

$$\max_u \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \left(q_s\alpha_s - \frac{\gamma}{2}\sigma(Y_s, Z_s)^2 q_s^2 - \frac{K}{2}u_s^2 \right) ds,$$

where the constant γ is a risk aversion parameter. We define the value function

$$v(q, x, y, z) = \sup_u \mathbb{E}_{q,x,y,z} \int_0^\infty e^{-\rho t} \left(q_t(\bar{\alpha} + x_t) - \frac{\gamma}{2}\sigma(Y_t, Z_t)^2 q_t^2 - \frac{K}{2}u_t^2 \right) dt,$$

where we have adopted the notation

$$\mathbb{E}_{q,x,y,z}[\cdot] = \mathbb{E}\left[\cdot \,|\, q_0 = q, x_0 = x, Y_0 = y, Z_0 = z\right],$$

and the supremum is taken over admissible strategies that are $\mathcal{F}_t$-progressively measurable, square-integrable (i.e., $\int_0^T u_t^2 dt < \infty$ a.s. for all $T > 0$), and such that (4) and (5) has a unique strong solution on $[0, \infty)$.

For simplicity of exposition and without loss of generality, we will take $\bar{\alpha} = 0$ throughout.[1] The usual dynamic programming principle leads to the HJB equation

$$\begin{aligned} 0 = \sup_u \Bigg\{ & qx - \frac{\gamma}{2}\sigma^2(y,z)q^2 - \frac{1}{2}Ku^2 - \rho v + u v_q - \kappa x v_x + \frac{1}{2}\eta v_{xx} \\ & + \frac{1}{\varepsilon}\mathcal{L}_0 v + \delta \mathcal{M}_2 v + \frac{1}{\sqrt{\varepsilon}}\rho_1 \sqrt{\eta} a(y) v_{xy} + \sqrt{\delta}\rho_2\sqrt{\eta} g(z) v_{xz} + \sqrt{\frac{\delta}{\varepsilon}}\rho_{12} a(y) g(z) v_{yz} \Bigg\}, \end{aligned} \tag{6}$$

where $\mathcal{L}_0$ and $\mathcal{M}_2$ are, respectively, the infinitesimal generators of the process $Y^{(1)}$ and $Z^{(1)}$:

$$\mathcal{L}_0 = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}, \qquad \mathcal{M}_2 = \frac{1}{2}g(z)^2\frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}.$$

Plugging in the optimal trading rate

$$u^* = \frac{1}{K}v_q, \tag{7}$$

we obtain

$$\begin{aligned} 0 = & qx - \frac{\gamma}{2}\sigma^2(y,z)q^2 + \frac{1}{2K}v_q^2 - \rho v - \kappa x v_x + \frac{1}{2}\eta v_{xx} + \frac{1}{\varepsilon}\mathcal{L}_0 v + \delta\mathcal{M}_2 v \\ & + \frac{1}{\sqrt{\varepsilon}}\rho_1\sqrt{\eta}a(y)v_{xy} + \sqrt{\delta}\rho_2\sqrt{\eta}g(z)v_{xz} + \sqrt{\frac{\delta}{\varepsilon}}\rho_{12}a(y)g(z)v_{yz}. \end{aligned} \tag{8}$$

We note that (8) is a nonlinear PDE which is not easily solved either analytically or numerically. Our approach is to view this problem as a perturbation around the special case of *constant* volatility problem studied by Gârleanu and Pedersen (2009, 2013).

3.4 Constant volatility solution

In the case of constant volatility σ, the value function v does not depend on the volatility factors y and z. The HJB equation simplifies to

$$0 = qx - \frac{\gamma}{2}\sigma^2 q^2 + \frac{1}{2K}v_q^2 - \rho v - \kappa x v_x + \frac{1}{2}\eta v_{xx}. \tag{9}$$

Gârleanu and Pedersen (2009) provide a closed-form solution

$$v(q,x) = -\frac{1}{2}A_{qq}q^2 + A_{qx}qx + \frac{1}{2}A_{xx}x^2 + A_0,$$

where

$$A_{qq} = \frac{K}{2}\left(\sqrt{\rho^2 + 4\gamma\frac{\sigma^2}{K}} - \rho\right), \qquad A_{qx} = \left(\kappa + \rho + \frac{A_{qq}}{K}\right)^{-1},$$

$$A_{xx} = \frac{A_{qx}^2}{K(2\kappa+\rho)}, \qquad A_0 = \frac{\eta}{2\rho}A_{xx}.$$

We will denote by $\mathcal{GP}(q,x;\sigma^2)$ the constant volatility solution. The optimal trading rate is given by

$$u^*(q,x) = \frac{1}{K}\left(-A_{qq}q + A_{qx}x\right).$$

[1] We note that the case of nonzero $\bar{\alpha}$ can be analyzed analogously, though with more cumbersome formulas which do not shed light on the structure of the optimal trading problem.

Remark 3.3. *The optimal trading rate can be written as*

$$u^*(q,x) = r^{\mathcal{GP}}\left(aim_t^{\mathcal{GP}} - q\right), \tag{10}$$

with

$$r^{\mathcal{GP}} = \frac{1}{2}\left(\sqrt{\rho^2 + 4\gamma\frac{\sigma^2}{K}} - \rho\right), \qquad aim_t^{\mathcal{GP}} = \frac{A_{qx}}{A_{qq}}x_t. \tag{11}$$

In words, the optimal portfolio q_t tracks aim_t with speed r. The tracking speed decreases with the transaction cost K and increases with the risk-aversion coefficient γ. The target portfolio shrinks to zero as variance σ^2 increases.

4 Fast Mean-Reverting Stochastic Volatility

We first analyze the optimal trading problem under fast mean-reverting stochastic volatility. We have the following dynamics for a stock or index price process P:

$$\begin{aligned} dP_t &= x_t\,dt + \sigma(Y_t)\,dB_t, \\ dx_t &= -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}, \\ dY_t &= \frac{1}{\varepsilon}b(Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t)\,dW_t^{(1)}, \end{aligned} \tag{12}$$

where $W^{(0)}$ and $W^{(1)}$ are Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ with instantaneous correlation coefficient between volatility and stock return shocks $\rho_1 \in (-1,1)$.

The investor chooses his optimal trading strategy to maximize the present value of the future stream of expected excess returns, penalized for risk and trading costs:

$$\max_{(u_t)_{t\geq 0}} \mathbb{E}\int_0^\infty e^{-\rho t}\left(q_t x_t - \frac{\gamma}{2}\sigma(Y_t)^2 q_t^2 - \frac{K}{2}u_t^2\right)dt.$$

We define the value function

$$v^\varepsilon(q,x,y) = \sup_u \mathbb{E}_{q,x,y}\left\{\int_0^\infty e^{-\rho t}\left(q_t x_t - \frac{\gamma}{2}\sigma(Y_t)^2 q_t^2 - \frac{K}{2}u_t^2\right)dt\right\},$$

where the supremum is taken over admissible strategies that are $\mathcal{F}_t$-progressively measurable. The associated HJB equation PDE for v^ε is

$$0 = qx - \frac{\gamma}{2}\sigma^2(y)q^2 + \frac{1}{2K}\left(v_q^\varepsilon\right)^2 - \rho v^\varepsilon - \kappa x v_x^\varepsilon + \frac{1}{2}\eta v_{xx}^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_0 v^\varepsilon + \rho_1\sqrt{\frac{\eta}{\varepsilon}}a(y)v_{xy}^\varepsilon, \tag{13}$$

which is simply (8) without the z-dependence. Analytically or numerically solving the nonlinear PDE (13) is a difficult problem. In the limit $\varepsilon \downarrow 0$, it is a singular perturbation problem, and our approach is to construct an asymptotic approximation of the solution.

4.1 Expansion of the value function

We look for an asymptotic expansion of the value function of the form

$$v^\varepsilon(q,x,y) = v^{(0)}(q,x,y) + \sqrt{\varepsilon}v^{(1)}(q,x,y) + \varepsilon v^{(2)}(q,x,y) + \varepsilon^{3/2}v^{(3)}(q,x,y)\cdots.$$

Inserting this expansion into (13), and collecting terms in successive powers of ε, we obtain at the highest order ε^{-1}:

$$\mathcal{L}_0 v^{(0)} = 0.$$

Since $\mathcal{L}_0$ takes derivatives in y, this equation is satisfied with $v^{(0)}(q,x)$ independent of y. With this choice, we have $v_y^{(0)} = 0$, so expanding the nonlinear term in (13) up to order ε gives:

$$\left(v_q^\varepsilon\right)^2 = \left(v_q^{(0)}\right)^2 + 2\sqrt{\varepsilon} v_q^{(0)} v_q^{(1)} + \epsilon \left(\left(v_q^{(1)}\right)^2 + 2 v_q^{(0)} v_q^{(2)} \right) + \cdots .$$

Therefore, at the next order $\varepsilon^{-1/2}$ in the expansion of the PDE, there is no contribution from the nonlinear term, and we have

$$\mathcal{L}_0 v^{(1)} + \rho_1 \sqrt{\eta} a(y) v_{xy}^{(0)} = 0.$$

With our choice $v^{(0)}(q,x)$, we obtain simply $\mathcal{L}_0 v^{(1)} = 0$. Again, we satisfy this equation with $v^{(1)} = v^{(1)}(q,x)$, independent of y.

Then, collecting the order one terms leads to:

$$\mathcal{L}_0 v^{(2)} + qx - \frac{\gamma}{2}\sigma^2(y) q^2 + \frac{1}{2K}\left(v_q^{(0)}\right)^2 - \rho v^{(0)} - \kappa x v_x^{(0)} + \frac{1}{2}\eta v_{xx}^{(0)} + \rho_1 \sqrt{\eta} a(y) v_{xy}^{(1)} = 0. \tag{14}$$

4.2 Zeroth order term $v^{(0)}$

Equation (14) is a Poisson equation for $v^{(2)}$ whose solvability condition (Fredholm Alternative) requires that

$$\left\langle qx - \frac{\gamma}{2}\sigma^2(y) q^2 + \frac{1}{2K}\left(v_q^{(0)}\right)^2 - \rho v^{(0)} - \kappa x v_x^{(0)} + \frac{1}{2}\eta v_{xx}^{(0)} \right\rangle = 0,$$

where $\langle\cdot\rangle$ is defined by the unique invariant distribution Φ of the ergodic process $Y^{(1)}$:

$$\langle g \rangle = \int g(y)\Phi(dy),$$

for any smooth function g. Using that $v^{(0)}(q,x)$ is independent of y, the solvability condition simplifies to

$$qx - \frac{\gamma}{2}\langle\sigma^2\rangle q^2 + \frac{1}{2K}\left(v_q^{(0)}\right)^2 - \rho v^{(0)} - \kappa x v_x^{(0)} + \frac{1}{2}\eta v_{xx}^{(0)} = 0. \tag{15}$$

Notice that (15) is the nonlinear PDE (9) for the optimal portfolio problem with constant volatility $\sqrt{\langle\sigma^2\rangle}$, and so,

$$v^{(0)}(q,x) = \mathcal{GP}(q,x;\langle\sigma^2\rangle). \tag{16}$$

4.3 First order term $v^{(1)}$

Combining Equations (14) and (15), we can write

$$\mathcal{L}_0 v^{(2)} = g, \tag{17}$$

where $g(y) = \frac{\gamma}{2}\left(\sigma^2(y) - \langle\sigma^2\rangle\right) q^2$. The solution of the Poisson equation (17) can be expressed as

$$v^{(2)} = -\int_0^\infty \mathbf{P}_t g(y)\, dt + C(q,x), \tag{18}$$

where $C(q,x)$ is some "constant" of integration in y, and the transition semigroup $\mathbf{P}_t$ is defined by its action on bounded measurable functions g:

$$\mathbf{P}_t g(y) = \mathbb{E}\left\{ g(Y_t^{(1)}) \middle| Y_0^{(1)} = y \right\}.$$

See, for instance, Fouque et al. (2011, Section 3.2).

Continuing, at order $\sqrt{\varepsilon}$ in the expansion of the PDE, we have

$$\frac{1}{K}v_q^{(0)}v_q^{(1)} - \rho v^{(1)} - \kappa x v_x^{(1)} + \frac{1}{2}\eta v_{xx}^{(1)} + \mathcal{L}_0 v^{(3)} + \rho_1\sqrt{\eta}a(y)v_{xy}^{(2)} = 0. \tag{19}$$

Equation (19) is a Poisson equation for $v^{(3)}$ whose solvability condition is

$$\frac{1}{K}v_q^{(0)}v_q^{(1)} - \rho v^{(1)} - \kappa x v_x^{(1)} + \frac{1}{2}\eta v_{xx}^{(1)} = 0. \tag{20}$$

Observe that (20) is a linear homogeneous PDE for $v^{(1)}$, we can choose $v^{(1)} = 0$ identically. With this choice, we have $\mathcal{L}_0 v^{(3)} = 0$. Again, we satisfy this equation with $v^{(3)} = v^{(3)}(q,x)$, independent of y.

4.4 Second order term $v^{(2)}$

At order ε in the expansion of the PDE, we have

$$\frac{1}{2K}\left(v_q^{(1)}\right)^2 + \frac{1}{K}v_q^{(0)}v_q^{(2)} - \rho v^{(2)} - \kappa x v_x^{(2)} + \frac{1}{2}\eta v_{xx}^{(2)} + \mathcal{L}_0 v^{(4)} + \rho_1\sqrt{\eta}a(y)v_{xy}^{(3)} = 0.$$

The solvability condition gives

$$0 = \frac{1}{K}v_q^{(0)}C_q - \rho C - \kappa x C_x + \frac{1}{2}\eta C_{xx}.$$

This is a linear equation without source term, we can choose C to be identically zero. Therefore, from (18) we see that the leading order correction to the value function is given by

$$v^{(2)}(q,y) = -\frac{\gamma}{2}q^2\int_0^\infty \mathbf{P}_t\left(\sigma^2(y) - \langle\sigma^2\rangle\right)dt =: -\frac{1}{2}q^2\varphi(y). \tag{21}$$

Example 1. Suppose that $\sigma^2(y) = y$ and the volatility factor $Y_t^{(1)}$ is the Cox-Ingersoll-Ross (1985) process, that is

$$b(y) = \theta(\mu - y), \quad a(y) = \widehat{\sigma}\sqrt{y}. \tag{22}$$

Applying the transition semigroup $\mathbf{P}_t$ on the function g gives

$$\begin{aligned}\mathbf{P}_t g(y) &= \mathbb{E}\left[g(Y_t^{(1)})\middle| Y_0^{(1)} = y\right]\\ &= \frac{\gamma}{2}\mathbb{E}\left[Y_t^{(1)} - \mu\middle| Y_0^{(1)} = y\right]q^2\\ &= \frac{\gamma}{2}e^{-\theta t}(y-\mu)q^2.\end{aligned}$$

Then Equation (18) immediately gives

$$v^{(2)} = -\frac{\gamma}{2\theta}(y-\mu)q^2.$$

One can readily check that the above does indeed solve the HJB equation (17).

Example 2. As an alternative example, let us consider the exponential Ornstein-Uhlenbeck stochastic volatility model (Masoliver and Perelló, 2006). In this case we have $\sigma(y) = me^y$ and the volatility factor $Y_t^{(1)}$ is the Ornstein-Uhlenbeck process, that is

$$b(y) = -\theta y, \quad a(y) = \widehat{\sigma}.$$

Applying the transition semigroup $\mathbf{P}_t$ on the function g gives

$$\begin{aligned}\mathbf{P}_t g(y) &= \mathbb{E}\left[g(Y_t^{(1)})\middle| Y_0^{(1)} = y\right]\\ &= \frac{\gamma}{2}q^2m^2e^{k^2/\alpha}\left(e^{2ye^{-\alpha t} - \frac{k^2}{\alpha}e^{-2\alpha t}} - 1\right).\end{aligned}$$

There does not appear to be a closed-form expression for the time-integral of the function $\mathbf{P}_t g(y)$, but we can compute explicitly the leading order correction when the volatility factor fluctuates around its mean level

$$\begin{aligned} v^{(2)} &= -\frac{\gamma}{2} q^2 m^2 e^{k^2/\alpha} \int_0^\infty \left(e^{2ye^{-\alpha t} - \frac{k^2}{\alpha} e^{-2\alpha t}} - 1 \right) dt \\ &\approx -\frac{\gamma}{4\alpha} q^2 m^2 e^{k^2/\alpha} \left(4y\Gamma\left(\frac{1}{2}, \frac{k}{\alpha^2}\right) - \left[\widehat{\gamma} + \Gamma\left(0, \frac{k^2}{\alpha}\right) + \log\left(\frac{k^2}{\alpha}\right) \right] \right), \end{aligned}$$

where $\widehat{\gamma} \approx 0.5772$ is the Euler-Mascheroni constant and Γ is the incomplete gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} \, dt.$$

4.5 Optimal Portfolio

We now analyze and interpret how the principal expansion terms $v^{(0)}$ and $v^{(2)}$ for the value function can be used in the expression for the optimal portfolio u^* in (7), which leads to an approximate feedback policy of the form

$$u^*(q, x, y) = u^{(0)}(q, x, y) + \varepsilon u^{(2)}(q, x, y) + \cdots . \tag{23}$$

The zeroth order trading rate is independent of y:

$$u^{(0)}(q, x) = \frac{1}{K} \left(A_{qx} x - A_{qq} q \right). \tag{24}$$

This is simply the Gârleanu and Pedersen (2009) constant volatility strategy evaluated at the long-term variance $\langle \sigma^2 \rangle$.

Differentiating (21) gives the principal correction to the optimal trading rate:

$$u^{(2)}(q, y) = -\frac{q}{K} \varphi(y). \tag{25}$$

In the case where the volatility factor is driven by a Cox-Ingersoll-Ross process (see Example 1), the expression simplifies to

$$u^{(2)}(q, y) = -\frac{\gamma}{\theta K} q \left(y - \mu \right). \tag{26}$$

Notice that $u^{(2)}$ and q have opposite signs when $\sigma^2(y) > \langle \sigma^2 \rangle$. The economic interpretation is that the investor should optimally deleverage his portfolio when the current volatility level is higher than the long-term average.

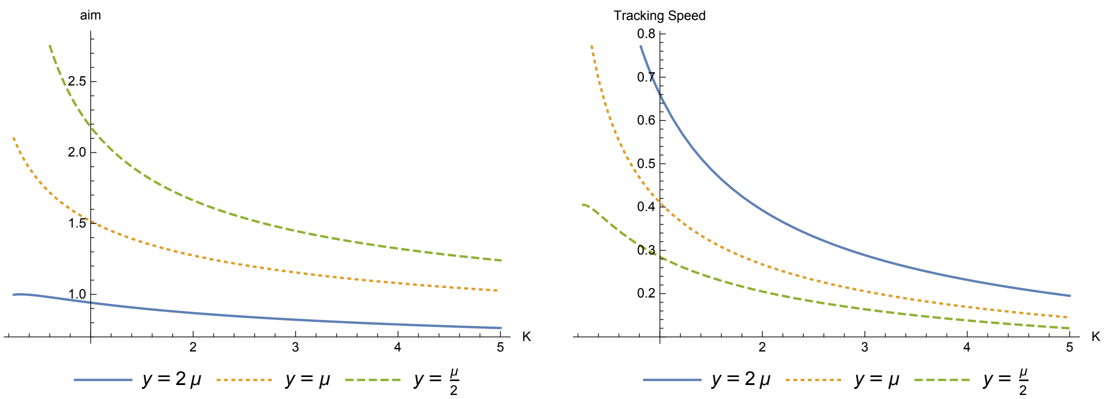
Remark 4.1. *As in Remark 3.3 for the constant volatility case, we can write the optimal trading rate in the "aim-speed" representation*

$$u^*(q, x) = r \left(aim_t - q \right).$$

From (23), (24) *and* (25), *we obtain*

$$\begin{aligned} r &= r^{\mathcal{GP}} + \varepsilon \frac{\varphi(y)}{K}, \\ aim_t &= \frac{A_{qx}}{A_{qq} + \varepsilon \varphi(y)} x = aim_t^{\mathcal{GP}} \left(1 - \varepsilon \frac{\varphi(y)}{A_{qq}} + \cdots \right), \end{aligned} \tag{27}$$

where $(r^{\mathcal{GP}}, aim_t^{\mathcal{GP}})$ *were introduced in Remark 3.3. We observe that both the aim portfolio and the tracking speed are affected under fast-scale stochastic volatility.*



(a) Target portfolio aim_t.

(b) Tracking speed r.

Figure 1: Target portfolio aim_t and optimal tracking speed r in the fast scale volatility model (22), under the "aim-speed" representation (10). Parameters used are $\rho = 0.2, \kappa = 1, \gamma = 0.5, \theta = 1, \mu = 0.5, x = 1, \varepsilon = 1$.

Example 1 (Continued)**.** In the case of a fast CIR volatility factor, we can derive explicitly

$$\begin{aligned} r &= r^{\mathcal{GP}} + \varepsilon \frac{\gamma}{\theta K} (y - \mu), \\ aim_t &= aim_t^{\mathcal{GP}} \left(1 - \varepsilon \frac{\gamma}{\theta A_{qq}} (y - \mu) \right). \end{aligned}$$

In words, the optimal tracking speed r increases with the short-term volatility factor, while this effect is dampened by higher transaction cost K or lower risk-aversion coefficient γ; the size of the target portfolio is also reduced when the current volatility is above its long-term average. Figure 1 illustrates the dependence of the target portfolio aim_t and the tracking speed r on the transaction cost K.

5 Slow scale volatility asymptotics

We now perform an asymptotic analysis under the assumption that stochastic volatility is slowly fluctuating. The model reads

$$\begin{aligned} dP_t &= x_t\, dt + \sigma(Z_t)\, dB_t, \\ dx_t &= -\kappa x_t\, dt + \sqrt{\eta}\, dW_t^{(0)}, \\ dZ_t &= \delta c(Z_t)\, dt + \sqrt{\delta} g(Z_t)\, dW_t^{(2)}, \end{aligned}$$

where $W^{(0)}$ and $W^{(2)}$ are Brownian motions with instantaneous correlation coefficient between volatility and stock return shocks $\rho_2 \in (-1, 1)$, and δ is the small time-scale parameter for expansion. The HJB equation for the value function

$$v^\delta(q, x, z) = \sup_u \mathbb{E}_{q,x,z} \left\{ \int_0^\infty e^{-\rho t} \left(q_t x_t - \frac{\gamma}{2} \sigma(Z_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) dt \right\},$$

is

$$0 = qx - \frac{\gamma}{2} \sigma^2(z) q^2 + \frac{1}{2K} \left(v_q^\delta \right)^2 - \rho v^\delta - \kappa x v_x^\delta + \frac{1}{2} \eta v_{xx}^\delta + \delta \mathcal{M}_2 v^\delta + \sqrt{\delta} \rho_2 \sqrt{\eta} g(z) v_{xz}^\delta, \tag{28}$$

which is simply (8) with the y-dependence removed.

5.1 Slow scale expansion

We look for expansion of the value function of the form

$$v^\delta(q,x,y) = v^{(0)}(q,x,z) + \sqrt{\delta}v^{(1)}(q,x,z) + \delta v^{(2)}(q,x,z) + \delta^{3/2}v^{(3)}(q,x,z) + \cdots . \tag{29}$$

Then it follows by setting $\delta = 0$ that $v^{(0)}$ solves

$$0 = qx - \frac{\gamma}{2}\sigma^2(z)q^2 + \frac{1}{2K}\left(v_q^{(0)}\right)^2 - \rho v^{(0)} - \kappa x v_x^{(0)} + \frac{1}{2}\eta v_{xx}^{(0)}.$$

Therefore, the principal term is the Gârleanu and Pedersen value function explicitly given by $\mathcal{GP}(q,x,\sigma^2(z))$. As with the fast factor zeroth order approximation to the value function given in (16), the zeroth order approximation in the slow factor model is the constant volatility value function, but with $\sigma^2(z)$, the current volatility, instead of the averaged quantity $\langle\sigma^2\rangle$.

5.2 Slow scale value function correction

Taking the order $\sqrt{\delta}$ term after inserting the expansion (29) into the PDE (28) leads to

$$0 = \frac{1}{K}v_q^{(0)}v_q^{(1)} - \rho v^{(1)} - \kappa x v_x^{(1)} + \frac{1}{2}\eta v_{xx}^{(1)} + \rho_2\sqrt{\eta}g(z)v_{xz}^{(0)}. \tag{30}$$

With a linear ansatz

$$v^{(1)}(q,x,z) = B_q(z)q + B_x(z)x,$$

we can write down the solution to the HJB equation

$$B_q(z) = \frac{\rho_2\sqrt{\eta}g(z)A'_{qx}(z)}{\rho + A_{qq}(z)/K}, \qquad B_x(z) = \frac{\rho_2\sqrt{\eta}g(z)A'_{xx}(z) + A_{qx}(z)B_q(z)/K}{\rho+\kappa}, \tag{31}$$

where $A_{qx}(z)$ and $A_{xx}(z)$ are the corresponding coefficients in $\mathcal{GP}(q,x,\sigma^2(z))$.

5.3 Optimal trading strategy

The optimal trading strategy in feedback form is given by

$$u^*(q,x,z) = \frac{1}{K}\left(A_{qx}(z)x - A_{qq}(z)q\right) + \frac{1}{K}\sqrt{\delta}B_q(z).$$

It follows from a straight-forward calculation that B_q and ρ_2 have opposite signs, provided that the function σ is monotonically increasing in the volatility factor Z. When the correlation ρ_2 between the volatility and return factors is positive, the investor optimally decreases his trading rate as he anticipates a higher return estimate is accompanied by a higher volatility. Conversely, if the return-volatility correlation ρ_2 is negative, the investor optimally increases his trading rate u.

Remark 5.1. *The optimal tracking speed r in the representation* (10) *in not affected in the case of slow volatility factor; the target portfolio, however, is given by*

$$aim_t = aim_t^{\mathcal{GP}} + \sqrt{\delta}\frac{B_q(z)}{A_{qq}(z)} + \cdots .$$

In the case of negative correlation ρ_2, the investor optimally increases the leverage for positive signal x_t, this is because positive return shock is correlated with lower volatility, and a higher target portfolio (than $aim_t^{\mathcal{GP}}$) captures this potential gain. The investor, however, deleverages the portfolio for negative signal x_t since the target portfolio is a short position on the stock, a negative return shock is correlated to higher volatility and hence higher risk. The two effects work against each other and the investor optimally shinks the target position closer to zero. The case of positive correlation ρ_2 is analyzed analogously.

6 Multiscale stochastic volatility

We return to the two-factor multiscale volatility model (5), introduced in Section 3, where there is one fast volatility, and one slow. We show that the separate fast and slow expansions to first order essentially combine, but with slight modification of the averaged parameters involved.

Under our simplifying assumption of zero expected stock return $\bar{\alpha}$, the stock price process follows

$$\begin{aligned} dP_t &= x_t\,dt + \sigma(Y_t, Z_t)\,dB_t, \\ dx_t &= -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}, \\ dY_t &= \frac{1}{\varepsilon} b(Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}} a(Y_t)\,dW_t^{(1)}, \\ dZ_t &= \delta c(Z_t)\,dt + \sqrt{\delta} g(Z_t)\,dW_t^{(2)}. \end{aligned}$$

The value function

$$v^{\varepsilon,\delta}(q,x,y,z) = \sup_u \mathbb{E}_{q,x,y,z} \left\{ \int_0^\infty e^{-\rho t} \left(q_t x_t - \frac{\gamma}{2}\sigma(Y_t, Z_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) dt \right\},$$

has the associated HJB equation

$$\begin{aligned} 0 = qx - \frac{\gamma}{2}\sigma^2(y,z)q^2 + \frac{1}{2K}\left(v_q^{\varepsilon,\delta}\right)^2 - \rho v^{\varepsilon,\delta} - \kappa x v_x^{\varepsilon,\delta} + \frac{1}{2}\eta v_{xx}^{\varepsilon,\delta} + \frac{1}{\varepsilon}\mathcal{L}_0 v^{\varepsilon,\delta} \\ + \delta \mathcal{M}_2 v^{\varepsilon,\delta} + \rho_1 \sqrt{\frac{\eta}{\varepsilon}} a(y) v_{xy}^{\varepsilon,\delta} + \sqrt{\delta}\rho_2\sqrt{\eta} g(z) v_{xz}^{\varepsilon,\delta} + \rho_{12}\sqrt{\frac{\delta}{\varepsilon}} a(y) g(z) v_{yz}^{\varepsilon,\delta}. \end{aligned} \tag{32}$$

The optimal strategy in feedback form is given by

$$u^*(q,x,y,z) = \frac{1}{K} v_q^{\varepsilon,\delta}. \tag{33}$$

6.1 Combined expansion in slow and fast scales

For expositional purposes, we focus on the leading order corrections to the value function from the fast and slow scale volatilities. Appendix A presents the full second order asymptotic expansion to the multiscale stochastic volatility problem. First we construct an expansion in powers of $\sqrt{\delta}$:

$$v^{\varepsilon,\delta} = v^{\varepsilon,0} + \sqrt{\delta} v^{\varepsilon,1} + \delta v^{\varepsilon,2} + \cdots, \tag{34}$$

so that $v^{\varepsilon,0}$ is obtained by setting $\delta = 0$ in the equation for $v^{\varepsilon,\delta}$:

$$0 = qx - \frac{\gamma}{2}\sigma^2(y,z)q^2 - \frac{1}{2K}\left(v_q^{\varepsilon,0}\right)^2 - \rho v^{\varepsilon,0} + u v_q^{\varepsilon,0} - \kappa x v_x^{\varepsilon,0} + \frac{1}{2}\eta v_{xx}^{\varepsilon,0} + \frac{1}{\varepsilon}\mathcal{L}_0 v^{\varepsilon,0} + \rho_1\sqrt{\frac{\eta}{\varepsilon}} a(y) v_{xy}^{\varepsilon,0}. \tag{35}$$

This is the same HJB problem (13) as for the value function v^ε except that the volatility depends on the current level z of the slow volatility factor, which enters as a parameter in the PDE (35). It is clear that when we construct an expansion for $v^{\varepsilon,0}$ in powers of $\sqrt{\varepsilon}$:

$$v^{\varepsilon,0} = v^{(0)} + \sqrt{\varepsilon} v^{(1,0)} + \varepsilon v^{(2,0)} + \cdots,$$

we will obtain, as in Section 4, that $v^{(0)}$ is the Gârleanu and Pedersen (2009) value function with constant volatility $\bar{\sigma}^2(z)$:

$$v^{(0)}(q,x,z) = \mathcal{GP}(q,x;\bar{\sigma}^2(z)), \tag{36}$$

where $\bar{\sigma}^2(z) = \langle \sigma^2(\cdot, z) \rangle$. That is, the variance is squared-averaged over the fast factor with respect to its invariant distribution, and evaluated at the current level of slow factor.

Following Section 4, the correction term $v^{(1,0)}$ is identically zero and $v^{(2,0)}$ is given by

$$v^{(2,0)} = -\frac{\gamma}{2} q^2 \int_0^\infty \mathbf{P}_t \left(\sigma^2(y,z) - \bar{\sigma}^2(z)\right) dt =: -\frac{1}{2} q^2 \varphi(y,z). \tag{37}$$

Next we return to the slow scale expansion (34) and extract the order $\sqrt{\delta}$ terms in (32) to obtain the following equation for $v^{\varepsilon,1}$:

$$0 = \frac{1}{K} v_q^{\varepsilon,0} v_q^{\varepsilon,1} - \rho v^{\varepsilon,1} - \kappa x v_x^{\varepsilon,1} + \frac{1}{2}\eta v_{xx}^{\varepsilon,1} + \frac{1}{\varepsilon}\mathcal{L}_0 v^{\varepsilon,1} + \rho_1 \sqrt{\frac{\eta}{\varepsilon}} a(y) v_{xy}^{\varepsilon,1} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{\varepsilon,0} + \frac{1}{\sqrt{\varepsilon}} \rho_{12} a(y) g(z) v_{yz}^{\varepsilon,0}. \tag{38}$$

We look for an expansion

$$v^{\varepsilon,1} = v^{(0,1)} + \sqrt{\varepsilon} v^{(1,1)} + \varepsilon v^{(2,1)} + \cdots, \tag{39}$$

where we are only interested here in the first term which will give the principal slow scale correction to the value function.

The order ε^{-1} terms in (38) give $\mathcal{L}_0 v^{(0,1)} = 0$ and we take $v^{(0,1)} = v^{(0,1)}(q,x,z)$ independent of y. At order $\varepsilon^{-1/2}$, we have $\mathcal{L}_0 v^{(1,1)} = 0$ and so again $v^{(1,1)} = v^{(1,1)}(q,x,z)$. At order one:

$$\begin{aligned} 0 = {} & \frac{1}{K} v_q^{(0)} v_q^{(0,1)} - \rho v^{(0,1)} - \kappa x v_x^{(0,1)} + \frac{1}{2}\eta v_{xx}^{(0,1)} + \mathcal{L}_0 v^{(2,1)} \\ & + \rho_1 \sqrt{\eta} a(y) v_{xy}^{(1,1)} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{(0)} + \rho_{12} a(y) g(z) v_{yz}^{(1,0)}. \end{aligned} \tag{40}$$

Viewed as a Poisson equation for $v^{(2,1)}$, this yields the following solvability condition for $v^{(0,1)}$:

$$0 = \frac{1}{K} v_q^{(0)} v_q^{(0,1)} - \rho v^{(0,1)} - \kappa x v_x^{(0,1)} + \frac{1}{2}\eta v_{xx}^{(0,1)} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{(0)}.$$

This is the same PDE problem (30) as for the slow scale correction in Section 5, except with $\sigma(z)$ replaced by $\bar{\sigma}(z)$. We conclude that $v^{(0,1)}(q,x,z) = B_q(z) q + B_x(z) x$, with

$$B_q(z) = \frac{\rho_2 \sqrt{\eta} g(z) A'_{qx}(z)}{\rho + A_{qq}(z)/K}, \qquad B_x(z) = \frac{\rho_2 \sqrt{\eta} g(z) A'_{xx}(z) + A_{qx}(z) B_q(z)/K}{\rho + \kappa}, \tag{41}$$

where $A_{qx}(z)$ and $A_{xx}(z)$ are the corresponding coefficients in $\mathcal{GP}(q,x,\bar{\sigma}^2(z))$.

In summary, the leading-order multiscale correction is given by

$$v^{\varepsilon,\delta}(q,x,y,z) = \mathcal{GP}(q,x;\bar{\sigma}^2(z)) - \frac{\varepsilon}{2} q^2 \varphi(y,z) + \sqrt{\delta}\left(B_q(z) q + B_x(z) x\right) + \cdots. \tag{42}$$

In Appendix A, we present the full second order asymptotic expansion; the results are conveniently summarized in Table 2.

	$\mathcal{O}(1)$	$\mathcal{O}(\sqrt{\varepsilon})$	$\mathcal{O}(\varepsilon)$
$\mathcal{O}(1)$	$\mathcal{GP}(q,x;\bar{\sigma}^2(z))$	0	$v^{(2,0)} = (37)$
$\mathcal{O}(\sqrt{\delta})$	$v^{(0,1)} = (41)$	$v^{(1,1)} = (49)$	
$\mathcal{O}(\delta)$	$v^{(0,2)} = (52)$		

Table 2: Summary of the full second order asymptotic expansion to the multiscale stochastic volatility problem (32).

6.2 Multiscale optimal portfolio

The optimal portfolio up to orders ε and $\sqrt{\delta}$ for the multiscale model is obtained by inserting the value function approximation (42) into the optimal strategy feedback function (33), which leads to

$$u^{\varepsilon,\delta} = \frac{1}{K}\left(A_{qx}(z)x - A_{qq}(z)q\right) - \frac{\varepsilon q}{K}\varphi(y,z) + \frac{\sqrt{\delta}}{K}B_q(z) + \cdots, \tag{43}$$

where

$$A_{qq}(z) = \frac{K}{2}\left(\sqrt{\rho^2 + 4\gamma\frac{\bar{\sigma}(z)^2}{K}} - \rho\right), \quad A_{qx}(z) = \left(\kappa + \rho + \frac{A_{qq}(z)}{K}\right)^{-1},$$

and

$$B_q(z) = \frac{\rho_2\sqrt{\eta}g(z)A'_{qx}(z)}{\rho + A_{qq}(z)/K}.$$

The formula (43) for the approximate optimal trading rate up to order ϵ and $\sqrt{\delta}$ highlights the contribution from the volatility factor-returns correlations. The principal (zero order) strategy

$$u^{(0)}(q,x,z) = \frac{1}{K}\left(A_{qx}(z)x - A_{qq}(z)q\right)$$

is a moving Gârleanu and Pedersen strategy with respect to the slow factor Z, as in the slow-only case (Section 5).

Remark 6.1. *Both the aim and tracking speed are affected by multiscale stochastic volatility in representation* (10). *Indeed, we have*

$$\begin{aligned} r &= r^{\mathcal{GP}} + \varepsilon\frac{\varphi(y,z)}{K} + \cdots, \\ aim_t &= \frac{A_{qx}(z)x + \sqrt{\delta}B_q(z)}{A_{qq}(z) + \varepsilon\varphi(y,z)} + \cdots = \left(aim_t^{\mathcal{GP}} + \sqrt{\delta}\frac{B_q(z)}{A_{qq}(z)}\right)\left(1 - \varepsilon\frac{\varphi(y,z)}{A_{qq}(z)} + \cdots\right). \end{aligned} \tag{44}$$

7 Examples & numerical solutions

We present numerical examples to demonstrate that the asymptotic approximation can be computed efficiently under a wide variety of models of practical interest. Then we demonstrate how the proposed algorithms improve our Profit&Loss using Monte-Carlo simulations.

7.1 Heston stochastic volatility model

Kraft (2005) considered the one-factor stochastic volatility model in which the volatility factor Z_t is a CIR process:

$$\sigma(z) = z^{1/2}, \qquad c(z) = m - z, \qquad g(z) = \beta\sqrt{z},$$

that is

$$\begin{aligned} dP_t &= x_t\,dt + \sqrt{Z_t}\,dB_t, \\ dx_t &= -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}, \\ dZ_t &= \delta(m - Z_t)\,dt + \sqrt{\delta}\beta\sqrt{Z_t}\,dW_t^{(2)}. \end{aligned} \tag{45}$$

We assume the standard Feller condition $\beta^2 < 2m$, which we note does not involve the time scale parameter δ.

In Figure 2, we show the value function over a range of the time scale parameter δ, for three different values of the volatility factor. The leading-order correction $v^{(1)}$ to the value function is proportional to $\sqrt{\delta}$

and ρ_2. When the correlation between the slow volatility and stock return shocks is positive, the principal impact of stochastic volatility is lowering of the value function. Intuitively, when the stock return x_t goes up, the optimal stock holding q_t also goes up; but in the case of positive correlation ρ_2, this is also followed by an increase in uncertainty, causing the investor to be more conservative and reduce leverage. Figure 3 shows the principal effect of a slow-scale stochastic volatility on the optimal trading rate u.

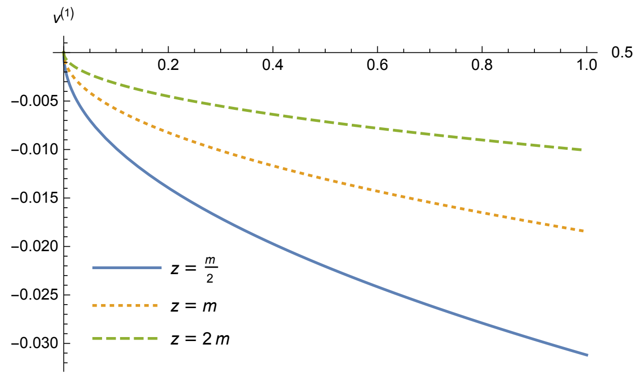


Figure 2: Value functions in the slow scale volatility model (45) for a range of δ, for three different values of the volatility factor. Parameters used are $\rho = 0.2, \gamma = 1, m = 0.5, \beta = 0.25, K = 1, \rho_2 = 0.5, \eta = 0.5, \kappa = 1$.

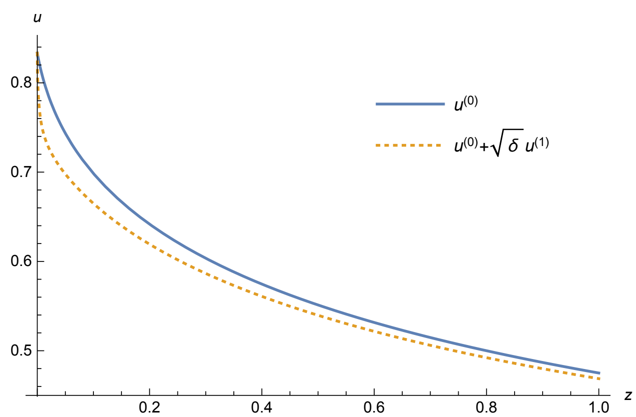


Figure 3: Optimal trading rate u in the slow scale volatility model (45). Parameters used are as in Figure 2 and $\delta = 0.5$.

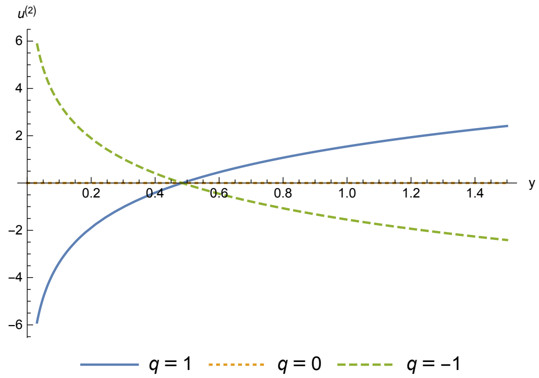
7.2 Chacko and Viceira (2005) model

As another example, we illustrate our approximation with a model considered in Chacko and Viceira (2005):

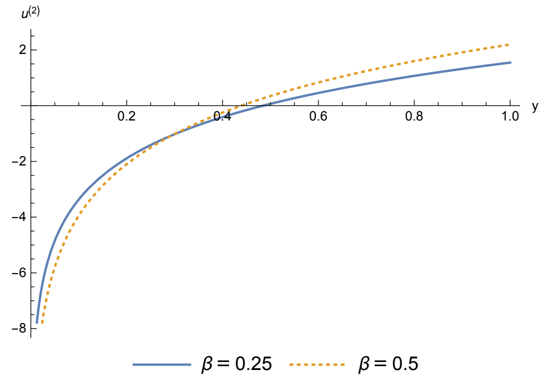
$$\sigma(z) = z^{-1/2}, \qquad c(z) = m - z, \qquad g(z) = \beta\sqrt{z},$$

that is

$$\begin{aligned} dP_t &= x_t\,dt + \sqrt{\frac{1}{Z_t}}\,dB_t, \\ dx_t &= -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}, \\ dZ_t &= \frac{1}{\varepsilon}(m - Z_t)\,dt + \frac{1}{\sqrt{\varepsilon}}\beta\sqrt{Z_t}\,dW_t^{(1)}. \end{aligned} \tag{46}$$



(a) Optimal trading rate u.



(b) Optimal trading rate u.

Figure 4: Optimal trading rate u in the fast scale volatility model (46). Parameters used are as in Figure 2.

Equation (21) applied to the current setting gives the leading order correction to the optimal trading rate in the presence of fast-scale stochastic volatility. Chou and Lin (2006) show that the probability transition density of the CIR process is

$$p_t(x, y) = \frac{2}{\beta^2\left(1 - e^{-t}\right)} \exp\left[\frac{2\left(x + ye^t\right)}{\beta^2\left(1 - e^t\right)}\right] \left(\frac{ye^t}{x}\right)^{\nu/2} I_\nu\left(-\frac{4\sqrt{xye^t}}{\beta^2\left(1 - e^t\right)}\right), \qquad \nu = \frac{2m}{\beta^2} - 1,$$

where I_ν is the modified Bessel function of the first kind of index ν. With this we can compute the expected variance at time t

$$\mathbf{P}_t\sigma^2(y) = \frac{\zeta_t e^{-y\mu_t}}{q}\, {}_1F_1(q, 1 + q, y\mu_t), \qquad \langle\sigma^2\rangle = \frac{2}{2m - \beta^2}, \tag{47}$$

where

$$\zeta_t = \frac{2}{\beta^2(1 - e^{-t})}, \qquad \mu_t = \zeta_t e^{-t}, \qquad q = \frac{2m}{\beta^2} - 1,$$

and ${}_1F_1\left(\cdot, \cdot, \cdot\right)$ is the Kummer confluent hypergeometric function. The correction to the optimal control under the fast-scale Chacko and Viceira volatility process can be computed with a single numerical integral:

$$u^{(2)}(q, y) = -\frac{q}{K}\varphi(y) = -\frac{\gamma}{K} q \int_0^\infty \mathbf{P}_t\left(\sigma^2(y) - \langle\sigma^2\rangle\right) dt.$$

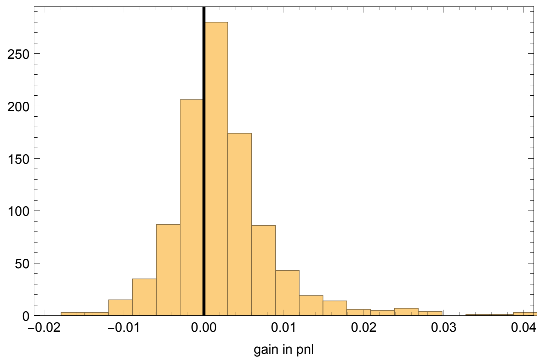
As shown in Figure 4, the correction term $u^{(2)}$ is nonlinear in y, in contrast to the Heston model in Example 1.

7.3 Monte Carlo simulation

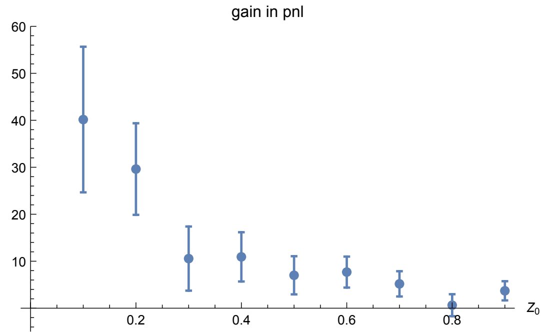
We have tested our trading strategy using a Monte Carlo simulation under the fast and slow-scale optimal trading algorithm described in Section 4. We simulate the Heston stochastic volatility model of Example 1 using the 3-stage Rossler Stochastic Runge-Kutta scheme. Figure 5 demonstrates the gain in P&L using the optimal trading strategy (26) over the constant volatility Gârleanu and Pedersen strategy.

For the fast-scale stochastic volatility, we compare our proposed strategy (26) with the zeroth order Gârleanu and Pedersen constant volatility strategy. With parameters $\rho = 0.2, \gamma = 5, m = 0.5, \beta = \sqrt{0.5}, K = 1, \rho_1 = 0.5, \eta = 0.5, \kappa = 1, \epsilon = 0.25$, we record a gain in P&L of 23.91 bps. See Figure 5 for the distribution of the difference in P&L between the proposed strategy (26) and the Gârleanu and Pedersen strategy.

In the case of slow-scale stochastic volatility, we compare the propsed leading-order correction to the optimal trading rate with the Gârleanu and Pedersen (2009) strategy with the *current* volatility $\sigma(Z_t)$. The



(a) Fast-scale.



(b) Slow-scale.

Figure 5: Gain in P&L in the fast and slow-scale stochastic volatility model. Left panel: distribution of the difference in P&L between the proposed strategy and the Gârleanu and Pedersen strategy; parameters chosen are $\gamma = 5, \beta = \sqrt{0.5}, \rho_1 = 0.5, \epsilon = 0.25$. Right panel: gain in P&L in the slow-scale Heston stochastic volatility model for different starting value of the volatility factor Z_0; parameters chosen are $\gamma = 1, \beta = 0.25, \rho_2 = 0.5, \sqrt{\delta} = 0.25$. Other parameters are as in Figure 2.

gain in P&L depends on the initial value of volatility factor Z_0. Table 7.3 demonstrates the gain in P&L of the proposed trading strategies under the slow-scale Heston stochastic volatility model for different starting value of the volatility factor Z_0. We note that the proposed algorithm provides an improvement on the trading P&L for all starting values Z_0; while the gain in P&L is most significant when the initial volatility factor Z_0 is below its long-term level m.

Table 3: Gain in P&L in the slow-scale Heston stochastic volatility model; parameters chosen are $\gamma = 1, \beta = 0.25, \rho_2 = 0.5, \sqrt{\delta} = 0.25$. Other parameters are as in Figure 2.

Z_0	mean (bps)	std error (bps)
0.1	40.157	15.497
0.2	29.617	9.750
0.3	10.560	6.826
0.4	10.928	5.242
0.5	7.005	4.061
0.6	7.686	3.298
0.7	5.183	2.686
0.8	0.637	2.339
0.9	3.705	2.041

8 Conclusion

The impact of stochastic volatility on the problem of dynamic trading can be studied and quantified through asymptotic approximation, which are tractable to compute. We have derived the first two terms of the approximations for the Gârleanu and Pedersen (2009, 2013) value function, when volatility is driven by a single fast or slow factor, and Section 6 shows how these can be combined to incorporate both long and short time scales of volatility fluctuations. Using numerical examples and Monte-Carlo simulations, we have demonstrated that our proposed strategies is efficient to compute and they offer improvement in the Profit&Loss when the volatility process is characterized by its time scales of fluctuation.

There are a number of directions where similar techniques may play an effective role and we mention a few.

1. The impact of stochastic liquidity on the optimal portfolio, first formulated by Almgren (2012) in continuous time and later extended by Cheridito and Sepin (2014) in discrete time, is clearly of interest and a challenge. We refer to Gatheral and Schied (2013) for modern developments and background. The joint asymptotics to study the impact on portfolio choice of friction from both stochastic liquidity and stochastic volatility will be considered in a future paper.

2. In the present paper, the trading signals are considered observable. In practice, they are often observed with high signal-to-noise ratio. It is therefore important to quantify the impact of partial observations on the optimal trading behavior. In similar spirit to the Black and Litterman (1991) model, one can incorporate investors' views on upcoming performance are incorporated along with any degree of uncertainty that the investor may have in these views. The treatment of optimal trading with partial observations and intermittent insertion of expert opinions is the subject of an upcoming paper.

A Full second order asymptotics

In this appendix we provide the full second order asymptotic expansion to the multiscale stochastic volatility model in Section 6. At order $\varepsilon^{1/2}$ of equation (38), we have

$$0 = \frac{1}{K} v_q^{(0)} v_q^{(1,1)} - \rho v^{(1,1)} - \kappa x v_x^{(1,1)} + \frac{1}{2}\eta v_{xx}^{(1,1)} + \mathcal{L}_0 v^{(3,1)} + \rho_{12} a(y) g(z) v_{yz}^{(2,0)}.$$

When viewed as a Poisson equation for $v^{(3,1)}$, this gives the solvability condition for $v^{(1,1)}$:

$$0 = \frac{1}{K} v_q^{(0)} v_q^{(1,1)} - \rho v^{(1,1)} - \kappa x v_x^{(1,1)} + \frac{1}{2}\eta v_{xx}^{(1,1)} + G(z) q^2, \tag{48}$$

where the source term G is given by

$$G(z) = -\frac{1}{2}\rho_{12} g(z) \left\langle a(y) \frac{\partial^2}{\partial y \partial z} \varphi(y,z) \right\rangle.$$

The equation (48) can be solved using a quadratic ansatz

$$v^{(1,1)}(q,x,z) = \frac{1}{2} C_{qq}(z) q^2 + C_{qx}(z) q x + \frac{1}{2} C_{xx}(z) x^2 + C_0(z), \tag{49}$$

where

$$C_{qq}(z) = \frac{G(z)}{\frac{A_{qq}(z)}{K} + \frac{\rho}{2}}, \quad C_{qx}(z) = \frac{A_{qx}(z) C_{qq}(z)}{\frac{A_{qq}(z)}{K} + \rho + \kappa}, \quad C_{xx}(z) = \frac{\frac{A_{qx}(z)}{K} C_{qx}(z)}{\kappa + \rho/2}, \quad C_0(z) = \frac{\eta}{2\rho} C_{xx}(z).$$

Returning to the slow scale expansion (34), we extract the order δ term in (32) to obtain the following equation for $v^{\epsilon,2}$:

$$\begin{aligned} 0 = &\frac{1}{2K}\left(\left(v_q^{\varepsilon,1}\right)^2 + 2 v_q^{\varepsilon,0} v_q^{\varepsilon,2}\right) - \rho v^{\varepsilon,2} - \kappa x v_x^{\varepsilon,2} + \frac{1}{2}\eta v_{xx}^{\varepsilon,2} + \frac{1}{\varepsilon}\mathcal{L}_0 v^{\varepsilon,2} \\ &+ \mathcal{M}_2 v^{\varepsilon,0} + \rho_1 \sqrt{\frac{\eta}{\varepsilon}} a(y) v_{xy}^{\varepsilon,2} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{\varepsilon,1} + \frac{1}{\sqrt{\varepsilon}} \rho_{12} a(y) g(z) v_{yz}^{\varepsilon,1}. \end{aligned} \tag{50}$$

The order ε^{-1} terms in (50) give $\mathcal{L}_0 v^{(0,2)} = 0$ and we take $v^{(0,2)} = v^{(0,2)}(q,x,z)$ independent of y. At order $\varepsilon^{-1/2}$, we have $\mathcal{L}_0 v^{(1,2)} = 0$ and so again $v^{(1,2)} = v^{(1,2)}(q,x,z)$. At order one:

$$0 = \frac{1}{2K}\left(v_q^{(0,1)}\right)^2 + \frac{1}{K}v_q^{(0)}v_q^{(0,2)} - \rho v^{(0,2)} - \kappa x v_x^{(0,2)} + \frac{1}{2}\eta v_{xx}^{(0,2)} + \mathcal{L}_0 v^{(2,2)} + \mathcal{M}_2 v^{(0)} + \rho_2\sqrt{\eta}g(z)v_{xz}^{(0,1)}. \quad (51)$$

When viewed as a Poisson equation for $v^{(2,2)}$, this yields the following solvability condition for $v^{(0,2)}$:

$$0 = \frac{1}{2K}\left(v_q^{(0,1)}\right)^2 + \frac{1}{K}v_q^{(0)}v_q^{(0,2)} - \rho v^{(0,2)} - \kappa x v_x^{(0,2)} + \frac{1}{2}\eta v_{xx}^{(0,2)} + \mathcal{M}_2 v^{(0)} + \rho_2\sqrt{\eta}g(z)v_{xz}^{(0,1)}.$$

This can again be solved using a linear-quadratic ansatz

$$v^{(0,2)}(q,x,z) = \frac{1}{2}D_{qq}(z)q^2 + D_{qx}(z)qx + \frac{1}{2}D_{xx}(z)x^2 + D_0(z), \quad (52)$$

where

$$\begin{aligned} D_{qq}(z) &= -\frac{\mathcal{M}_2 A_{qq}(z)}{\rho + \frac{2}{K}A_{qq}(z)}, \quad D_{qx}(z) = \frac{\mathcal{M}_2 A_{qx}(z) + \frac{1}{K}A_{qx}(z)D_{qq}(z)}{\frac{1}{K}A_{qq}(z) + \rho + \kappa}, \\ D_{xx}(z) &= \frac{\frac{1}{K}A_{qx}(z)D_{qx}(z) + \frac{1}{2}\mathcal{M}_2 A_{xx}(z)}{\kappa + \rho/2}, \quad D_0(z) = \frac{1}{\rho}\left(H(z) + \frac{1}{2}\eta D_{xx}(z)\right), \end{aligned} \quad (53)$$

and

$$H(z) = \frac{1}{2K}B_q(z)^2 + \mathcal{M}_2 A_0(z) + \rho_2\sqrt{\eta}g(z)B_x'(z).$$

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