# FRACKING, RENEWABLES & MEAN FIELD GAMES

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**Abstract.** The dramatic decline in oil prices, from around \$110 per barrel in June 2014 to less than \$40 in March 2016, highlights the importance of competition between different energy sources. Indeed, the sustained price drop has been primarily attributed to OPEC's strategic decision not to curb its oil production in the face of increased supply of shale oil in the US, spurred by the technological innovation of "fracking". We study how continuous time Cournot competitions, in which firms producing similar goods compete with one another by setting quantities, can be analyzed as continuum dynamic mean field games. In this context, we illustrate how the traditional oil producers may react in counter-intuitive ways in face of competition from alternative energy sources.

**1. Introduction.** The recent rapid fall in the price of oil is arguably the biggest energy story of the past two years. Back in June 2014, the price of Brent crude was up around \$115 per barrel. As of January 23, 2015, it had fallen by more than half, down to \$49 per barrel, and fell into the \$30 range in March 2016 (see Figure 1). The
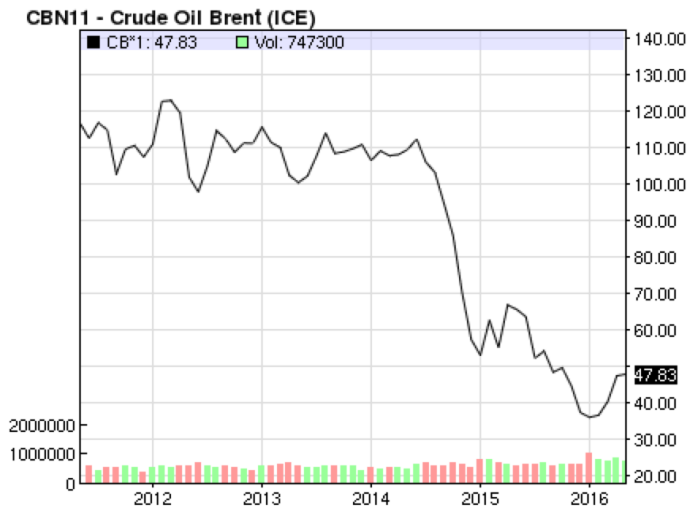


Fig. 1: End of day Commodity Futures Price Quotes for Crude Oil. Source: `www.nasdaq.com`

dramatic decline in oil prices illustrates the evolution of the global energy market as competition between different energy sources expands. Indeed, the sustained price drop was prompted in large part by OPEC's decision not to curb its oil production in the face of increased supply of shale gas and oil in the US, itself arising from technological advances such as hydraulic fracturing and horizontal drilling, collectively referred to as fracking.

The goal of the present paper is to explain how dynamic game theory, in particular mean field games proposed by Lasry and Lions [17] and Huang *et al.* [14, 15], can be used to explain some of the strategic interactions between various energy producers.

*How OPEC sets production.* The Organization of Petroleum Exporting Countries (OPEC) is a cartel of oil-producing nations that accounts for about 40% of the world's oil production. Comprising of twelve member countries (including key oil nations like Saudi Arabia, Iran, Iraq, and the UAE), OPEC mandates to "coordinate and unify the petroleum policies" of its members and to "ensure the stabilization of oil markets in order to secure an efficient, economic and regular supply of petroleum to consumers, a steady income to producers, and a fair return on capital for those investing in the petroleum industry."[1] OPEC typically meets twice a year to set production quotas. As with most commodities, the price of oil is mainly dictated by supply and demand. Since the supply of oil was determined in large part by OPEC, the higher they set their quotas, the lower the oil price.

Oil prices had been high since 2010 through the middle of 2014, bouncing around $110 per barrel because of escalating oil consumption in countries like China and political instability in key oil nations like Iraq. Given the high oil prices, many energy companies (most notably Chevron Corporation, Exxon Mobil Corp and ConocoPhillips Co) found it profitable to begin extracting oil from difficult-to-drill places. In the United States, companies began using techniques like hydraulic fracturing and horizontal drilling to extract oil from shale formations in North Dakota and Texas.

*Plummeting oil price.* Hydraulic fracturing, or "fracking", is the process through which oil and gas are released from shale deposits deep underground by means of drilling and injecting pressurized liquid made of water, sand, and chemicals. According to the US Energy Information Administration, there are over 500,000 active natural gas wells in the US as of 2011, adding significantly to the world oil supply. To put this in context, the US fracking industry has added nearly 4 million extra barrels of crude oil per day to the global market since 2008 (compared to global production of about 75 million barrels per day). This surge in supply, together with a lack of demand due to sluggish global economic growth, led to a fall in oil price of nearly 50% over the second half of 2014. As oil prices tumbled, most observers expected to see OPEC, the world's largest oil cartel, cut back on production to push prices back up.

*OPEC's war on fracking.* This brings us to the OPEC Conference in Vienna on 27 November 2014. Some countries, like Venezuela and Iran, wanted the cartel to cut back on production in order to boost the price. On the other side of the debate, Saudi Arabia didn't want to give up market share and refused to reduce production – in the hopes that lower oil prices would help impede expansion of the fracking industry. In the end, despite the oversupply on the world market, OPEC failed to agree on a response and ended up keeping production unchanged. So the price of oil began declining even further.

The price of oil has hovered in the $40-55/barrel range for most of the first half of 2015, but it is estimated that many fracking companies need prices above $60-80 to break even. There is now speculation that many fracking operations may be forced into closure. The theory is that OPEC is now engaged in a "price war" with the US frackers. Led by powerful oil nations such as Saudi Arabia, OPEC is seeking to drive the fracking industry out of business, once again regain its place as the world's

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[1]Source: `http://www.opec.org/opec_web/en/publications/345.htm`

pre-eminent source of oil, and stabilize oil prices well above the present level. This is the central issue of *blockading* we want to model using dynamic game theory.

**1.1. Competitive oligopolistic view.** We take a competitive oligopolistic view of an idealized global energy market, in which game theory describes the outcome of competition. Oligopoly models of markets with a small number of competitive players go back to the classical works of Cournot [5] and Bertrand [2] in the 1800s. The Cournot and Bertrand models differ on the assumptions about the strategic variables a firm chooses to compete with its rivals. The Bertrand model assumes that firms compete on price while the Cournot model assumes that the competition is on output quantity. The Cournot framework of oligopoly is appropriate for energy production in which major players determine their output relative to their production costs, as in the expected scenario that OPEC will cut production in order to increase the market price of oil.

In the context of nonzero-sum dynamic games between $N$ players, each with their own resources, the computation of a Nash equilibrium is a challenging problem, typically involving coupled systems of $N$ nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs), with one value function per player. This is further complicated by the fact that the players' resources are exhaustible and the market structure changes over time as players deplete their reserves and exit the market. Harris *et al.* [12] study a Cournot version of the problem, and Ledvina and Sircar [18] study a similar problem in the Bertrand framework.

Meanwhile, mean field games proposed by Lasry and Lions [17] and independently by Huang *et al.* [14, 15] allow one to handle certain types of competition in the continuum limit of an infinity of small players by solving a coupled system of *two* PDEs. The interaction here is such that each player only sees and reacts to the statistical distribution of the states or actions of other players. Optimization against the distribution of other players leads to a backward HJB equation; and in turn their actions determine the evolution of the state distribution, encoded by a forward Kolmogorov equation. This continuum approximation allows for analytical and computational results which are hard to obtain from the $N$-player system.

Our goal is to extend the basic Cournot mean field game (MFG) model in [4], and study the competition between the traditional energy producers and alternative sources. In this setting, the economy is framed as a *Cournot competition* where the market model is specified by inverse demand functions, which give prices as a function of quantities produced. In the context of a global energy market, we model OPEC by a continuum of oil producers with low costs of production, but each member nation has a finite reserve. The other side of the economy is represented by an alternative energy producer (e.g. the fracking industry in the US or renewable production such as from solar technology) with relatively costly production. However, the alternative energy producer is distinguished from the traditional oil producers by its relative abundance of production capacity. Throughout this paper, we make the simplifying assumption that the alternative energy source is inexhaustible relative to the traditional source.

*"The Stone Age did not end for lack of stone, and the Oil Age will end long before the world runs out of oil."* With recent technological advances such as renewables and fracking, this intriguing quote of former Saudi oil minister Sheikh Zaki Yamani may not be far-fetched and the end of the traditional oil age may be upon us. In this paper, our central innovation is to consider the interaction and competition between traditional and alternative energy producers. We do so by considering three distinct time scales representing different idealizations of the global energy market. Table

1 describes the three time scales under consideration and the main features of each horizon.

Table 1: Three distinct time scales representing different idealizations of the global energy market.

| Horizon | Global Energy Market |
| --- | --- |
| **Long-term** | Over a longer horizon, alternative energy sources become more competitive as their production costs decrease further due to technological advances. While traditional fossil fuels are not necessarily depleted, renewable energy gains considerable market share due to increasing (scarcity) costs of fossil fuel extraction. We study how the global economy transitions from the traditional/exhaustible energy production to its renewable counterpart (Section 3). |
| **Intermediate** | Over an intermediate horizon, the production of traditional energy (fossil fuels e.g. oil) is still the cheapest. However, alternative energy sources (shale oil or solar power) are gaining market share due to decreasing production costs. Traditional energy producers may strategically increase their production rate to compete for market share with the alternative energy producers. This may be the logic behind OPEC's decision not to cut crude oil output. Section 1.3 below illustrates the issue of blockading in the simplest setting of static games. Competition between exhaustible fossil fuels and renewable alternatives is studied in a dynamic model in Section 4. |
| **Short-term** | Over a short time frame, traditional energy sources are not going to run out. The major determinant of the energy price level is supply shocks due to exploration successes, itself arising from investment in research and development. We model the joint strategic decision of (costly) exploration effort and production rate in this context (Section 5). |

**1.2. Market Structure and Participants.** Typically we are interested in competition between traditional oil producers and an alternative energy producer. In this setting there is a continuum of traditional oil producers labelled by "position" $x$ and density $m(x)$. We denote the quantity of the traditional (resp. alternative) producers by $q(x)$ (resp. $\widehat{q}$), and the average production of the traditional producers by $Q$:

$$Q = \int q(x)m(x)\,dx.$$

The (Cournot) market structure is defined by a decreasing inverse demand function $P$.

The price $p = P(q + \epsilon Q + \delta\widehat{q})$ received by an oil producer is decreasing in his own production quantity $q$, the average quantity $Q$ produced by the other players, and the quantity $\widehat{q}$ of the alternative energy producer. Here $\epsilon, \delta \geq 0$ are interaction parameters that measure the impact of the other players. Similarly, the price $\widehat{p} = P(\widehat{q} + \delta Q)$ received by the alternative energy producer is decreasing in his own production quantity $\widehat{q}$ as well as the average production quantity $Q$ of the oil producers.

Often we will take $P$ to be linear and $\epsilon$ and $\delta$ to be equal to one to make various formulas easier to read and for illustration. We note that $Q$ is the average production

so the impact of the other players on the representative player's price is different from the impact of his own price $q$ even when $\epsilon = 1$.

**1.3. Static MFG and blockading.** To illustrate the effect of blockading in the simplest setting, we consider a static (one-period) competition between traditional oil producers and an alternative energy producer. We consider linear inverse demand function $P(\xi) = 1 - \xi$. The producer at position $x$ has cost of production $c(x)$. In addition, there is an alternative energy producer with cost $c_0$.

In a Nash equilibrium $(q^*(x), \widehat{q}^*)$ for the "$\infty + 1$" players, each one maximizes profit as a best response to the other players' equilibrium strategies:

$$\sup_{q \geq 0} q(1 - q - Q - \widehat{q}^* - c(x)), \qquad \sup_{\widehat{q} \geq 0} \widehat{q}(1 - \widehat{q} - Q - c_0),$$

where now $Q = \int q^* m$. If there is an interior maximum (i.e. each player having positive equilibrium production), then we have

(1.1) $$q^*(x) = \frac{1}{2}\left(1 - Q - \widehat{q}^* - c(x)\right), \qquad \widehat{q}^* = \frac{1}{2}(1 - Q - c_0).$$

Integrating $q^*$ against $m$ and solving for $Q$ using the above expression for $\widehat{q}^*$ yields

$$Q = \frac{1}{3}\left(1 - \widehat{q}^* - \langle c \rangle\right) = \frac{1}{5}\left(1 + c_0 - 2\langle c \rangle\right), \quad \text{where} \quad \langle c \rangle = \int_{\mathbb{R}_+} c(x) m(x)\, dx.$$

Consequently, from (1.1) we derive

$$\widehat{q}^* = \frac{1}{5}\left(2 - 3c_0 + \langle c \rangle\right), \qquad q^*(x) = \frac{1}{5}\left(1 - \frac{5}{2}c(x) + c_0 + \frac{1}{2}\langle c \rangle\right).$$

Blockading of the alternative producer occurs when $\widehat{q}^* \leq 0$, or equivalently when $c_0 \geq (2 + \langle c \rangle)/3$ in terms of production costs. The interpretation is that the alternative energy producer is blockaded when his cost $c_0$ is too high compared to the average production cost $\langle c \rangle$ of the traditional producers. In this case the alternative producer produces nothing $\widehat{q}^* = 0$, and the traditional producers take over the market

$$Q = \frac{1}{3}\left(1 - \langle c \rangle\right), \quad \text{which leads to} \quad q^*(x) = \frac{1}{3}\left(1 - \frac{3}{2}c(x) + \frac{1}{2}\langle c \rangle\right).$$

In this case, we say that the alternative energy producer is *blockaded* from production. Figure 2 shows that as $c_0$ decreases (representing increased competitiveness of the costly alternative energy source), the traditional low-cost producer may strategically choose not the reduce production in an attempt to keep the alternative producer blockaded. In our context, this may be OPEC holding back on cuts in production to drive shale oil producers out of the market and into bankruptcy.

We study in Section 4 a dynamic version of this game incorporating exhaustibility of the traditional fuel.

**1.4. Related Literature.** Our paper is related to two different strands of literature: the literature of dynamic oligopoly with exhaustibility, and the literature on the use of mean field games in economic applications.
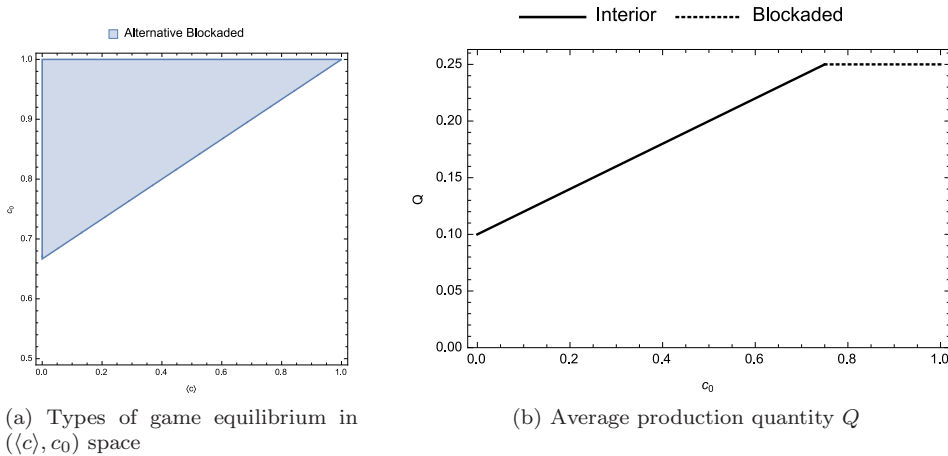
(a) Types of game equilibrium in $(\langle c \rangle, c_0)$ space

(b) Average production quantity $Q$

Fig. 2: Static Cournot duopoly with linear demand $P(\xi) = 1 - \xi$. When $c_0$ is large relative to fixed $\langle c \rangle$, the alternative energy producer is blockaded from production, and the traditional oil producers take over the entire market.

*Dynamic Oligopoly.* The study of static oligopoly models of markets with a small number of competitive players goes back to the classical works of Cournot [5] and Bertrand [2] in the 1800s. More recently, energy markets have been modeled through dynamic games. Harris *et al.* [12] characterize a dynamic Cournot game in an oligopoly market by systems of nonlinear Hamilton-Jacobi PDEs. Ledvina and Sircar [18] study the corresponding Bertrand game in which firms compete with one another by setting prices. We refer to Dockner [8] for an introduction to the applications of dynamic games in economics and management science.

Hotelling [13] introduced one of the first models for the management of an exhaustible resource. In a monopoly setting, Hotelling solved a calculus-of-variations problem and showed that the marginal value of reserves grows at the discount rate along the optimal extraction path, which is now referred to as Hotelling's rule. The competition between a single exhaustible producer and $N-1$ renewable producers has been considered by Ledvina and Sircar [19]. This simplified setup provides insights into the effect of blockading: how low must oil reserves go before it becomes profitable for the renewable producers to enter the market. It also leads to a modified piecewise version of Hotelling's rule.

Other aspects of the exhaustibility issue are renewability and exploration. For example, while fossil fuels are ultimately exhaustible, they are also replenishable by (costly) exploration efforts. The optimal planning of exploration effort has been considered by Pindyck [25] and many others in the monopoly context, and Ludkovski and Sircar [21] in a dynamic duopoly. Dynamic Cournot games when the demand function is stochastic are studied in Ludkovski and Yang [24]. For a recent survey of game theoretic models for energy production, we refer to Ludkovski and Sircar [22].

*Mean Field Games.* Second, there is also a literature on the use of mean field games to economic applications. Since the seminal papers by Lasry and Lions [17] and Huang *et al.* [14, 15], this approximation technique has attracted considerable interest recently as the corresponding $N$-player dynamic games are almost always

intractable using PDE methods. In the context of energy production, Guéant *et al.* [10, 11] have considered a mean field version of a Cournot game with a quadratic cost function; while Chan and Sircar [4] apply asymptotic and numerical methods to study how substitutability affects the market equilibrium in Bertrand and Cournot mean field games.

There are many other applications of mean field games in economics, and we list only a few. Lucas and Moll [20] study knowledge growth in an economy with many agents of different productivity levels. In particular, what they call a "balanced growth path" resembles our sustainable economy in Section 5. Carmona *et al.* [3] present a mean field game model for analyzing systemic risk. Mean field games analysis has been adopted to study the optimal execution problem in algorithmic trading by Jaimungal and Nourian [16]. For a comprehensive study of the uniqueness and existence of equilibrium strategies of a general class of mean field games, including the linear-quadratic framework, we refer to Bensoussan *et al.* [1]. We explain the differences between the type of problem considered in the bulk of this literature and our type of problem in Section 2.1, and what is known about existence and uniqueness for this kind of system in Appendix A.

**1.5. Organization and Results.** We study the interaction between the traditional and alternative energy producers from three perspectives: *competition*, *transition*, and *exploration*.

In Section 2, we revisit the basic framework for dynamic Cournot mean field games with exhaustible resources and extend the results of [4] to include nonlinear demand functions. Section 3 considers an economy in which the exhaustible producers can transition to an alternative energy source (e.g. solar or hydroelectric power) when they run out of reserves. This essentially introduces a Neumann boundary condition to the PDE problem. We provide explicit leading-order correction to the value function in the regime of small exhaustibility.

Section 4 investigates the competitive interaction between the exhaustible oil producers with an alternative energy producer, who has marginal cost of production $c > 0$, but inexhaustible supplies. We shall see the blockading of the renewable producer when his production cost $c$ is high enough and when the exhaustible resource is still abundant. Section 5 deals with exploration and exhaustibility. Incorporating the stochastic effect of resource exploration into the dynamic Cournot framework, we study the equilibrium production rate and exploration effort in a sustainable economy. This corresponds to a steady-state solution to the MFG PDE problem. We conclude in Section 6. Appendix A discusses an existence and uniqueness theorem for a classical solution to the type of MFG system considered here.

**2. Dynamic Cournot model.** The basic mean field game model in [4] will serve as a baseline for our analysis of competition between the traditional energy producers and the alternative sources. While the exposition there focuses on a Bertrand (price-setting) model, it is shown in their Appendix B that in the continuum mean field setting, the dynamic Cournot and Bertrand games are identical. Since our focus in this paper is the global energy market in which the Cournot framework is more appropriate, we elaborate on and extend the dynamic Cournot mean field game model to nonlinear demand functions in this section. To introduce notation and ideas, we concentrate in this section only on the exhaustible oil producers without competition from the alternative producers, which are introduced in Section 3.

**2.1. Dynamic continuum mean field games.** In the dynamic problem, firms produce energy by depleting their reserves of a fossil fuel, and different producers have different levels of initial reserves. When they exhaust their reserves, they no longer participate and the market shrinks. There is an infinity of players labelled by their reserves $x > 0$, with initial density of reserves $M(x)$. They choose production rates $q_t$ which deplete the remaining reserve $X_t$; moreover the reserve level may be subject to random fluctuations (e.g. due to noisy seismic estimation of the oil or gas well). The reserve level $X_t$ follows the dynamics

$$dX_t = -q_t \, dt + \sigma \, dW_t,$$

as long as $X_t > 0$, and $X_t$ is absorbed at zero. Here $W$ is a standard Brownian motion, and $\sigma \geq 0$ is a constant.

A firm that starts with reserve $x > 0$ at time $t \geq 0$ sets quantities to maximize the lifetime profit discounted at constant rate $r > 0$ over Markov controls $q_t = q(t, X_t)$, with the corresponding price $p_t = p(t, X_t)$ given by the inverse demand to be specified below. Hence the value function of the firm is defined by

$$(2.1) \qquad v(t,x) = \sup_q \mathbb{E}\left\{ \int_t^\infty e^{-r(s-t)} p_s q_s \mathbb{1}_{\{X_s>0\}} \, ds \,\middle|\, X_t = x \right\}, \quad x > 0.$$

The game runs till some exhaustion time $T$ (which may be infinite) when all producers have exhausted their reserves, and $T$ has to be determined endogenously as part of the problem.

The price received by the player depends on his own production quantity $q_t$ as well as the mean production rate $Q(t)$, according to

$$(2.2) \qquad p_t = P(q_t + \varepsilon Q(t)), \qquad Q(t) = \int_{\mathbb{R}_+} q(t,x) m(t,x) \, dx,$$

where $m(t,x)$ denotes the density of producers' reserves at time $t > 0$, and $P$ is a decreasing inverse demand function. We note that the price received by a representative producer depends on the other players through their *mean* production rate $Q$, and so the interaction is of mean field type. The parameter $\varepsilon$ measures the degree of interaction or product substitutability, in the sense that the price received by an individual firm decreases as the other firms increase production of their goods.

In [4], the inverse demand function is taken to be linear: $P(Q) = 1 - Q$. In this section, we extend their results to nonlinear inverse demand function $P$ of power type:

$$(2.3) \qquad P(Q) = \begin{cases} \frac{\eta}{1-\rho}\left(1 - \left(\frac{Q}{\eta}\right)^{1-\rho}\right), & \rho \neq 1, \\ \eta\left(\log \eta - \log Q\right), & \rho = 1. \end{cases}$$

The parameter $\rho$ is known as the relative prudence. Notice that we recover the special case of linear demand by setting $\rho = 0$ and $\eta = 1$. Following [18], we focus on the case where $\rho < 1$ in which the choke price $P(0^+) = \eta/(1-\rho)$ is finite. This family of pricing function is shown in [12] to be particularly tractable for the computation of Nash equilibrium in the static Cournot game. It is further analyzed in [23].

**2.2. Dynamic programming and the HJB equation.** The HJB equation associated to (2.1) is

$$(2.4) \qquad \partial_t v + \frac{1}{2}\sigma^2 \partial_{xx}^2 v - rv + \max_{q \geq 0} q \left( P(q + \varepsilon Q(t)) - \partial_x v \right) = 0, \quad x > 0,$$

where the inverse demand $P$ is given by (2.3). Notice that the embedded optimization can be interpreted as a *static* profit maximization problem for a producer with (shadow) cost $\partial_x v$ facing competitors' production $Q(t)$. The optimal production quantity $q^*$ is given (implicitly) by the first order condition

$$(2.5) \quad P'(q^*+\varepsilon Q(t))q^*+P(q^*+\varepsilon Q(t))-\partial_x v(t,x) = 0, \quad Q(t) = \int_{\mathbb{R}_+} q^*(t,x)m(t,x)\, dx.$$

When a player runs out of reserves, he no longer produces or makes income, and so we have the boundary condition $v(t,0) = 0$. At the exhaustion time $T$, $v(T,x) = 0$, but as mentioned before, the terminal time $T$ when all oil runs out has to be determined endogenously.

Given the rate of depletion $q^*(t,x)$, we can determine the 'population dynamics' of the producers by the forward Kolmogorov equation

$$(2.6) \qquad \partial_t m - \frac{1}{2}\sigma^2\partial^2_{xx}m - \partial_x\left(q^*m\right) = 0,$$

with $m(0,x) = M(x)$. The system (2.4) and (2.6) is an example what Lasry and Lions [17] have called a mean field game. The backward evolution equation (2.4) represents the firms' decisions based on anticipating how the game unfolds in the future; and the forward evolution equation (2.6) represents where they actually end up, based on their strategic decision and initial distribution. The forward/backward system of PDEs is coupled through the dependence of $Q$ in the HJB equation (2.4) on $m$, and the dependence of $q^*$ in the forward Kolmogorov equation (2.6) on $\partial_x v$.

REMARK 2.1. *After solving for the internal maximization provblem in the HJB equation (2.4), the system (2.4) and (2.6) is of the form*

$$(2.7) \qquad \partial_t v + \frac{1}{2}\sigma^2\partial^2_{xx}v - rv + H(t,\partial_x v,[h(\partial_x v)m]) = 0,$$
$$\partial_t m - \frac{1}{2}\sigma^2\partial^2_{xx}m - \partial_x G\left(t,\partial_x v,[h(\partial_x v)m]\right) = 0,$$

*for some functions $H$, $G$ and $h$. Here, $H$ and $G$ depend on $h(\partial_x v)m$ nonlocally, specifically $[h(\partial_x v)m] = \int h(\partial_x v)m\, dx$.*

*The vast majority of the MFG literature studies mean field game models of the form*

$$(2.8) \qquad \partial_t v + \frac{1}{2}\sigma^2\partial^2_{xx}v - rv + H(t,x,\partial_x v) = V[m],$$
$$\partial_t m - \frac{1}{2}\sigma^2\partial^2_{xx}m - \partial_x\left(G(t,x,\partial_x v)\right) = 0,$$

*where $V[m]$ is a monotone operator. But in the systems (2.7) arising in this paper, the coupling between $\partial_x v$ and $m$ is non-local.*

*The models leading to systems of the form (2.8) have interaction through the states, namely $\int xm$, whereas for Cournot oligopoly models, interaction is through the controls, that is $\int qm$, which leads to equations of the form (2.7). However, motivated by our previous paper [4], the recent preprint of Graber & Bensoussan [9] addresses existence and uniqueness of a classical solution to systems of the type (2.7), specifically when the inverse demand function $P$, and hence $h$, is linear. We discuss in Appendix A how their theorem can be applied to validate the numerical work in the model we consider in Section 4.*

**2.3. Small competition asymptotics.** Solving this coupled system of equations is highly non-trivial. In this section, we focus on the deterministic setting $\sigma = 0$. The main goal is to determine the principal effect of competitiveness on the market equilibrium.

From the inverse demand function (2.2), we see that $\varepsilon$ parameterizes the degree of interaction among firms; the limit $\varepsilon = 0$ corresponds to independent markets, where each firm has a monopoly in his own market. In the small competition regime, one can formally look for an approximation to the PDE system of the form

$$v(t, x) = v_0(t, x) + \varepsilon v_1(t, x) + \varepsilon^2 v_2(t, x) + \cdots,$$
$$m(t, x) = m_0(t, x) + \varepsilon m_1(t, x) + \varepsilon^2 m_2(t, x) + \cdots.$$

Solving for the value function $v$ and the density $m$ perturbatively leads to approximations to $q^*$ and $Q$ in (2.5):

$$q^*(t, x) = q_0(t, x) + \varepsilon q_1(t, x) + \cdots, \quad Q(t) = Q_0(t) + \varepsilon Q_1(t) + \cdots.$$

**2.3.1. Monopoly value function and production rate.** The monopoly value function $v_0$ is determined by setting $\varepsilon = 0$ in the HJB equation (2.4), and after performing the maximization, we have:

$$(2.9) \qquad \partial_t v_0 - r v_0 + C \left( \frac{\eta}{1 - \rho} - \partial_x v_0 \right)^\beta = 0, \quad v_0(t, 0) = 0,$$

where $C = \beta^{-\beta} \eta^{-\frac{\rho}{1-\rho}}$ and $\beta = \dfrac{2 - \rho}{1 - \rho}$, and we recall that we only consider the cases where $\rho < 1$. The solution to (2.9) is given in the following proposition, which can be verified by direct substitution.

PROPOSITION 2.2. *The leading order (monopoly) value function is time-independent $v_0(t, x) = v_0(x)$, and it is implicitly given by*

$$(2.10) \qquad \frac{\beta C}{r} \left( \frac{1 - \rho}{\eta} \right)^{1-\beta} \mathcal{B} \left( \frac{1 - \rho}{\eta} \left( \frac{r}{C} \right)^{1/\beta} v_0^{1/\beta}; \beta, 0 \right) = x,$$

*where the incomplete beta function $\mathcal{B}(z; a, b)$ is defined by*

$$\mathcal{B}(z; a, b) = \int_0^z t^{a-1} (1 - t)^{b-1} \, dt.$$

In particular, we recover the case of linear inverse demand studied in [4] if we set $\rho = 0$ since in this case $\beta = 2$, $\mathcal{B}(z; 2, 0) = -z - \log(1 - z)$, and so $v_0$ can be written in terms of the Lambert-W function.

*Production trajectory and Hotelling's rule.* In the monopoly case, the optimal production quantity $q_0$ for each individual firm is given by setting $\varepsilon = 0$ in (2.5):

$$(2.11) \qquad P'(q_0)q_0 + P(q_0) - v_0' = 0.$$

It follows from (2.11) that

$$(2.12) \qquad q_0(x) = \left( \frac{P(0) - v_0'(x)}{\beta \eta^\rho} \right)^{\frac{1}{1-\rho}}.$$

It is easy to show that $v_0(x)$ is an increasing concave function, with $v_0(0) = 0$ and $v_0(\infty) = \eta^2(2-\rho)^{-\beta}/r$. Therefore, $v_0'(x)$ is a decreasing function, with $v_0'(0) = P(0) = \frac{\eta}{1-\rho}$ and $v_0'(\infty) = 0$. Consequently, $q_0(x)$ is an increasing function of $x$, with $q_0(0) = 0$ and $q_0(\infty) = \eta(2-\rho)^{-\frac{1}{1-\rho}}$. Therefore, players produce at a finite rate, and as they run out of reserves, they decrease their production rate to zero at $x = 0$.

Let $x_0(t; x)$ denote the optimal monopoly production trajectory starting from $x$:

$$x_0'(t) = -q_0(x_0(t)), \quad x_0(0) = x, \quad \text{and define} \quad \mathcal{S}(t) = v_0'(x_0(t)),$$

so $\mathcal{S}(t)$ is the shadow cost along the optimal production trajectory $x_0(t)$.

PROPOSITION 2.3. *The classical Hotelling's rule holds also for the continuum mean field monopoly, i.e. the shadow cost grows at the discount rate $r$ along the optimal production trajectory: $\mathcal{S}'(t) = r\mathcal{S}(t)$. It follows that the market price $\mathcal{P}(t) = P(q_0(x_0(t)))$ satisfies the following linear ODE:*

$$(2.13) \qquad \mathcal{P}'(t) = r\left(\mathcal{P}(t) - \frac{\eta}{2-\rho}\right).$$

*Proof.* First we write the monopoly ODE (2.9) as $rv_0 = q_0(P(q_0) - v_0')$, where $q_0$ satisfies the first order condition (2.11). Then differentiating the ODE with respect to $x$, and using (2.11), we obtain $rv_0' = -q_0 v_0''$. Now we compute the growth rate of the shadow cost $\mathcal{S}(t)$ along the optimal production trajectory:

$$\mathcal{S}'(t) = \frac{d}{dt}v_0'(x_0(t)) = -v_0''(x_0(t))\,q_0(x_0(t)) = r\mathcal{S}(t).$$

By direct calculation, the market price is a linear function of $v_0'$: $P(q_0(x)) = \frac{\eta + v_0'(x)}{2-\rho}$. It follows easily that $\mathcal{P}$ also satisfies a linear ODE, which is given by (2.13). □

**2.3.2. Monopoly exhaustion times.** We define the hitting time $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ to be the time to exhaustion in the deterministic monopoly market starting at initial reserve $x$:

$$(2.14) \qquad \tau(x) = \inf\{t \geq 0 \mid x_0(t; x) = 0\}.$$

Even though there does not seem to be an explicit expression for the reserve trajectory, the exhaustion time $\tau(x)$ can be given explicitly in the following proposition.

PROPOSITION 2.4. *The exhaustion time $\tau(x)$ is given explicitly by*

$$(2.15) \qquad \tau(x) = \frac{1}{r}\log\left(\frac{P(0)}{v_0'(x)}\right).$$

*Moreover, the exhaustion time $\tau$ can be inverted in closed-form to give*

$$(2.16) \qquad \tau^{-1}(t) = \frac{\beta C}{r}P(0)^{\beta-1}\mathcal{B}\left(1 - e^{-rt}; \beta, 0\right).$$

*Proof.* From the definition (2.14) of $\tau(x)$, and using that $v_0'$ is a monotonic function with $v_0'(0) = P(0)$, we write

$$\tau(x) = \inf\{t \geq 0 \mid v_0'(x_0(t)) = P(0)\} = \inf\{t \geq 0 \mid v_0'(x)e^{rt} = P(0)\},$$

where we have used Hotelling's rule given in Proposition 2.3. The expression (2.15) follows. From the HJB equation (2.9), we derive

$$v_0(x) = \frac{C}{r} P(0)^\beta \left(1 - e^{-r\tau(x)}\right)^\beta.$$

Plugging this into the expression (2.10), and identifying $x = \tau^{-1}(t)$ lead to (2.16). $\square$

In the special case of linear inverse demand (where $\rho = 0$ and $\eta = 1$), we have $\tau(x) = -r^{-1} \log \left[-\mathbb{W}(\theta(x))\right]$, where $\mathbb{W}$ is the Lambert-W function, and $\theta(x) = -e^{-2rx-1}$, which recovers the one found in [4].

When the initial density $M(x)$ has compact support $[0, x_{max}]$, all oil reserves are exhausted at finite time $T = \tau(x_{max})$; otherwise $T = \infty$.

**2.3.3. Monopoly density function.** In the deterministic monopoly setting, the forward Kolmogorov equation (2.6) reads

$$(2.17) \qquad \partial_t m_0 - \partial_x[q_0 m_0] = 0, \quad m_0(t,x) = M(x).$$

PROPOSITION 2.5. *The monopoly density is given by*

$$(2.18) \quad m_0(t,x) = \frac{q_0\left(\tau^{-1}\left(t + \tau(x)\right)\right)}{q_0(x)} M\left(\tau^{-1}\left(t + \tau(x)\right)\right) = \frac{d}{dx} F(\tau^{-1}(t + \tau(x))),$$

*where $F$ denotes the cumulative distribution function (CDF) of the initial density $M$. Moreover, the proportion $\eta_0 : \mathbb{R}_+ \to [0,1]$ of remaining firms is given by*

$$\eta_0(t) = \int_{\mathbb{R}_+} m_0(t,x) \, dx = 1 - F(\tau^{-1}(t)).$$

*Proof.* The explicit solution to (2.17) follows from the method of characteristics; while the second expression for $m_0$ in (2.18) follows from straightforward manipulation. The computation of $\eta_0$ follows similarly to [4, Proposition 6]. $\square$

**2.3.4. First order asymptotics.** Let $G$ be the supremum in the PDE (2.4): $G(\varepsilon) = \max_{q \geq 0} q \left(P(q + \varepsilon Q) - \partial_x v\right)$. The first order condition is given in (2.5), and so, to first order in $\varepsilon$, we have $G \approx G(0) + \varepsilon G'(0)$, where

$$G'(0) = (P(q_0) - v_0')q_1 + q_0(q_1 + Q_0)P'(q_0) - q_0 \partial_x v_1 = q_0 \left(Q_0 P'(q_0) - \partial_x v_1\right).$$

This leads to the equation satisfied by the first order correction $v_1$:

$$\partial_t v_1 - r v_1 - q_0 \partial_x v_1 = -q_0 P'(q_0) Q_0, \qquad v_1(t,0) = 0.$$

PROPOSITION 2.6. *The first-order correction to the value function $v_1$ is given by*

$$v_1(t,x) = -\frac{\eta}{2 - \rho} \int_0^{\tau(x)} \left(e^{-rs} - e^{-r\tau(x)}\right) Q_0(t + s) \, ds.$$

*Proof.* The proof follows similarly to [4, Proposition 8]. $\square$

Clearly $v_1 < 0$, so that competition reduces the value function from the monopoly limit $v_0$, as is expected. The first order correction $q_1$ to the optimal production quantity is

(2.19) $$q_1 = \frac{\partial_x v_1 - Q_0\left(P'(q_0) + q_0 P''(q_0)\right)}{2P'(q_0) + q_0 P''(q_0)} = \frac{\partial_x v_1 - (1-\rho)Q_0 P'(q_0)}{(2-\rho)P'(q_0)}.$$

The first-order correction to density $m_1$ satisfies the following equation

$$\partial_t m_1 - \partial_x \left[q_0 m_1 + q_1 m_0\right] = 0, \quad m_1(0, x) = 0.$$

PROPOSITION 2.7. *The first-order correction to density $m_1$ is given by*

$$m_1(t, x) = \int_0^t \frac{q_0\left(x_0(s - t; x)\right)}{q_0(x)} g\left(s, x_0(s - t; x)\right) ds,$$

*where the inhomogeneous term is given by $g = \partial_x q_1 m_0 + q_1 \partial_x m_0$.*

*Proof.* The proof follows similarly to [4, Proposition 9]. ☐

The following proposition demonstrates that the principal effect of competitive interaction is that firms slow down production and increase the exhaustion time.

PROPOSITION 2.8. *For concave pricing functions $P$ (i.e. $\rho < 0$), the first order correction $q_1$ to the equilibrium production rate is negative.*

*Proof.* From the expression (2.19) for $q_1$, it suffices to show that

$$\partial_x v_1 - (1-\rho)Q_0 P'(q_0) \geq 0.$$

Indeed, defining $F = \beta q_0 \left[\partial_x v_1 - (1-\rho)Q_0 P'(q_0)\right]$, a straightforward calculation leads to

$$F(t, x) = -r v_0'(x) \int_0^{\tau(x)} Q_0(t + s) \, ds + (1-\rho)Q_0(t)\left(P(0) - v_0'(x)\right).$$

One can readily check that $F(t, 0) = 0$ and $\partial_x F(t, x) \geq 0$ for concave $P$, and so from (2.19), $q_1 \leq 0$. ☐

**3. Transition to renewable resources.** In this section, we consider a model in which the exhaustible producers can switch to a more expensive alternative energy source (e.g. solar or hydroelectric power) when they run out of reserves. This model corresponds to the long horizon in Table 1. This yields a continuum mean field version of the continuous-time Cournot model of Harris *et al.* [12]. In the context of energy production, resources, such as oil or natural gas, have finite supply, and exhaustibility enters as boundary conditions for the PDEs. As we shall see, this essentially introduces a Neumann boundary condition to the PDE problem. We suppose there is an alternative, but costly, technology (for example solar power), and study the system using asymptotic approximation. In the regime of small costs for the alternative, the first order correction to the value function satisfies a partial integro-differential equation which is explicitly solvable.

**3.1. Deterministic model setup.** The energy market is modeled by a Cournot game which is specified by the inverse demand: $P(q, Q) = 1 - q - Q$, where $Q$ is the mean energy production (from both traditional and alternative sources). A firm with

reserves $x > 0$ at time $t \geq 0$ sets quantities to maximize lifetime discounted profits. Its value function is given by

$$(3.1) \qquad v(t,x) = \sup_q \int_t^\infty e^{-r(s-t)} p_s q_s \mathbb{1}_{\{X_s > 0\}} \, ds, \qquad X_t = x,$$

subject to the deterministic dynamics $dX_t = -q_t \, dt$.

The HJB equation associated to (3.1) reads

$$\partial_t v - rv + \max_{q \geq 0} q \left(1 - q - Q - \partial_x v\right) = 0.$$

Plugging in the optimal strategy (in feedback form)

$$(3.2) \qquad q^*(t,x) = \frac{1}{2}\left(1 - Q(t) - \partial_x v(t,x)\right),$$

the HJB equation becomes

$$(3.3) \qquad \partial_t v - rv + \frac{1}{4}\left(1 - Q(t) - \partial_x v(t,x)\right)^2 = 0.$$

On hitting the boundary $x = 0$, the player switches to an alternative inexhaustible source at marginal cost of production $c$. Playing against the mean production $Q$, his equilibrium strategy is $q^*(t,0) = \frac{1}{2}(1 - Q - c)$. Assuming continuity of the equilibrium production rates $q_t$ up to the boundary $x = 0$ leads to the Neumann boundary condition $\partial_x v(t,0) = c$. The interpretation is that, on running out of the exhaustible resource, the shadow cost $\partial_x v(t,x)$ of the player turns into the real cost $c$.

The mean production $Q(t)$ comes from two sources: the exhaustible and inexhaustible parts. We assume that no player produces from the more expensive source as long as the cheaper one is available. Thus we can write

$$(3.4) \qquad Q(t) = \frac{1}{2}\left(1 - Q(t) - c\right)\left(1 - \eta(t)\right) + \int_{\mathbb{R}^+} q^*(t,x) m(t,x) \, dx,$$

where $\eta$ is the proportion of players using the cheap energy source:

$$(3.5) \qquad \eta(t) = \int_{\mathbb{R}^+} m(t,x) \, dx,$$

and so $(1 - \eta)$ is the fraction of firms who have exhausted their traditional fuel reserves and are now producing from the alternative inexhaustible source at marginal cost $c$. Solving for $Q$ in (3.4), and using (3.2), we obtain

$$(3.6) \qquad Q = \frac{1}{3}\left(1 - c(1 - \eta) - \int_{\mathbb{R}^+} m \partial_x v \, dx\right).$$

The forward Kolmogorov equation for $m$ is

$$(3.7) \qquad \partial_t m - \frac{1}{2}\partial_x \left((1 - Q - \partial_x v) \, m\right) = 0,$$

with $m(0,x) = M(x)$.

### 3.2. Small cost expansion.

*Inexhaustible limit.* In the limit where $c = 0$, the firms are indifferent to using oil or alternative energy sources, and so costless energy is effectively inexhaustible. Consequently, the value function $v_0$ is a constant. From (3.6) with $c = 0$ and $\partial_x v_0 = 0$, we derive a constant mean production $Q_0 = 1/3$, which from the HJB equation (3.3) gives $v_0(t, x) = (9r)^{-1}$, also satisfying the boundary condition $\partial_x v_0(t, 0) = 0$. From (3.7), the density $m$ is transported at constant speed $m_0(t, x) = M(x + t/3)$.

We are interested in the case when $c$ is small but non-zero. To this end we formally look for an expansion in the small $c$ regime:

$$
\begin{aligned}
v(t, x) &= v_0(t, x) + cv_1(t, x) + \mathcal{O}(c^2), \\
m(t, x) &= m_0(t, x) + cm_1(t, x) + \mathcal{O}(c^2).
\end{aligned}
$$
(3.8)

Of course, we have just found that $v_0$ is independent of $t$ and $x$ and is given by $(9r)^{-1}$.

*First order correction: value function.* Plugging the formal expansion (3.8) into the HJB equation (3.3) we get

$$
\partial_t v_1 - rv_1 - \frac{1}{3}(Q_1 + \partial_x v_1) = 0,
$$

with $\partial_x v_1(t, 0) = 1$. Here $Q_1$ is the first order correction to the mean production, given by

$$
Q_1 = -\frac{1}{3}\left(1 - \eta_0 + \int_{\mathbb{R}^+} m_0 \partial_x v_1 \, dx\right).
$$

Therefore, we obtain the partial integro-differential equation (PIDE) for $v_1$

$$
\partial_t v_1 - rv_1 - \frac{1}{3}\partial_x v_1 + \frac{1}{9}\left(1 - \int_{\mathbb{R}^+} m_0(1 - \partial_x v_1) \, dx\right) = 0, \quad \partial_x v_1(t, 0) = 1.
$$
(3.9)

By considering an additively separable solution of the form $v_1(t, x) = f(x) + g(t)$, the solution to the above PIDE can be readily computed:

$$
v_1(t, x) = -\frac{1}{3r}e^{-3rx} - \int_0^t I(s)e^{r(t-s)} \, ds,
$$
(3.10)

where $I(t) = \frac{1}{9}\left(1 - \int_0^\infty M(x + t/3)\left(1 - e^{-3rx}\right) dx\right)$. That $v_1$ is additively separable simplifies the expression for $Q_1$ considerably:

$$
Q_1(t) = -\frac{1}{3}\left(1 - \eta_0 + \int_{\mathbb{R}^+} m_0 \partial_x v_1 \, dx\right) = -3I(t).
$$
(3.11)

In particular, we see that $Q_1$ is always negative. The economic interpretation is that increasing the costs of inexhaustible energy source slows down production of the exhaustible one. From (3.2), we can compute the equilibrium production rate:

$$
q^*(t, x) = \frac{1}{3} - \frac{c}{2}\left(Q_1(t) + e^{-3rx}\right) + \mathcal{O}(c^2).
$$

*First order correction: density.* The first order correction to density $m_1$ satisfies

$$
\partial_t m_1 - \frac{1}{3}\partial_x m_1 + \frac{1}{2}\partial_x\left((Q_1 + \partial_x v_1) m_0\right) = 0, \quad m_1(0, x) = 0.
$$
(3.12)

Having already solved for $v_1$, we can write down the solution analytically

$$
m_1(t, x) = \int_0^t h\left(s, x + \frac{t-s}{3}\right) ds, \qquad h(t, x) = -\frac{1}{2}\partial_x\left((Q_1 + \partial_x v_1) m_0\right).
$$
(3.13)

**3.3. Numerical illustration.** We illustrate the effect of exhaustibility using a numerical example. Suppose the initial distribution of reserves is given by an exponential distribution with parameter $\lambda$ (i.e. $M(x) = \lambda e^{-\lambda x}$). Then we can compute the value function correction

$$v_1(t,x) = -\frac{1}{9}\left(e^{rt}-1\right)\left(\frac{1}{r} - \frac{3e^{-\lambda t/3}}{\lambda + 3r}\right) - \frac{e^{-3rx}}{3r},$$

and the density correction

$$m_1(t,x) = \frac{1}{2}\lambda e^{-\lambda(x+t/3)}\left(\frac{r\left(3 - 3e^{-\lambda t/3}\right)}{\lambda + 3r} + \frac{(\lambda + 3r)\left(1 - e^{-rt}\right)e^{-3rx}}{r} - \frac{\lambda t}{3}\right).$$

The leading order correction to the mean production rate is

$$Q_1(t) = -\frac{1}{3}\left(1 - \frac{3re^{-\lambda t/3}}{\lambda + 3r}\right).$$

Observe that the correction to the mean quantity $Q_1$ is always negative, in other words, exhaustibility shows down production, as shown in Figure 3.



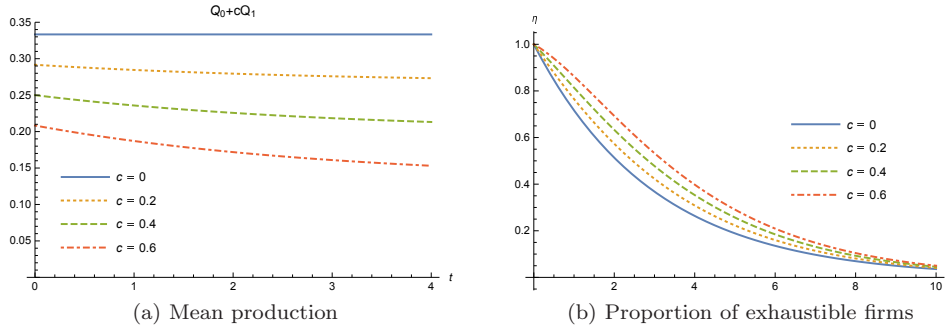(a) Mean production                        (b) Proportion of exhaustible firms

Fig. 3: Effects of exhaustibility when the exhaustible producers can transition to production of renewable resources, parameters used for numerical simulation are $\lambda = 1$ and $r = 0.2$.

**3.4. Variable production costs.** We illustrate that the technique of asymptotic expansion can be applied to the variable production cost model in [6]. In their setting, instead of an abrupt transition to the alternative technology, a firm's marginal production cost $\widehat{c}(x)$ gradually increases up to $c$ as the traditional energy reserve runs out $x \to 0$. For illustrative purpose, we will take the variable production costs to be $\widehat{c}(x) = ce^{-\gamma x}$. The interpretation is that the exhaustible producer has non-zero cost of extraction; in particular, as reserves begin to run out, costs for exhaustible producers often increase (deeper drilling, more expensive extraction technology required), and the exhaustible producer may choose to invest in R&D (research and development, including exploration) which adds to the marginal production costs.

*Model setup.* With non-zero production cost $\widehat{c}(x)$, the value function of a representative firm is

$$(3.14) \qquad v(t,x) = \sup_q \int_t^\infty e^{-r(s-t)}(p_s - \widehat{c}(X_s))q_s \mathbb{1}_{\{X_s>0\}}\, ds, \qquad X_t = x,$$

subject to the dynamics $dX_t = -q_t\, dt$. The associated HJB equation is

$$(3.15) \qquad \partial_t v - rv + \max_{q \geq 0} q\left(1 - q - Q - \partial_x v - \widehat{c}(x)\right) = 0,$$

with $\partial_x v(t, 0) = 0$. The forward Kolmogorov equation for $m$ is

$$(3.16) \qquad \partial_t m - \frac{1}{2}\partial_x \left((1 - Q - \partial_x v - \widehat{c}(x))\, m\right) = 0,$$

with $m(0, x) = M(x)$, where the mean production $Q(t)$ comes from both the exhaustible and inexhaustible producers:

$$(3.17) \qquad Q = \frac{1}{3}\left(1 - c(1 - \eta) - \int_{\mathbb{R}^+} m(\partial_x v + \widehat{c}(x))\, dx\right).$$

*Small exhaustibility expansion.* We formally look for an asymptotic expansion (3.8) in the small $c$ regime. The inexhaustible limit $c = 0$ in the present model admits a solution of a constant mean production $Q_0 = 1/3$ with $v_0(t, x) = (9r)^{-1}$. The density $m_0$ is transported at constant speed $m_0(t, x) = M\left(x + t/3\right)$. The leading-order corrections to the value function and density in the expansion (3.8) are given by the following proposition.

PROPOSITION 3.1. *The leading-order correction to the value function $v_1$ is additively separable:*

$$(3.18) \qquad v_1(t, x) = f(x) + g(t), \qquad where$$

$$(3.19) \qquad f(x) = \frac{3re^{-\gamma x} + \gamma e^{3rx}}{9r^2 + 3\gamma r}, \qquad g(t) = \frac{1}{3}\int_0^t Q_1(s)e^{r(t-s)}\, ds,$$

$$(3.20) \qquad Q_1(t) = -\frac{1}{3}\left(1 - \int_{\mathbb{R}^+} M(x + t/3)\left(1 - e^{-\gamma x} - f'(x)\right) dx\right).$$

*Moreover, the leading-order correction to the density $m_1$ is given explicitly by*
$$(3.21)$$
$$m_1(t, x) = \int_0^t h\left(s, x + \frac{t-s}{3}\right) ds, \qquad h(t, x) = -\frac{1}{2}\partial_x\left(\left(Q_1(t) + f'(x) + e^{-\gamma x}\right) m_0\right).$$

*Proof.* Plugging the formal expansion (3.8) into the HJB equation (3.15) we get

$$\partial_t v_1 - rv_1 - \frac{1}{3}\left(Q_1 + \partial_x v_1 + e^{-\gamma x}\right) = 0,$$

with $\partial_x v_1(t, 0) = 0$. Here $Q_1 = -\frac{1}{3}\left(1 - \eta_0 + \int_{\mathbb{R}^+} m_0(\partial_x v_1 + e^{-\gamma x})\, dx\right)$ is the first order correction to the mean production. Therefore, we obtain the PIDE for $v_1$:

$$\partial_t v_1 - rv_1 - \frac{1}{3}\partial_x v_1 + \frac{1}{9}\left(1 - \int_{\mathbb{R}^+} m_0(1 - \partial_x v_1 - e^{-\gamma x})\, dx\right) = \frac{1}{3}e^{-\gamma x}, \quad \partial_x v_1(t, 0) = 0.$$

By considering the additively separable solution of the form (3.18), the solution to the above PIDE can be readily computed to give (3.19) and (3.20).

As for the density, the first order correction $m_1$ satisfies

$$\partial_t m_1 - \frac{1}{3}\partial_x m_1 + \frac{1}{2}\partial_x\left(\left(Q_1 + \partial_x v_1 + e^{-\gamma x}\right) m_0\right) = 0, \quad m_1(0, x) = 0.$$

Using the method of characteristics, we obtain the solution (3.21) analytically. □

Notice in particular that as the transition rate $\gamma$ goes to infinity, the total production rate $Q_1$ in this case reduces to (3.11) as $1 - e^{-\gamma x} - f'(x) \xrightarrow[\gamma \to \infty]{} 1 - e^{-3rx}$.

*Numerical illustration.* We illustrate the effect of exhaustibility using a numerical example, where we assume that the initial reserves have an exponential density with parameter $\lambda$. Figure 4 shows the effect of the transition rate $\gamma$ on the mean production rate $Q \approx Q_0 + cQ_1$. Notice that a smoother transition (i.e. decreasing $\gamma$) leads to higher production rate.
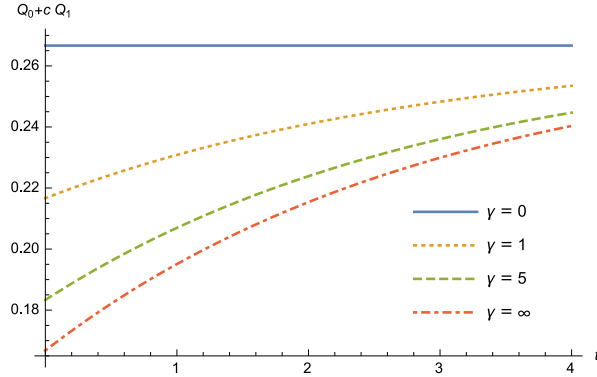


Fig. 4: Effect of the transition rate $\gamma$ on the mean production rate $Q \approx Q_0 + cQ_1$. Parameters used are $r = 0.2, \lambda = 1, c = 0.2$.

## 4. Competition with a renewable Producer.

In this section we consider the competition between the traditional oil producers with an alternative energy producer. This model corresponds to the intermediate time horizon in Table 1. The economy consists of a continuum of firms depleting a non-renewable energy source with zero marginal cost, and an alternative producer with inexhaustible reserves, but higher cost of production $c > 0$. This corresponds to sustainable production from "green" sources (e.g. solar power, or to leading order approximation, the fracking industry in the US). The two classes of producers compete against each other through the Cournot game equilibrium.

For the exhaustible producers, the remaining reserves $(X_t)$ follow the dynamics

$$dX_t = -q_t \, dt + \sigma \, dW_t,$$

as long as $X_t > 0$, where $W$ is a standard Brownian motion and $q_t = q(t, X_t)$ is his rate of production at time $t$. Each exhaustible producer has oil resources which he extracts at zero costs, and which is subject to random fluctuation (e.g. due to noisy seismic estimates of oil reserves). When he runs out, he cannot produce anymore and we have $q(t, 0) = 0$. The renewable player produces from an alternative source which is expensive but abundant: his marginal cost of production is $c > 0$. His rate of production is denoted by $\widehat{q}(t)$.

The Cournot market is specified by linear inverse demand functions. The inverse demand faced by an exhaustible producer producing at rate $q$ unit is given by $P(q, \widehat{q}, Q) = 1 - q - \delta\widehat{q} - \varepsilon Q$, where $Q$ is the mean production rate of the exhaustible producers, and $\widehat{q}$ is the production rate of the renewable producer. The inverse demand faced by the renewable producer is similarly given by $\widehat{P}(\widehat{q}, Q) = 1 - \widehat{q} - \delta Q$. Here $\varepsilon$ is the interaction parameter between exhaustible producers, and $\delta$ is the interaction parameter between exhaustible producers and the renewable producer.

The value functions for the traditional and alternative producers are their discounted lifetime profit, respectively given by:

(4.1)

$$v(t,x) = \sup_{q \geq 0} \mathbb{E} \left\{ \int_t^\infty e^{-r(s-t)} q_s p_s \mathbb{1}_{\{X_s > 0\}} \, dt \, \middle| \, X_t = x \right\},$$

$$g(t) = \sup_{\widehat{q} \geq 0} \int_t^\infty e^{-r(s-t)} \widehat{q}_s (\widehat{p}_s - c) \mathbb{1}_{\{\eta(s) > 0\}} \, ds + \int_t^\infty e^{-r(s-t)} \frac{1}{4} (1-c)^2 \mathbb{1}_{\{\eta(s)=0\}} \, ds,$$

where $\eta$ is the fraction of exhaustible producers with reserves remaining, given by (3.5). The second term in the definition of $g$ expresses that the renewable producer has a monopoly when all the exhaustible producers are out of reserves.

We also stress that $\widehat{q}$ must be non-negative: for large enough $c$, we will see that the renewable player is blockaded in that his cost of producer is so high and his competitors' reserves of the cheaper resource are so plentiful, that his equilibrium strategy is not to produce anything until the exhaustible producers have run down their reserves some more. When $c = 1$, the renewable player never participates in the game, and the above model reduces to the standard Cournot mean field game studied in Section 2.

*Dynamic programming and HJB equations.* The HJB equations associated to (4.1) read

(4.2)

$$\partial_t v + \frac{1}{2}\sigma^2 \partial_{xx}^2 v + \sup_q \left[ q \left( 1 - q - \varepsilon Q(t) - \delta \widehat{q}(t) - \partial_x v \right) \right] = rv,$$

$$g'(t) + \sup_{\widehat{q}} \left[ \widehat{q} \left( 1 - \widehat{q} - \delta Q(t) - c \right) \right] = rg.$$

From the optimal feedback control $q^*(t, X_t)$ of the exhaustible producer, the density $m$ of reserves $X_t$ follows the forward Kolmogorov equation

(4.3)
$$\partial_t m - \frac{1}{2}\sigma^2 \partial_{xx}^2 m - \partial_x (q^* m) = 0,$$

with $m(0, x) = M(x)$. The total production by exhaustible producer is given by

(4.4)
$$Q(t) = \int_{\mathbb{R}_+} q^*(t, x) m(t, x) \, dx.$$

(a) If the renewable producer is not blockaded, the feedback production rates are

$$q_{nb}^*(t, x) = \frac{1}{4} \left( 2 - \delta - (2\varepsilon - \delta^2) Q(t) + \delta c - 2\partial_x v \right), \quad \widehat{q}^*(t) = \frac{1}{2} \left( 1 - \delta Q(t) - c \right),$$

and the HJB equations become

$$\partial_t v + \frac{1}{2}\sigma^2 \partial_{xx}^2 v + \frac{1}{16} \left( 2 - \delta - (2\varepsilon - \delta^2) Q(t) + \delta c - 2\partial_x v \right)^2 = rv,$$

$$g'(t) + \frac{1}{4} \left( 1 - \delta Q(t) - c \right)^2 = rg.$$

(b) If the renewable producer is blockaded, we have $\widehat{q}^* = 0$ and

$$q_b^*(t, x) = \frac{1}{2} \left( 1 - \varepsilon Q(t) - \partial_x v \right).$$

In this case the HJB equation becomes

$$\partial_t v + \frac{1}{2}\sigma^2 \partial_{xx}^2 v + \frac{1}{4} \left( 1 - \varepsilon Q(t) - \partial_x v \right)^2 = rv, \qquad g'(t) = rg.$$

*Forward Kolmogorov equation.* Combining the two cases, we can write the forward Kolmogorov equation of the reserves density as

$$(4.5) \qquad 0 = \partial_t m - \frac{1}{2}\sigma^2 \partial_{xx}^2 m - \partial_x \left[ m \left( \mathbb{1}_B q_b^*(t,x) + \mathbb{1}_B^c q_{nb}^*(t,x) \right) \right],$$

where $\mathbb{1}_B$ is the blockading indicator function.

**4.1. Numerical solutions.** To study the full MFG equation system (4.2) and (4.5), we need to solve a coupled system of forward/backward PDE system with a free boundary. We propose an iterative algorithm to calculate the MFG solution and optimal production rate. Starting with an initial guess $Q^0$ for the total production, we follow for $n = 1, 2, \ldots$

*Step 1.* Given the mean production $Q^{n-1}$ from the previous iteration, solve the optimal control problem by numerically solving the HJB equations:
(a) The optimal strategy of the renewable producer is simply

$$\widehat{q}^n(t) = \frac{1}{2} \left( 1 - \delta Q^{n-1}(t) - c \right)^+.$$

The renewable producer is "blockaded" whenever $1 - \delta Q(t) - c < 0$.
(b) The exhaustible producer solves the optimal control problem

$$\partial_t v^n + \frac{1}{2}\sigma^2 \partial_{xx}^2 v^n + \frac{1}{4} \left( 1 - \varepsilon Q^{n-1}(t) - \partial_x v^n \right)^2 \mathbb{1}_B$$

$$+ \frac{1}{16} \left( 2 - \delta - (2\varepsilon - \delta^2)Q^{n-1}(t) + \delta c - 2\partial_x v^n \right)^2 \mathbb{1}_B^c = rv^n.$$

The feedback production strategy of the exhaustible producer is

$$q^n(t,x) = \frac{1}{2} \left( 1 - \varepsilon Q^{n-1}(t) - \partial_x v^n \right) \mathbb{1}_B + \frac{1}{4} \left( 2 - \delta - (2\varepsilon - \delta^2)Q^{n-1}(t) + \delta c - 2\partial_x v^n \right) \mathbb{1}_B^c.$$

*Step 2.* Given the optimal production strategy $q^n$, we can solve the forward Kolmogorov equation

$$\partial_t m^n - \frac{1}{2}\sigma^2 \partial_{xx}^2 m^n - \partial_x \left[ m^n q^n \right] = 0.$$

and determine the $n$th-iteration mean production $Q^n(t) = \int_{\mathbb{R}^+} q^n(t,x) m^n(t,x) \, dx$.

As observed in [4], it is computationally convenient to consider the tail distribution function

$$(4.6) \qquad\qquad \eta(t,x) = \int_x^\infty m(t,y) \, dy.$$

The forward Kolmogorov equation in terms of $\eta$ reads

$$\partial_t \eta(t,x) - \frac{1}{2}\sigma^2 \partial_{xx}^2 \eta(t,x) - q(t,x)\partial_x \eta(t,x) = 0,$$

with initial condition (4.6) evaluated at $t = 0$, which allows us to handle point masses in the initial distribution $M$.

**4.2. Results and discussion.** We take the deterministic setting $\sigma = 0$ and choose $\varepsilon = \delta = 1$ for the following numerical illustrations.

*Initialization and convergence of algorithm.* The initial guess of the mean production $Q^0$ is taken to be the explicit result (2.10) derived in the context of monopoly Cournot competition (without the renewable producer). We observe that our iterative algorithm converges rapidly, typically within 10 iterations. From the left panel of Figure 5, we notice that the exhaustible producers slow down production in the presence of a renewable competitor.

*Blockading of renewable producer.* If the constant marginal cost of production $c$ is high enough, the renewable producer can be blockaded and produces nothing. The idea is that when the marginal cost of production is high, and when the exhaustible resource is plentiful, it may be advantageous for the renewable producer to hold back production and wait until the exhaustible players diminish their reserves. We see numerical evidence that blockading occurs when the marginal cost of production $c = 0.9$.



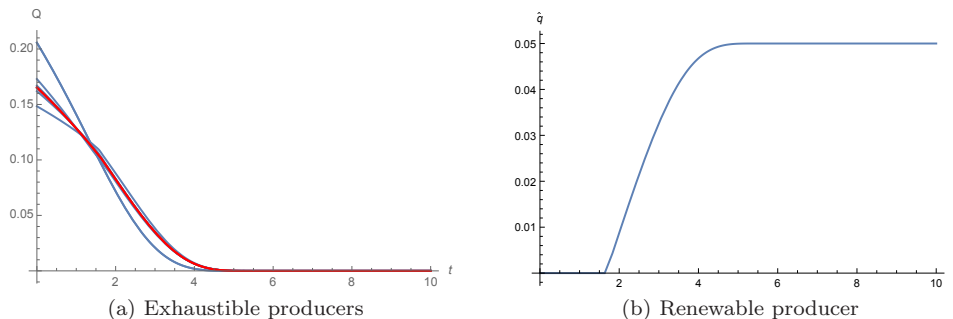(a) Exhaustible producers    (b) Renewable producer

Fig. 5: Production rates for the exhaustible (left) and renewable (right) producers. Notice the renewable producer is blockaded until about $t = 1.5$. The parameter are $\varepsilon = \delta = 1, r = 0.2, M \sim \text{Beta}(2, 4)$ and $c = 0.9$.

**4.3. Strategic blockading entry of renewable resources.** We now return to OPEC's strategic decision not to curb its oil production in face of increased supply of shale gas and oil in the US. In Figure 6, we consider the mean production rate $Q$ of the exhaustible producers when they are rivaled with an alternative source of different marginal costs $c$. The left panel shows the production profile $Q(t)$ over time; while the right panel plots the short term production $Q(0)$ as a function of the renewable energy cost $c$. When $c = 1$, we know that the alternative energy producer does not participate and the traditional energy producers have the entire market to themselves. However, we see that as $c$ decreases from 1, the exhaustible producers may strategically increase their mean production in the short run (and hence driving energy price down) to keep the renewable energy out of the market. Therefore, our model is capable of providing a dynamical explanation to OPEC's decision to maintain oil production in order to compete for market share with the fracking industry in the US.

**5. Resource Discovery.** In this section, we study the stochastic effect of resource exploration in dynamic Cournot mean field game models of exhaustible resources. This model corresponds to the short horizon in Table 1. The exhaustible producers may invest in exploration, with effort level indicated by $a_t \geq 0$ and cost $\mathcal{C}(a_t)$. The production capacity $X_t$ decreases at a (controlled) production rate $q_t \geq 0$, and increases through jumps thanks to discrete new discoveries. Exploration successes

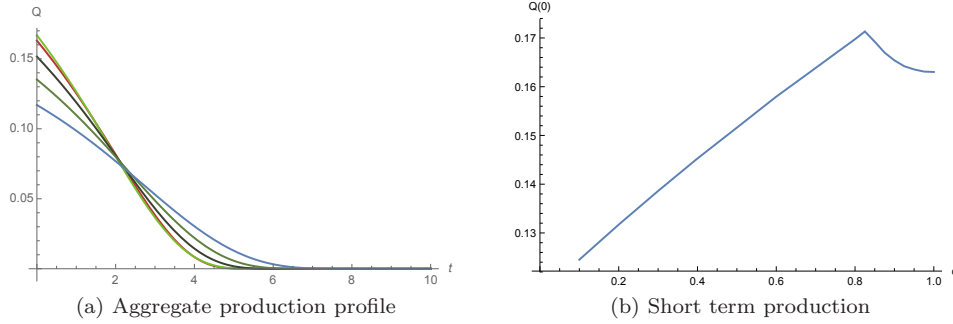(a) Aggregate production profile          (b) Short term production

Fig. 6: Left panel: the mean production rate $Q$ for 5 different values of production costs $c = 0, 0.2, 0.4, 0.6, 0.8$. Right panel: initial production rate of the exhaustible producers. Notice the strategic blockading of entry for large $c$. Other parameters are as in Figure 5.

are represented by a point process $N_t$ with (controlled) intensity $\lambda a_t$, where $\lambda$ is given model parameter. Suppose that each discovery leads to an increase in reserves by a fixed amount $\delta > 0$, then we have the following dynamics:

$$dX_t = -q_t \, dt + \delta \, dN_t.$$

Similar models with resource exploration have been considered by Deshmukh and Pliska [7] for monopolies and Ludkovski and Sircar [21] for duopolies.

The cost of exploration is captured by a positive, non-decreasing function $\mathcal{C}(\cdot)$. We will further assume that $\mathcal{C}$ is strictly convex to guarantee that optimal effort levels are finite. The economic interpretation is based on a spatial model of the deposits of non-renewable resources (e.g. fossil fuels in different geographical regions). In the simplest case of a Poisson random measure with constant rate $\lambda$, exploration of a region $A$ yields amount $\nu(A) \sim \text{Poisson}(\lambda|A|)$. In this model, the exploration effort $a$ mimics the speed at which one sweeps through areas searching for deposits. The convex cost $\mathcal{C}$ comes from diseconomies of scale at higher sweeping speeds.

The objective function is now the discounted lifetime revenue minus exploration cost:

$$(5.1) \qquad v(t,x) = \sup_{q,a} \mathbb{E} \left\{ \int_t^\infty e^{-r(s-t)} \left\{ q_s p_s \mathbb{1}_{\{X_s > 0\}} - \mathcal{C}(a_s) \right\} ds \,\bigg|\, X_t = x \right\}.$$

With inverse demand $1 - (q + \varepsilon Q)$, the HJB equation corresponding to the value function is

$$(5.2) \qquad \partial_t v + \sup_{q \geq 0} \left\{ q \left(1 - q - \varepsilon Q - \partial_x v \right) \right\} + \sup_{a \geq 0} \left\{ a\lambda \Delta v - \mathcal{C}(a) \right\} - rv = 0,$$

where the delay term $\Delta v(t,x) = v(t, x+\delta) - v(t,x)$ and the mean production is given by (4.4). The optimal production rate and effort level are given by

$$a^*(t,x) = (\mathcal{C}')^{-1} (\lambda \Delta v(t,x)), \qquad q^*(t,x) = \frac{1}{2} \left(1 - \varepsilon Q(t) - \partial_x v(t,x)\right).$$

Given the optimal controls, the population dynamics $m(t, x)$ is governed by the forward Kolmogorov equation:

(5.3)
$$\partial_t m(t, x) - \partial_x \left( q^*(t, x) m(t, x) \right) - \lambda \left\{ a^*(t, x - \delta) m(t, x - \delta) - a^*(t, x) m(t, x) \right\} = 0.$$

Note that the introduction of random jumps leads to a system of *non-local* PDEs.

**5.1. Sustainable economy.** Motivated by what Lucas and Moll [20] call a "balanced growth path", we look for stationary solution to the above mean field game equation system. The interpretation is a sustainable energy market in which resource extraction is balanced by the exploration successes. The stationary equations are

(5.4)
$$rv(x) = \sup_{q \geq 0} \left\{ q \left( 1 - q - \varepsilon Q - v'(x) \right) \right\} + \sup_{a \geq 0} \left\{ a \lambda \Delta v - \mathcal{C}(a) \right\},$$

$$0 = -\frac{d}{dx} \left( q^*(x) m(x) \right) - \lambda \left\{ a^*(x - \delta) m(x - \delta) - a^*(x) m(x) \right\},$$

$$a^*(x) = \left( \mathcal{C}' \right)^{-1} \left( \lambda \Delta v(x) \right), \quad q^*(x) = \frac{1}{2} \left( 1 - \varepsilon Q - v'(x) \right), \quad Q = \int_{\mathbb{R}_+} q^*(x) m(x) \, dx.$$

**5.2. Computational algorithm.** Following [21], we take power costs

$$\mathcal{C}(a) = \frac{1}{\beta} a^\beta + \kappa a, \qquad \beta > 1, \kappa > 0.$$

Since $\mathcal{C}'(0) = \kappa$, a strictly positive $\kappa$ guarantees a finite saturation point $x_{sat} < \infty$ such that $a^*(x) = 0$ for $x > x_{sat}$, and $(X_t)$ does become arbitrarily large infinitely often. In this case, the optimal effort is given by $a^*(x) = (\lambda \Delta v(x) - \kappa)_+^{\gamma - 1}$, where $\gamma = \beta / (\beta - 1)$. The HJB equation can be written as

$$rv(x) = \frac{1}{4} \left( 1 - \varepsilon Q - v'(x) \right)^2 + \frac{1}{\gamma} \left( \lambda \Delta v(x) - \kappa \right)_+^\gamma.$$

The boundary condition $v(0)$ is determined by optimizing the level of exploration effort $a$ while the producer is stuck at $x = 0$ waiting for his first exploration success, which his waiting time exponentially distributed with mean $(\lambda a)^{-1}$. This leads to

(5.5) $$v(0) = \sup_{a \geq 0} \mathbb{E} \left[ e^{-r\tau} v(\delta) - \int_0^\tau e^{-rt} \mathcal{C}(a) \, dt \right] = \sup_{a \geq 0} \frac{a \lambda v(\delta) - \mathcal{C}(a)}{\lambda a + r}.$$

Numerically solving for the value function $v$ is challenging due to the implicit boundary condition and the presence of a "forward" delay term on the semi-infinite domain $\mathbb{R}_+$. We resolve this difficulty by using an iterative scheme. Starting with an initial guess of the value function $v_0$ and mean production rate $Q_0$, for $n \geq 1$ we numerically solve the following inductively.

*Value function.* We replace the forward delay term by the its counterpart from the previous iteration:

(5.6) $$rv_n(x) = \frac{1}{4} \left( 1 - \varepsilon Q_{n-1} - v'_n(x) \right)^2 + \frac{1}{\gamma} \left( \lambda \left( v_{n-1}(x + \delta) - v_n(x) \right) - \kappa \right)_+^\gamma,$$

with boundary condition

$$v_n(0) = \sup_{a \geq 0} \frac{a \lambda v_{n-1}(\delta) - \mathcal{C}(a)}{\lambda a + r}.$$

Observe that (5.6) is a standard first-order nonlinear ordinary differential equation with "source" term $v_{n-1}(\cdot + \delta)$ and can be solved using standard tools, such as Runge-Kutta methods.

*Density.* Given the value function $v_n$ we can determine the optimal production rate $q_n^*(x)$ and optimal exploration level $a_n^*(x)$:

$$a_n^*(x) = (\lambda \Delta v_n(x) - \kappa)_+^{\gamma-1}, \qquad q_n^*(x) = \frac{1}{2}\left(1 - \varepsilon Q_{n-1} - v_n'(x)\right).$$

The stationary solution to the forward Kolmogorov equation (5.3) is determined by

$$0 = -\frac{d}{dx}\left(q_n^*(x)m_n(x)\right) - \lambda\left\{a_n^*(x-\delta)m_{n-1}(x-\delta) - a_n^*(x)m_n(x)\right\}.$$

We obtain the stationary solution by solving the time-dependent problem (5.3) and take the large time limit. By using the finite volume method we ensure that the density integrates to one. Now we integrate $m_n$ over the optimal feedback production rate $q_n^*$ to update the mean production $Q_n = \int_{\mathbb{R}_+} q_n^*(x)m_n(x)\,dx.$

**5.3. Numerical illustration.** Figure 7 illustrates the numerical solution for the sustainable economy (5.4). We observe in Figure 7a that while the production rate $q^*$ is monotone increasing in $x$, the exploration level $a^*$ is monotone decreasing. Figures 7b, 7c and 7d show the sample path for the evolution of the game solution over time. The system state is described by $(X_t)$ in the top right panel which drives the feedback controls $q^*(X_t)$ and $a^*(X_t)$ in the bottom panels. One can readily observe that higher reserves lower exploration rates and increase production. The recurrent behavior of $(X_t)$ is apparent, as the resource is repeatedly exhausted until a new discovery replenishes reserves and allows to restart production.

**6. Conclusion.** In this paper, we apply the Cournot mean field game model to the global energy market. We focus on the interaction between traditional oil producers and alternative sources (e.g. solar, hydroelectric power, or fracking). Specifically, we investigate the issue from three perspectives: competition, transition, and exploration. This leads to three extensions of the basic Cournot MFG model.

**Transition** As the traditional oil producers run out of reserves, they can transition to energy production with alternative sources. This essentially introduce a Neumann boundary condition to our PDE problem. In the regime of small exhaustibility, we find explicit correction to the value function and optimal production rate by solving a partial integro-differential equation.

**Competition** We find that if the alternative energy source has a high enough cost of production, the traditional energy producers may strategically increase production rate (and hence lowering the energy price ever further) to keep the alternative energy producer blockaded. This explains OPEC's strategic decision not to reduce production quotas in the face of falling oil prices due in large part to the US fracking boom.

**Exploration** We have studied the impact of exploration and discovery in Cournot models of exhaustible resources. We characterize a sustainable energy market as a stationary solution to the forward Kolmogorov equation. We find that higher reserves lower exploration rates and increase production.

**Appendix A. Existence and Uniqueness.** We demonstrate existence and uniqueness of the PDE system (4.2) and (4.3) in the case $\delta = 0$ thanks to the remarkable results by Graber and Bensoussan [9], which were motivated by our prior paper
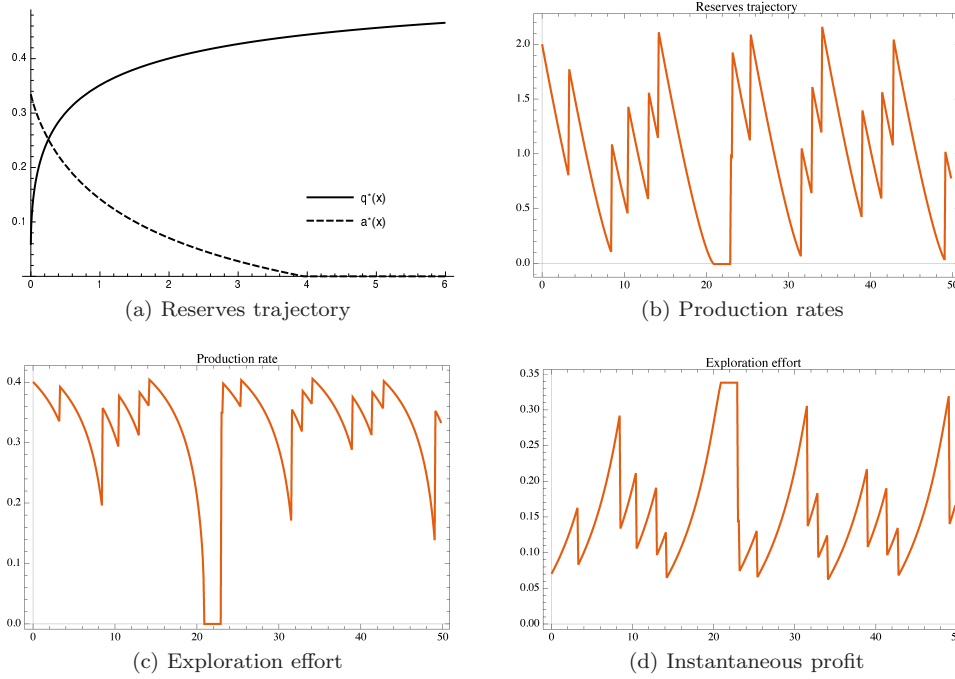
(a) Reserves trajectory

(b) Production rates

(c) Exploration effort

(d) Instantaneous profit

Fig. 7: Trajectory of the game solution over time. Top panel: reserves $(X_t)$ of a representative player; middle panel: production rate $q^*(X_t)$; bottom panel: exploration rate $a^*(X_t)$. The parameters are $\delta = 1, \lambda = 1, r = 0.1, \mathcal{C}(a) = 0.1a + a^2/2$ and $\varepsilon = 0.25$.

[4] on mean field games of Bertrand-type. We first state their thorem, and then show how the Bertrand problem can be transformed to the Cournot system that we study in Section 4.

THEOREM A.1 (Graber and Bensoussan). *Assuming the following on the data:*

1. *$u_T(x)$ and $m_0(x)$ are functions in $C^{2+\gamma}([0, L])$ for some $\gamma > 0$;*
2. *$u_T$ and $m_0$ satisfy compatible boundary conditions: $u_T(0) = u_T'(L) = 0$ and $m_0(0) = m_0(L) = m_0'(L) = 0$;*
3. *$m_0 \geq 0$ and $\int_0^L m_0(x)\, dx = 1$, i.e. $m_0$ is a probability density;*
4. *$u_T \geq 0$ and $u_T' \geq 0$, i.e. $u_T$ is non-negative and non-decreasing.*

*Then there exists a classical solution to the system*

(A.1)
$$\partial_t u + \frac{1}{2}\sigma^2 \partial_{xx} u - ru + H(t, \partial_x u, [\varphi \partial_x u]) = 0, \quad 0 < t < T, 0 < x < L,$$
$$\partial_t \varphi - \frac{1}{2}\sigma^2 \partial_{xx}\varphi - \partial_x \left(G(t, \partial_x u, [\varphi \partial_x u])\varphi\right) = 0, \quad 0 < t < T, 0 < x < L,$$

*subject to boundary conditions*

(A.2)
$$\varphi(0, x) = \varphi_0(x), \quad u(T, x) = u_T(x), \quad 0 \leq x \leq L,$$
$$u(t, 0) = \varphi(t, 0) = 0, \quad \partial_x u(t, L) = 0, \quad 0 \leq t \leq T,$$
$$\frac{1}{2}\sigma^2 \partial_x \varphi(t, L) + G(t, \partial_x u(t, L), [\varphi \partial_x u])\varphi(t, L) = 0, \quad 0 \leq t \leq T,$$

*where*

$$H(t, \partial_x u, [\varphi \partial_x u]) = \frac{1}{4} \left( a(\eta(t)) + c(\eta(t))\bar{p}(t) - \partial_x u \right)^2,$$

$$G(t, \partial_x u, [\varphi \partial_x u]) = \frac{1}{2} \left( a(\eta(t)) + c(\eta(t))\bar{p}(t) - \partial_x u \right),$$

(A.3)
$$a(\eta) = \frac{1}{1 + \varepsilon \eta}, \quad c(\eta) = \frac{\varepsilon \eta}{1 + \varepsilon \eta},$$

$$\bar{p}(t) = \frac{1}{2 - c(\eta(t))} \left( a(\eta(t)) + \frac{1}{\eta(t)} \int_0^L \partial_x u(t, x)\varphi(t, x)\, dx \right),$$

$$\eta(t) = \int_0^L \varphi(t, x)\, dx, \quad 0 \le \eta(t) \le 1.$$

*Moreover, there exists $\varepsilon_0 > 0$ sufficiently small such that for any $\varepsilon \le \varepsilon_0$, the above PDE system has at most one classical solution.*

**Equivalence of the two PDE problems.** If we define

$$\widehat{Q}(t) = \frac{1}{2 + \varepsilon \eta(t)} \left( \eta(t) - \int \partial_x u(t, x)\varphi(t, x)\, dx \right),$$

then it follows that $(1 + \varepsilon \eta(t))\widehat{Q}(t) = \eta(t)(1 - \bar{p}(t))$ and hence

$$a(\eta(t)) + c(\eta(t))\bar{p}(t) = 1 - \varepsilon \widehat{Q}(t).$$

Then equations (A.1) can be written as

(A.4)
$$0 = \partial_t u + \frac{1}{2}\sigma^2 \partial_{xx} u - ru + \frac{1}{4} \left( 1 - \varepsilon \widehat{Q}(t) - \partial_x u \right)^2,$$

$$0 = \partial_t \varphi - \frac{1}{2}\sigma^2 \partial_{xx} \varphi - \frac{1}{2}\partial_x \left( (1 - \varepsilon \widehat{Q}(t) - \partial_x u)\varphi \right),$$

and these are precisely the MFG equations (4.2) and (4.3), with the identifications $(u, \varphi) \mapsto (v, m)$, and in the case $\delta = 0$ (no competition from the alternative producer, so the $g$ equation in (4.2) is not needed). As the boundary conditions are the same as when we truncate the domain for the numerical solution in Section 4.1, Theorem A.1 applies, providing existence and uniqueness of a classical solution to the MFG equation system in Section 4, and validates the numerical findings.

REFERENCES

[1] ALAIN BENSOUSSAN, JENS FREHSE, AND PHILLIP YAM, *Mean field games and mean field type control theory*, Springer, 2013.
[2] JL BERTRAND, *Théorie mathématique de la richesse sociale*, Journal des Savants, (1883).
[3] RENE CARMONA, JEAN-PIERRE FOUQUE, AND LI-HSIEN SUN, *Mean field games and systemic risk*, Available at SSRN 2307814, (2013).
[4] PATRICK CHAN AND RONNIE SIRCAR, *Bertrand & Cournot mean field games*, Applied Mathematics & Optimization, 71 (2014), pp. 533–569.
[5] ANTOINE COURNOT, *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Chez L. Hachette, 1838.
[6] ANIRUDH DASARATHY AND RONNIE SIRCAR, *Variable costs in dynamic Cournot energy markets*, in Energy, Commodities and Environmental Finance, R. Aid, M. Ludkovski, and R. Sircar, eds., Fields Insititute Communications, Fields Institute, 2015.

[7]  Sudhakar D Deshmukh and Stanley R Pliska, *Optimal consumption and exploration of nonrenewable resources under uncertainty*, Econometrica, (1980), pp. 177–200.

[8]  Engelbert Dockner, *Differential games in economics and management science*, Cambridge University Press, 2000.

[9]  P Jameson Graber and Alain Bensoussan, *Existence and uniqueness of solutions for Bertrand and Cournot mean field games*, arXiv preprint arXiv:1508.05408, (2015).

[10]  Olivier Guéant, *Mean field games and applications to economics*, PhD thesis, PhD thesis, Université Paris-Dauphine, 2009.

[11]  Olivier Guéant, Jean-Michel Lasry, and Pierre-Louis Lions, *Mean field games and applications*, in Paris-Princeton Lectures on Mathematical Finance 2010, Springer, 2011, pp. 205–266.

[12]  Chris Harris, Sam Howison, and Ronnie Sircar, *Games with exhaustible resources*, SIAM Journal on Applied Mathematics, 70 (2010), pp. 2556–2581.

[13]  H. Hotelling, *The economics of exhaustible resources*, The Journal of Political Economy, 39 (1931), pp. 137–175.

[14]  Minyi Huang, Peter E Caines, and Roland P Malhamé, *Individual and mass behaviour in large population stochastic wireless power control problems: centralized and nash equilibrium solutions*, in Decision and Control, 2003. Proceedings. 42nd IEEE Conference on, vol. 1, IEEE, 2003, pp. 98–103.

[15]  Minyi Huang, Roland P Malhamé, Peter E Caines, et al., *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Communications in Information & Systems, 6 (2006), pp. 221–252.

[16]  Sebastian Jaimungal and Mojtaba Nourian, *Mean-field game strategies for a major-minor agent optimal execution problem*, Available at SSRN 2578733, (2015).

[17]  Jean-Michel Lasry and Pierre-Louis Lions, *Mean field games*, Japanese Journal of Mathematics, 2 (2007), pp. 229–260.

[18]  Andrew Ledvina and Ronnie Sircar, *Dynamic Bertrand oligopoly*, Applied Mathematics & Optimization, 63 (2011), pp. 11–44.

[19]  ———, *Oligopoly games under asymmetric costs and an application to energy production*, Mathematics and Financial Economics, 6 (2012), pp. 261–293.

[20]  Robert E Lucas and Benjamin Moll, *Knowledge growth and the allocation of time*, tech. report, National Bureau of Economic Research, 2011.

[21]  Michael Ludkovski and Ronnie Sircar, *Exploration and exhaustibility in dynamic Cournot games*, European Journal of Applied Mathematics, 23 (2012), pp. 343–372.

[22]  ———, *Game theoretic models for energy production*, in Energy, Commodities and Environmental Finance, R. Aid, M. Ludkovski, and R. Sircar, eds., Fields Insititute Communications, Fields Institute, 2015.

[23]  ———, *Technology ladders and R&D in dynamic Cournot markets*, J. Economic Dynamics & Control, (2016). To appear.

[24]  Michael Ludkovski and Xuwei Yang, *Dynamic Cournot models for production of exhaustible commodities under stochastic demand*, in Energy, Commodities and Environmental Finance, R. Aid, M. Ludkovski, and R. Sircar, eds., Fields Insititute Communications, Fields Institute, 2015.

[25]  Robert S Pindyck, *The optimal exploration and production of nonrenewable resources*, The Journal of Political Economy, (1978), pp. 841–861.