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Construction of a class of forward performance processes in stochastic factor models and an extension of Widder's theorem

Levon Avanesyan [1] · **Mykhaylo Shkolnikov** [1] ·
Ronnie Sircar [1]

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Abstract We consider the problem of optimal portfolio selection under forward investment performance criteria in an incomplete market. Given multiple traded assets, the prices of which depend on multiple observable stochastic factors, we construct a large class of forward performance processes, as well as the corresponding optimal portfolios, with power-utility initial data and for stock-factor correlation matrices with eigenvalue equality (EVE) structure, which we introduce here. This is done by solving the associated non-linear parabolic partial differential equations (PDEs) posed in the "wrong" time direction. Along the way we establish on domains an explicit form of the generalized Widder's theorem of Nadtochiy and Tehranchi [27, Theorem 3.12] and rely hereby on the Laplace inversion in time of the solutions to suitable linear parabolic PDEs posed in the "right" time direction.

Keywords Factor models · Forward performance processes · Generalized Widder's theorem · Hamilton-Jacobi-Bellman equations · Ill-posed partial differential equations · Incomplete markets · Merton problem · Optimal portfolio selection · Positive eigenfunctions · Time-consistency

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L. Avanesyan
levon.avanesyan23@gmail.com

M. Shkolnikov
mykhaylo@princeton.edu

R. Sircar
sircar@princeton.edu

[1] Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA

1 Introduction

In this paper we study the optimal portfolio selection problem under forward investment criteria of power-utility form in incomplete markets, specifically stochastic factor models with a stock-factor correlation structure named EVE, which we introduce here. Our setup is that of a continuous-time market model with multiple stocks whose returns and volatilities are functions of multiple observable stochastic factors following jointly a diffusion process. The incompleteness arises hereby from the imperfect correlation between the Brownian motions driving the stock prices and the factors. The factors themselves can model various market inputs, including stochastic interest rates, stochastic volatility and major macroeconomic indicators, such as inflation, GDP growth or the unemployment rate.

The optimal portfolio problem in continuous time was originally considered by Merton in his pioneering work [19], [20], and is commonly referred to as the Merton problem. In this framework an investor looks to maximize her expected terminal utility from wealth acquired in the investment process within a geometric Brownian motion market model. Good compilations of classical results can be found in the books by Duffie [5], and Karatzas and Shreve [16]. As fundamental as this setup is, it has two important drawbacks. First, the investor must decide on her terminal utility function before entering the market, and thereby cannot adapt it to changes in market conditions. Second, before settling on an investment strategy, the investor must firmly set her time horizon. That is, the portfolio derived in this framework is optimal only for one specific utility function over one time horizon.

External factors such as the economic cycle, natural disasters, and the political climate can lead to dynamic changes in one's preferences. This would change the terminal utility function, thereby affecting the optimal portfolio allocation. Moreover, the investor might want to alter the terminal time itself. In order to solve portfolio optimization problems with an uncertain investment horizon, *forward investment performance criteria* were introduced and developed by Musiela and Zariphopoulou [22, 23], as well as Henderson and Hobson [12]. Instead of looking to optimize the expectation of a deterministic utility function at a single terminal point in time, this approach looks to maximize the expectation of a stochastic utility function at every single point in time. *Forward performance processes* (FPPs), as defined in Musiela and Zariphopoulou [24], capture the time evolutions of such stochastic utility functions.

A comprehensive description of all FPPs remains a challenging open problem. Much work towards this goal has been carried out throughout the last ten years, see Berrier et al. [2], El Karoui and Mrad [6, 7], Henderson and Hobson [12], Musiela and Zariphopoulou [26], and Zitkovič [33] for some important results. In [26], Musiela and Zariphopoulou proposed a construction of FPPs by means of solutions to a stochastic partial differential equation (SPDE). To find all the FPPs characterized by the SPDE, one would have to find all forward volatility processes, along with initial utility functions, for which the SPDE has a classical solution. The case of zero forward volatility yields time-monotone FPPs, and was extensively discussed in

Musiela and Zariphopoulou [24, 25], as well as more recently in Källblad et al. [13] in the presence of model uncertainty.

We consider factor-driven market models and FPPs into which the randomness enters only through the underlying stochastic factors. Assuming such a form, with a compatible forward volatility process, the SPDE mentioned above reduces to an HJB equation set in the “wrong” time direction. In a complete market one can use the Fenchel-Legendre transform to linearize the HJB equation, and arrive at a linear second-order parabolic PDE set in the “wrong” time direction (see Nadtochiy and Tehranchi [27]). In an incomplete market no such linearizing transformation is available in general. To the best of our knowledge, the only exception is the special case of power utility in a one-factor market model, where a linearization is possible through a distortion transformation, as discovered in Zariphopoulou [32] for the Merton problem, and used for the construction of FPPs in Nadtochiy and Tehranchi [27], Nadtochiy and Zariphopoulou [28], and Shkolnikov et al. [30]. Construction and representation of FPPs in multi-factor incomplete market models have recently been addressed in Shkolnikov et al. [30] and Liang and Zariphopoulou [18]. The former deals with a two-factor case, and provides asymptotic results for different time scales. The latter allows for an arbitrary number of factors and trading constraints, and gives backward stochastic differential equation (BSDE) representations of FPPs. All of these papers assume power-type (or homothetic) utility structure, as we will do also in this paper.

We introduce a new class of multi-factor market models, which we will call *EVE correlation models* (see Definition 2.3). In this framework we reduce the fully nonlinear HJB equation to a linear second-order parabolic PDE. Thereby, we obtain explicit characterizations of FPPs in such models. Our analysis also applies to the Merton problem, whose value function solves the same HJB equation posed in the “right” time direction.

In one-factor market models, Nadtochiy and Tehranchi [27, Theorem 3.12] exhibited a characterization of all positive solutions to the linear parabolic equations posed in the “wrong” time direction, that arise in the construction of FPPs of power-utility type. Their theorem constitutes a generalization of the celebrated Widder’s theorem (see Widder [31]), which describes all positive solutions of the heat equation set in the “wrong” time direction. The generalized Widder’s theorem reveals that positive solutions of a linear second-order parabolic equation set in the “wrong” time direction must be linear combinations of exponentially scaled positive eigenfunctions for the corresponding elliptic operator according to a positive finite Borel measure. Moreover, each solution is uniquely identified with a pairing of the eigenfunctions and the measure.

In our first main theorem (Theorem 2.11) we give a new version of [27, Theorem 3.12] on domains in the multi-stock multi-factor EVE setup with an initial utility function of power type to describe a new class of FPPs. Note that generalized Widder’s theorems do not provide a way to construct the pairings of the eigenfunctions and the measure. Our second set of results (see Theorem 2.14 and Remark 2.15) addresses this issue: in Theorem 2.14 we give the Laplace transform of the measure in terms of the solution to a linear parabolic equation set in the “right” time direction, and we provide a method (see Remark 2.15) of finding the only possible corresponding eigenfunctions as well. Thus, we indeed obtain a large explicit class of FPPs.

The rest of the paper is structured as follows. In Section 2 we state our main results, postponing their proofs to later sections. In Section 3 we introduce relevant facts about FPPs and subsequently prove Theorem 2.11. In Section 4 we show Theorem 2.14, summarize some results from the theory of linear elliptic operators, and use them to establish Propositions 2.17, 2.19, 2.21 and 2.22. In Section 5 we discuss the Merton problem within the framework of our market model. Lastly, in Section 6 we discuss EVE correlation models and construct explicit FPPs in affine multi-stock multi-factor market models of EVE type.

2 Main results

2.1 Model

Consider an investor with initial capital $X_0 = x > 0$ aiming to invest in a market with $n \geq 1$ stocks, the prices of which follow a process S, and a riskless bank account with zero interest rate. The stock prices depend on an observable k-dimensional stochastic factor process Y taking values in $D \subseteq \mathbb{R}^k$, and are driven by a d_W-dimensional standard Brownian motion W. The factor process Y is itself driven by a d_B-dimensional standard Brownian motion B, whose correlation with W is given by a matrix $\mathrm{corr}(W,B) = (\rho_{ij})_{i,j=1}^{d_W,d_B} := \rho$, where $\rho_{ij} \in [-1, 1]$. Without loss of generality we assume that $d_W \geq n$ (see [14, Remark 0.2.6]). The investor's filtration $(\mathscr{F}_t)_{t\geq 0}$ is generated by a pair (S,Y) of processes satisfying

$$\begin{aligned}
&\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_i(Y_t)\,\mathrm{d}t + \sum_{j=1}^{d_W} \sigma_{ji}(Y_t)\,\mathrm{d}W_t^j, \quad i = 1, 2, \ldots, n, \\
&\mathrm{d}Y_t = \alpha(Y_t)\,\mathrm{d}t + \kappa(Y_t)^\top\,\mathrm{d}B_t, \\
&B_t = \rho^\top W_t + A^\top W_t^\perp,
\end{aligned}$$

where the superscript T denotes transposition and $W^\perp$ is a $d_{W^\perp}$-dimensional standard Brownian motion independent of W. We write μ for $(\mu_1, \mu_2, \ldots, \mu_n)^\top$ and σ for $(\sigma_{ij})_{i,j=1}^{d_W,n}$ throughout.

Remark 2.1 It is straightforward to show that the positive semidefiniteness of the correlation matrix of the Brownian motion (W, B) implies that the singular values of ρ are in $[0,1]$.

For the convenience of the reader we summarize the dimensions of all the quantities we have introduced thus far

$$\begin{aligned}
&\mu(\cdot) - n\times 1, \quad \sigma(\cdot) - d_W \times n, \quad W_t - d_W \times 1, \quad \alpha(\cdot) - k\times 1, \quad \kappa(\cdot) - d_B \times k, \\
&B_t - d_B \times 1, \quad \rho - d_W \times d_B, \quad A - d_{W^\perp} \times d_B, \quad W_t^\perp - d_{W^\perp} \times 1.
\end{aligned}$$

Note that there is no loss of generality in using the representation (2.3) for the standard Brownian motion B, since we can let A be the square root of the positive semidefinite matrix $I_{d_B} - \rho^\top \rho$ (recall that the singular values of ρ belong to $[0,1]$), and $d_{W^\perp} = d_B$.

Assumption 2.1 *The functions $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^{d_W \times n}$ are continuous, and the stochastic differential equation (SDE) (2.2) possesses a unique weak solution. Moreover, the columns of ρ belong to the range of left-multiplication by $\sigma(y)$ for all $y \in D$.*

Remark 2.2 The last condition in Assumption 2.1 holds only if the column rank of ρ is less than or equal to the column rank of σ, and implies $\sigma(y)\sigma(y)^{-1}\rho = \rho$ for all $y \in D$, where $\sigma(y)^{-1}$ is the Moore-Penrose pseudoinverse of $\sigma(y)$. Indeed, $\sigma(y)\sigma(y)^{-1}\sigma(y) = \sigma(y)$, so that the columns of $\sigma(y)$ (and consequently the vectors in their span, that is, the range of the left-multiplication by $\sigma(y)$) are invariant under the left-multiplication by $\sigma(y)\sigma(y)^{-1}$.

Our main result is for a particular class of multi-factor models, which we define next.

Definition 2.3 We will call a market model an eigenvalue equality (EVE) correlation model if for some $p \in [0,1]$,

$$\rho^\top \rho = p I_{d_B}. \tag{2.4}$$

Note that in Definition 2.3, p has to be between 0 and 1 since the singular values of ρ are in $[0,1]$ (see Remark 2.1).

Remark 2.4 Note that for EVE correlation models, since ρ is a $d_W \times d_B$-matrix, at least one of the following two has to hold:

(i) $d_W \geq d_B$,
(ii) $p = 0$.

Finally, we remark that when $d_B = 1$, ρ is a vector and $p := \rho^\top \rho \in [0,1]$, so that (2.4) holds automatically.

The name EVE comes from the fact that the only restriction is on the eigenvalues of the matrix $\rho^\top \rho$. For any orthonormal $d_B \times d_B$ matrix O, we may replace $\kappa(\cdot)$ by $O\kappa(\cdot)$ and B by $\widetilde{B} = OB$ in (2.2) without changing the dynamics of the pair (S,Y). Since $\widetilde{B}$ is a d_B-dimensional standard Brownian motion and $\mathrm{corr}(W, \widetilde{B}) = O^\top \rho^\top \rho O$ is diagonal for an appropriate choice of O, we could have assumed without loss of generality from the very beginning that $\rho^\top \rho$ is diagonal.

Section 6 is devoted to a further discussion of EVE correlation models.

2.2 Forward Performance Processes

The investor dynamically allocates her wealth in the market using a self-financing trading strategy that at any time $t \geq 0$ yields a portfolio allocation $\pi_t = (\pi_t^1, \ldots, \pi_t^n)$ among the n stocks with the associated wealth process

$$\frac{\mathrm{d}X_t^\pi}{X_t^\pi} = \Big(\sigma(Y_t)\pi_t\Big)^\top \lambda(Y_t)\,\mathrm{d}t + \Big(\sigma(Y_t)\pi_t\Big)^\top \mathrm{d}W_t, \quad X_0^\pi = x, \tag{2.5}$$

where $\lambda(Y_t) = (\sigma(Y_t)^\top)^{-1}\mu(Y_t)$ is the Sharpe ratio. Apart from the self-financeability, we impose additional conditions on the trading strategies to ensure that their wealth processes X^π are well-defined by (2.5).

Definition 2.5 An $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable self-financing trading strategy is called admissible if its portfolio allocation π among the n stocks fulfills

$$\forall t \geq 0: \quad \int_0^t \left|\pi_s^\top \sigma(Y_s)^\top \lambda(Y_s)\right| \mathrm{d}s < \infty \quad \text{and} \quad \int_0^t \left|\sigma(Y_s)\pi_s\right|^2 \mathrm{d}s < \infty$$

with probability one. In this case, we write $\pi \in \mathscr{A}$.

Next we recall the definition of FPPs given in Musiela and Zariphopoulou [24]. These capture how the utility function of an investor evolves over time as she continues to invest in the financial market.

Definition 2.6 An $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable process $U_\cdot(\cdot): [0,\infty)\times(0,\infty) \to \mathbb{R}$ is referred to as a (local) forward performance process (FPP) if

(i) with probability one, all functions $x \mapsto U_t(x)$, $t \geq 0$ are strictly concave and increasing,
(ii) for each $\pi \in \mathscr{A}$, the process $U_t(X_t^\pi)$, $t \geq 0$ is an $(\mathscr{F}_t)_{t\geq 0}$ (local) supermartingale,
(iii) there exists an optimal $\pi^* \in \mathscr{A}$ for which $U_t(X_t^{\pi^*})$, $t \geq 0$ is an $(\mathscr{F}_t)_{t\geq 0}$ (local) martingale.

We refer to Musiela and Zariphopoulou [24, 26], and Nadtochiy and Zariphopoulou [28] for motivation and explanation of this definition. We consider (local) FPPs of factor-form into which the randomness enters only through the stochastic factor process, that is,

$$U_t(x) = V(t,x,Y_t), \quad t \geq 0 \tag{2.6}$$

for a deterministic function $V: [0,\infty)\times(0,\infty)\times D \to \mathbb{R}$. Throughout the paper, we look for FPPs where the initial utility function is of product form, and a power function in the wealth variable:

$$U_0(x) = V(0,x,Y_0) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} h(Y_0) \quad \text{for some } \gamma \in (0,\infty)\backslash\{1\}. \tag{2.7}$$

The crucial simplification arising from the structure in (2.7) lies in its propagation to positive times. In this paper we will construct (local) FPPs of the following form.

Definition 2.7 We will say a (local) FPP $U_\cdot(\cdot)$ is of separable power factor form if

$$U_t(x) = V(t,x,Y_t) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} g(t,Y_t),$$

for some g that is continuously differentiable in t (its first argument) and twice continuously differentiable in y (the second argument).

In this paper, we characterize *all* separable power factor form local FPPs for EVE correlation models introduced in Definition 2.3.

2.3 Characterizing separable power factor form FPPs

In order to describe our construction of separable power factor form FPPs, we need to introduce some quantities related to linear elliptic operators of the second order. Consider on $C^2(D)$ such an operator

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(y)\partial_{y_i y_j} + \sum_{i=1}^{k} b_i(y)\partial_{y_i} + P(y) \tag{2.8}$$

under the following assumption.

Assumption 2.2 *There exists a C^3-diffeomorphism $\Xi : D \to \mathbb{R}^k$ so that the functions*

$$\begin{aligned}
\overline{a}_{ij}(z) &:= \Big((\nabla\Xi_i)^\top a \nabla\Xi_j\Big)\Big(\Xi^{-1}(z)\Big),\\
\overline{b}_i(z) &:= \Big((\nabla\Xi_i)^\top b\Big)\Big(\Xi^{-1}(z)\Big) + \frac{1}{2}\mathrm{trace}\Big(\mathrm{Hess}(\Xi_i)\, a\Big)\Big(\Xi^{-1}(z)\Big),\\
\overline{P}(z) &:= P\Big(\Xi^{-1}(z)\Big)
\end{aligned} \tag{2.9}$$

are uniformly bounded and uniformly η-Hölder continuous over $\mathbb{R}^k$ and the matrices $\overline{a}(z) := (\overline{a}_{ij}(z))_{i,j=1}^k$ are non-degenerate uniformly in $z \in \mathbb{R}^k$. That is, with the notation $\overline{b}(\cdot) = (\overline{b}_1(\cdot), \overline{b}_2(\cdot), \ldots, \overline{b}_k(\cdot))^\top$ and for some $\eta \in (0,1)$,

(i) $\sup_{z\in\mathbb{R}^k} |\overline{a}(z)|$, $\sup_{z\in\mathbb{R}^k} |\overline{b}(z)|$, $\sup_{z\in\mathbb{R}^k} |\overline{P}(z)| < \infty$,
(ii) $\|\overline{a}\|_{\eta,\mathbb{R}^k}$, $\|\overline{b}\|_{\eta,\mathbb{R}^k}$, $\|\overline{P}\|_{\eta,\mathbb{R}^k} < \infty$, *where* $\|f\|_{\eta,\mathbb{R}^k} = \sup_{z,z'\in\mathbb{R}^k, z\neq z'} \frac{|f(z)-f(z')|}{|z-z'|^\eta}$,

and

(iii) $\inf_{z\in\mathbb{R}^k, |v|=1} v^\top \overline{a}(z) v > 0$.

Remark 2.8 Assumption 2.2 entails that the domain D is C^3-diffeomorphic to $\mathbb{R}^k$ and that the operator $\mathscr{L}$ on D can be obtained as a pushforward under a C^3-diffeomorphism of a uniformly elliptic operator

$$\overline{\mathscr{L}} = \frac{1}{2}\sum_{i,j=1}^{k} \overline{a}_{ij}(z)\partial_{z_i z_j} + \sum_{i=1}^{k} \overline{b}_i(z)\partial_{z_i} + \overline{P}(z) \tag{2.10}$$

on $\mathbb{R}^k$ with uniformly bounded and uniformly η-Hölder continuous coefficients. For a star-shaped domain D, it is well-known (see, e.g., [8, Subsection 10.1]) that one can find C^∞-diffeomorphisms Ξ^{-1} mapping $\mathbb{R}^k$ onto D. However, whether a locally uniformly elliptic operator $\mathscr{L}$ with locally bounded and locally η-Hölder continuous coefficients is a pushforward under Ξ^{-1} of an operator $\overline{\mathscr{L}}$ with coefficients satisfying the conditions (i)-(iii) of Assumption 2.2 needs to be checked on a case-by-case basis. As an example, consider an operator $\frac{1}{2}\sum_{i=1}^{k} y_i^{c_i}(1-y_i)^{c_i'}\partial_{y_i y_i} + P(y)$ on $(0,1)^k$ with constants $c_i, c_i' \in (4,\infty)$ and a bounded η-Hölder continuous potential P. Then, for the C^∞-diffeomorphism $\Xi : (0,1)^k \to \mathbb{R}^k$, $y \mapsto (\tan(\pi y_i - \pi/2))_{i=1}^k$ it is elementary to verify that the coefficients of the resulting

$$\overline{\mathscr{L}} = \frac{1}{2}\sum_{i=1}^k \frac{\pi^2 y_i^{c_i}(1-y_i)^{c_i'}}{\cos(\pi y_i - \pi/2)^4}\bigg|_{y_i=\arctan z_i} \partial_{z_i z_i} + \sum_{i=1}^k \frac{\pi^2 y_i^{c_i}(1-y_i)^{c_i'}\sin(\pi y_i - \pi/2)}{\cos(\pi y_i - \pi/2)^3}\bigg|_{y_i=\arctan z_i} \partial_{z_i} + P\Big((\arctan z_i)_{i=1}^k\Big)$$

fulfill the conditions (i)-(iii) of Assumption 2.2.

Remark 2.9 Whenever $D = \mathbb{R}^k$ (as for instance in [27, Section 3.1]) it is standard in the literature to assume that the conditions (i)-(iii) of Assumption 2.2 hold for $a(\cdot)$, $b(\cdot)$, $P(\cdot)$. This implies the set of conditions in Assumption 2.2 by taking Ξ to be the identity map. Moreover, in this case, the SDE (2.2) admits a unique weak solution (see [15, Chapter 5, Remarks 4.17 and 4.30]).

We define the Hölder space $C^{2,\eta}(D) \subset C^2(D)$ as the subspace consisting of functions whose second-order partial derivatives are η-Hölder continuous (in the same sense as in condition (ii) of Assumption 2.2) on compact subsets of D. Next, we introduce the sets of positive eigenfunctions for the operator $\mathscr{L}$, which correspond to eigenvalues $\zeta \in \mathbb{R}$, and are normalized at some fixed $y_0 \in D$

$$C_{\mathscr{L}-\zeta}(D) = \Big\{\psi \in C^{2,\eta}(D) : \psi(\cdot) > 0,\, \psi(y_0) = 1,\, (\mathscr{L}-\zeta)\psi = 0\Big\}.$$

Moreover, we let $\mathbb{S}_{\mathscr{L}}(D)$ be the spectrum of $\mathscr{L}$ associated with positive eigenfunctions:

$$\mathbb{S}_{\mathscr{L}}(D) = \Big\{\zeta \in \mathbb{R} : C_{\mathscr{L}-\zeta}(D) \neq \emptyset\Big\}.$$

In subsection 4.2, we provide some well-known results about the structure of the eigenfunction spaces $C_{\mathscr{L}-\zeta}(D)$ and the set of eigenvalues $\mathbb{S}_{\mathscr{L}}(D)$. In particular, Proposition 4.4 yields that in our setup $\mathbb{S}_{\mathscr{L}}(D)$ is a half-line.

Finally, we call a functional $\Psi : \mathbb{S}_{\mathscr{L}}(D) \times D \to (0,\infty)$ such that $\Psi(\zeta,\cdot) \in C_{\mathscr{L}-\zeta}(D)$ for all $\zeta \in \mathbb{S}_{\mathscr{L}}(D)$, a *selection of positive eigenfunctions*, and recall the definition of Bochner integrability in this setting.

Definition 2.10 Given a positive finite Borel measure ν on $\mathbb{S}_{\mathscr{L}}(D)$, we refer to a selection of positive eigenfunctions $\Psi : \mathbb{S}_{\mathscr{L}}(D) \times D \to (0,\infty)$ as ν-Bochner integrable if, for all compact $K \subset D$, $\int_{\mathbb{S}_{\mathscr{L}}(D)} \|\Psi(\zeta,\cdot)\|_K\, \nu(\mathrm{d}\zeta) < \infty$, where $\|f\|_K = \sup_{y\in K}|f(y)|$.

In preparation for our main result we define

$$a(\cdot) = \kappa(\cdot)^\top \kappa(\cdot), \quad b(\cdot) = \alpha(\cdot) + \Gamma\kappa(\cdot)^\top \rho^\top \lambda(\cdot), \quad P(\cdot) = \frac{\Gamma}{2q}\lambda(\cdot)^\top\lambda(\cdot), \tag{2.11}$$

where $\Gamma = \frac{1-\gamma}{\gamma}$ and $q = \frac{1}{1+\Gamma p}$.

Theorem 2.11 *Consider an EVE correlation model (2.1)-(2.3) with a correlation matrix ρ satisfying Assumption 2.1. Suppose the second-order linear elliptic operator $\mathscr{L}$ in (2.8) with coefficients provided in (2.11) satisfies Assumption 2.2. Then, given a function $h : D \to (0,\infty)$, there exists a local FPP of separable power factor form with the initial condition*

$$U_0(x) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} h(Y_0)^q$$

if and only if there exists a positive finite Borel measure ν on $\mathbb{S}_{\mathscr{L}}(D)$ and a ν-Bochner integrable selection of positive eigenfunctions $\Psi : \mathbb{S}_{\mathscr{L}}(D) \times D \to (0,\infty)$ such that

$$h(y) = \int_{\mathbb{S}_{\mathscr{L}}(D)} \Psi(\zeta, y)\, \nu(\mathrm{d}\zeta). \tag{2.12}$$

Furthermore, each local FPP of separable power factor form is uniquely identified by such a pairing (Ψ,ν), and is given by

$$U_t(x) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} \left(\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta} \Psi(\zeta, Y_t)\, \nu(\mathrm{d}\zeta) \right)^q. \tag{2.13}$$

Any π^* that solves

$$\sigma(Y_t)\pi_t^* = \frac{1}{\gamma} \left(\lambda(Y_t) + q\rho\kappa(Y_t) \frac{\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta} (\nabla_y \Psi)(\zeta, Y_t)\, \nu(\mathrm{d}\zeta)}{\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta} \Psi(\zeta, Y_t)\, \nu(\mathrm{d}\zeta)} \right) \tag{2.14}$$

is an associated optimal portfolio.

Remark 2.12 We note that the equation (2.14) for optimal portfolios π^* does not involve the initial wealth x. This is a consequence of the local FPP being of separable power factor form. In the setting of the Merton problem, the same statement is true (and well-known) for terminal utility functions of power form.

Remark 2.13 A solution to the optimal portfolio equation (2.14) can be obtained as follows. Since $\sigma(\cdot)^{-1} = (\sigma(\cdot)^\top \sigma(\cdot))^{-1} \sigma(\cdot)^\top$, one can write $\lambda(\cdot) = (\sigma(\cdot)^\top)^{-1} \mu(\cdot)$ as $\sigma(\cdot)(\sigma(\cdot)^\top \sigma(\cdot))^{-1} \mu(\cdot)$. In addition, by Assumption 2.1 and the Borel selection result of [4, Theorem 6.9.6], one can find a measurable $\varsigma : D \to \mathbb{R}^{n\times d_B}$ satisfying $\sigma(\cdot)\varsigma(\cdot) = \rho$, which renders

$$\pi_t^* = \frac{1}{\gamma} \left(\left(\sigma(Y_t)^\top \sigma(Y_t) \right)^{-1} \mu(Y_t) + q\varsigma(Y_t)\kappa(Y_t) \frac{\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta} (\nabla_y \Psi)(\zeta, Y_t)\, \nu(\mathrm{d}\zeta)}{\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta} \Psi(\zeta, Y_t)\, \nu(\mathrm{d}\zeta)} \right) \tag{2.15}$$

a solution of (2.14).

The above theorem shows that for given admissible initial conditions, one can construct separable factor-form FPPs in EVE correlation models with general factor domains $D \subseteq \mathbb{R}^k$, while also providing necessary and sufficient conditions for such admissibility. In particular, an investor with risk-aversion γ and dependence $h(Y_0)$ of her current utility function on the initial value of the factor process, can extrapolate the future values of her utility function according to (2.13) and acquire a portfolio fulfilling (2.14) (e.g. the portfolio in (2.15)), provided h is of the form (2.12). It is therefore crucial to understand which functions h admit the representation (2.12) and to be able to determine the pairings (Ψ,ν) for such.

Note that condition (iii) in Assumption 2.2 and the invertibility of the Jacobian matrix of Ξ yield that $\kappa(y)$ has full column rank k at each point y, and thus $k \le d_B$. If $p \neq 0$, this combined with the observations in Remarks 2.2 and 2.4, implies the dimensional relationship $k \le d_B \le n \le d_W$ in our model (2.1)-(2.3).

To the best of our knowledge, the only other paper addressing explicitly FPPs in multi-factor models is Liang and Zariphopoulou [18]. In the models they consider, the factors are exponentially ergodic and live on the full space $D = \mathbb{R}^k$, and the following dimensional relationship holds: $n \le d_W = d_B = k$. In addition, the form of the SPDE in [18] (compare [18, equation (10)] to e.g. [27, equation (3)]) implies that $\sigma\sigma^{-1} = I_{d_W}$, and thereby $n = d_W$. Moreover, in [18], $\rho = \mathrm{corr}(W,B) = I_{d_W}$, and thus their model fits into the EVE framework with $p = 1$. The main difference of the setup in [18] from ours is the possibility of constraints on the set of admissible portfolios. Without constraints, it is possible to linearize the semi-linear PDE in [18, equation (13)] through the exact same steps as in the proof of our Proposition 3.3 below. For general constraints this is not possible. The authors circumvent this issue by representing FPPs as functions of the solutions to appropriate infinite-horizon BSDEs instead.

Another major difference from our results is in the set of allowable initial conditions from which FPPs can be constructed. In the absence of constraints, the results in [18] require the measure ν (in the terminology of our Theorem 2.11) to be a multiple of a Dirac mass on an element of the set of eigenvalues $\mathbb{S}_{\mathscr{L}}(\mathbb{R}^k)$, thereby restricting the function h to be a positive eigenfunction of the elliptic operator $\mathscr{L}$. Our Theorem 2.11 characterizes all admissible initial conditions through the equation (2.12). In addition, our factors live on general domains $D \subseteq \mathbb{R}^k$ and are not required to be ergodic.

2.4 Finding selections of positive eigenfunctions Ψ and measures ν

The next set of results addresses the problem of solving the equation (2.12) for the pairing (Ψ,ν), when it exists. The equation (2.12) stems from a new variant of the generalized Widder's theorem of Nadtochiy and Tehranchi [27, Theorem 3.12] (see Theorem 3.4 below) and, thus, our results can be viewed as yielding explicit versions of such theorems. The following theorem is also of independent interest, as it relates the pairing (Ψ,ν) arising in the positive solution of a linear second-order parabolic PDE posed in the "wrong" time direction to the solution of the same PDE posed in the "right" time direction.

Theorem 2.14 *Let $\mathscr{L}$ satisfy Assumption 2.2 and let $h \in C^{2,\eta}(D)$ be a positive function such that*

$$(t,y) \mapsto \mathbb{E}\Big[h(Z_t)\,\mathbf{1}_{\{\tau>t\}}\,\Big|\,Z_0 = y\Big] \quad \text{is locally bounded on} \quad [0,\varepsilon]\times D \tag{2.16}$$

for the weak solution Z of the SDE associated with $\mathscr{L}_0 := \mathscr{L} - P(y)$ *and* $\varepsilon > 0$*, where* τ *is the first exit time of Z from D. Then, there is a positive classical solution of*

$$\partial_t u + \mathscr{L} u = 0 \;\textit{ pointwise on }\; [-\varepsilon, 0] \times D, \quad \textit{with }\; u(0, \cdot) = h. \tag{2.17}$$

For a positive finite Borel measure ν *on* $\mathbb{S}_{\mathscr{L}}(D)$ *and a* ν*-Bochner integrable selection of positive eigenfunctions* $\Psi : \mathbb{S}_{\mathscr{L}}(D) \times D \to (0, \infty)$*, the function h can be expressed as* $\int_{\mathbb{S}_{\mathscr{L}}(D)} \Psi(\zeta, \cdot)\, \nu(\mathrm{d}\zeta)$ *if and only if the problem* (2.17) *has a unique positive classical solution u, so that for every* $y \in D$*, the function* $u(\cdot, y)$ *on* $(-\varepsilon, 0]$ *is the Laplace transform of the measure* $\Psi(\zeta, y)\, \nu(\mathrm{d}\zeta)$*, that is,*

$$u(t, y) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \Psi(\zeta, y)\, \nu(\mathrm{d}\zeta), \quad t \in (-\varepsilon, 0]. \tag{2.18}$$

In this case, it holds, in particular,

$$u(t, y_0) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \nu(\mathrm{d}\zeta), \quad t \in (-\varepsilon, 0]. \tag{2.19}$$

Remark 2.15 Theorem 2.14 reveals that, whenever a pairing (Ψ, ν) exists, it can be inferred by finding the measure ν through a one-dimensional Laplace inversion of $u(\cdot, y_0)$ (recall that the values of the Laplace transform on a non-trivial interval determine the underlying positive finite Borel measure, see e.g. [3, Section 30]) and then the functions $\Psi(\cdot, y)$, $y \in D \backslash \{y_0\}$ from $u(\cdot, y)$, $y \in D \backslash \{y_0\}$ through additional one-dimensional Laplace inversions.

As a by-product we obtain the following uniqueness result for linear second-order parabolic PDEs posed in the "wrong" time direction by combining the generalized Widder's theorem on domains (Theorem 3.4 below) with Theorem 2.14 and the uniqueness of the Laplace transform ([3, Section 30]).

Corollary 2.16 *For any operator* $\mathscr{L}$ *satisfying Assumption* 2.2 *and positive* $h \in C^{2,\eta}(D)$ *such that the function in* (2.16) *is locally bounded on a non-trivial cylinder* $[0, \varepsilon] \times D$*, there is at most one positive solution* $\tilde{u}$ *of the problem*

$$\partial_t \tilde{u} + \mathscr{L} \tilde{u} = 0 \;\textit{ on }\; [0, \infty) \times D, \quad \tilde{u}(0, \cdot) = h. \tag{2.20}$$

We stress that Corollary 2.16 is not an immediate consequence of the generalized Widder's theorem on domains (Theorem 3.4) by itself. The latter does ensure that every pairing (Ψ, ν) corresponds to exactly one positive solution $\tilde{u}$ of (2.20). However, it is not clear a priori whether the representation $h = \int_{\mathbb{S}_{\mathscr{L}}(D)} \Psi(\zeta, \cdot)\, \nu(\mathrm{d}\zeta)$ is unique for all functions h with the property (2.16). Theorem 2.14 and the uniqueness of the Laplace transform ([3, Section 30]) show that this representation is, indeed, unique.

For arbitrary operators relatively little is known about the sets of positive eigenfunctions $C_{\mathscr{L}-\zeta}(D)$. Nevertheless, in certain situations additional information on the sets $C_{\mathscr{L}-\zeta}(D)$ is available and can be exploited to find the selection of positive eigenfunctions Ψ for a given function h by a *finite* number of Laplace inversions.

Proposition 2.17 *Let $\mathscr{L}$ satisfy Assumption 2.2, then*

$$\zeta_c(D) := \inf\Big\{\zeta \in \mathbb{R} : \zeta \in \mathbb{S}_{\mathscr{L}}(D)\Big\} \in \mathbb{S}_{\mathscr{L}}(D). \tag{2.21}$$

If, in addition, the potential P is constant and $\mathscr{L}_0 := \mathscr{L} - P$ is such that the corresponding solution of the generalized martingale problem on D (see Pinsky [29, Section 1.13]) is recurrent, then $\zeta_c(D) = -P$ and $|C_{\mathscr{L}-\zeta_c(D)}(D)| = 1$.

Remark 2.18 The quantity $\zeta_c(D)$ of (2.21) is commonly referred to as the *critical eigenvalue* of the operator $\mathscr{L}$ on D.

The structure of the eigenspaces $C_{\mathscr{L}-\zeta}(D)$ can differ widely depending on the choice of the dimension k, the restrictions on the operator $\mathscr{L}$, and the domain D. The case $k = 1$ corresponds to having a single factor and leads to eigenspaces of dimension at most 2.

Proposition 2.19 *Suppose $\mathscr{L}$ satisfies Assumption 2.2 on a domain $D \subseteq \mathbb{R}$. Then, the number of extreme points of the convex set $C_{\mathscr{L}-\zeta}(D)$ is 2 for all $\zeta > \zeta_c(D)$ and belongs to $\{1,2\}$ for $\zeta = \zeta_c(D)$.*

Proposition 2.19 reveals that, in the setting of Theorem 2.14 with $k = 1$, one can determine the pairing (Ψ, ν) via a three-step procedure: first, one recovers ν by a one-dimensional Laplace inversion of $u(\cdot, y_0)$; second, one finds $\Psi(\zeta, y_1)\,\nu(\mathrm{d}\zeta)$ by a one-dimensional Laplace inversion of $u(\cdot, y_1)$ for an arbitrary $y_1 \in D\backslash\{y_0\}$; third, for all $\zeta \geq \zeta_c(D)$ in the support of ν, one solves the second-order linear *ordinary* differential equation for $\Psi(\zeta, \cdot)$ with the obtained boundary conditions at y_0 and y_1 to end up with the selection Ψ.

When $k \geq 2$, the variability in the dimensionality of the eigenspaces is illustrated by the following two scenarios, in which the eigenspaces have dimensions 1 and ∞, respectively.

Definition 2.20 A potential $P(\cdot)$ on $\mathbb{R}^k$, $k \geq 2$, is called *principally radially symmetric* if

$$P = P_0 + P_1,$$

where the functions P_0 and P_1 are locally integrable to power d for some $d > k/2$, with P_0 being radially symmetric ($P_0(y) = \widetilde{P}_0(|y|)$ for some $\widetilde{P}_0$), and P_1 vanishing outside of a compact set.

Proposition 2.21 *Let $k \geq 2$ and consider a positive $\phi \in C^{2,\eta}(\mathbb{R}^k)$ with bounded $\frac{\nabla\phi}{\phi}$ and $\frac{\Delta\phi}{\phi}$, as well as an operator $\widetilde{\mathscr{L}} := \Delta + P(y)$ on $\mathbb{R}^k$ with a locally η-Hölder continuous bounded principally symmetric potential $P(\cdot)$. Then, $\mathscr{L} := \frac{1}{\phi}\widetilde{\mathscr{L}}\phi$ has the property $|C_{\mathscr{L}-\zeta}(\mathbb{R}^k)| = 1$ for any $\zeta \geq \zeta_c(\mathbb{R}^k)$ such that*

$$\int_1^\infty t^{k-3} g_0(t)^2 \left(\int_t^\infty s^{1-k} g_0(s)^{-2}\,\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{2.22}$$

where g_0 is the unique solution of

$$g_0''(r)+\frac{k-1}{r}g_0'(r)-\left(\zeta-\widetilde{P}_0(r)\right)g_0(r)=0 \ \text{ on } \ (0,\infty), \quad g_0(r)=1+o(r) \ \text{ as } \ r\downarrow 0.$$

In the situation of Proposition 2.21, we must pick $\Psi(\zeta,\cdot)$ as the unique element of $C_{\mathscr{L}-\zeta}(\mathbb{R}^k)$. On the other hand, in the case of a multidimensional factor process on a bounded domain D with a Lipschitz boundary, the eigenspaces are infinite-dimensional.

Proposition 2.22 *Let $D\subset\mathbb{R}^k$, $k\geq 2$ be a bounded domain with a Lipschitz boundary and the coefficients $a(\cdot)$, $b(\cdot)$, $P(\cdot)$ of $\mathscr{L}$ obey (i)-(iii) in Assumption 2.2 on D. Then, the convex $C_{\mathscr{L}-\zeta}(D)$ has infinitely many extreme points for all $\zeta>\zeta_c(D)$.*

Thus, one cannot assert that the number of extreme points of $C_{\mathscr{L}-\zeta}(D)$ is finite in general. Therefore, the procedure of Remark 2.15 cannot always be reduced to a finite number of one-dimensional Laplace inversions. In such cases, we propose to determine the FPP on a finite number of grid points $y\in D$. First, one computes the Borel measure ν by applying the inverse Laplace transform to the left-hand side of (2.19). Next, for each y on the grid, one calculates the selection of eigenfunctions $\Psi(\cdot,y)$ by taking the inverse Laplace transform of the left-hand side in (2.18). From here, one can identify the value of the FPP on the grid by plugging the obtained values into the equation (2.13).

3 Proof of Theorem 2.11 and a new Widder's theorem

The goal of this section is to prove Theorem 2.11. Recall that we are interested in separable power factor form local FPPs as in Definition 2.7. We start by focusing on the function V in equation (2.6), and give a sufficient condition for $V(t,x,Y_t)$ to be a local FPP.

Proposition 3.1 *Under Assumption 2.1, let $V:[0,\infty)\times(0,\infty)\times D\to\mathbb{R}$ be continuously differentiable in t (its first argument) and twice continuously differentiable in x and y (the second and third arguments). Suppose further that V is strictly concave and increasing in x and a classical solution of the HJB equation*

$$\partial_t V+\mathscr{L}_y V-\frac{1}{2}\frac{|\lambda\,\partial_x V+\rho\kappa\partial_x\nabla_y V|^2}{\partial_{xx}V}=0 \quad \text{on} \quad [0,\infty)\times(0,\infty)\times D, \tag{3.1}$$

where $\mathscr{L}_y$ is the generator of the factor process Y. Then, $V(t,x,Y_t)$ is a local FPP. Moreover, the corresponding optimal portfolio allocations π^* among the n stocks are of a feedback form and characterized by

$$\sigma(Y_t)\pi_t^*=-\frac{\lambda(Y_t)\,\partial_x V(t,X_t^{\pi^*},Y_t)+\rho\kappa(Y_t)\,\partial_x\nabla_y V(t,X_t^{\pi^*},Y_t)}{X_t^{\pi^*}\,\partial_{xx}V(t,X_t^{\pi^*},Y_t)}. \tag{3.2}$$

Proof For the former statement, one only needs to repeat the derivation of [30, equation (1.6)] mutatis mutandis and to use $\sigma(\cdot)\sigma(\cdot)^{-1}\rho = \rho$ (see Remark 2.2). For the latter statement, we apply Itô's formula to $V(t, X_t^\pi, Y_t)$ and substitute $\frac{1}{2}\frac{|\lambda\,\partial_x V + \rho\kappa\partial_x\nabla_y V|^2}{\partial_{xx}V}$ for $\partial_t V + \mathscr{L}_y V$ to conclude that the drift coefficient of $V(t, X_t^\pi, Y_t)$ is $\frac{1}{2}\partial_{xx}V(t, X_t^\pi, Y_t)$ multiplied by

$$\left|\frac{\lambda(Y_t)\partial_x V(t,X_t^\pi,Y_t)+\rho\kappa(Y_t)\partial_x\nabla_y V(t,X_t^\pi,Y_t)}{\partial_{xx}V(t,X_t^\pi,Y_t)} + X_t^\pi\sigma(Y_t)\pi_t\right|^2. \tag{3.3}$$

The process $V(t, X_t^\pi, Y_t)$ is a local martingale if and only if the expression in (3.3) vanishes, which happens if and only if (3.2) holds.

Remark 3.2 The process $V(t,x,Y_t)$ of Proposition 3.1 is a true FPP if $V(t, X_t^\pi, Y_t)$ is a true supermartingale for every $\pi \in \mathscr{A}$ and a true martingale for every optimal portfolio allocation π^* of (3.2). In view of Fatou's lemma, the supermartingale property is fulfilled if $\inf_{s\in[0,t]} V(s, X_s^\pi, Y_s)$ is integrable for all $t \geq 0$ and $\pi \in \mathscr{A}$. The martingale property is valid if the diffusion coefficients $\partial_x V(t, X_t^{\pi^*}, Y_t) X_t^{\pi^*}(\sigma(Y_t)\pi_t^*)^\top$, $\nabla_y V(t, X_t^{\pi^*}, Y_t)\kappa(Y_t)^\top$ of $V(t, X_t^{\pi^*}, Y_t)$ are $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-square integrable on each $[0,t]\times\Omega$.

The HJB equation (3.1) is a fully non-linear PDE and one does not expect to find explicit formulas for its solutions in general. However, in EVE correlation market models, for initial conditions of separable power type, the HJB equation (3.1) can be linearized.

Proposition 3.3 *Let ρ be an EVE correlation matrix as in Definition 2.3, and let $\Gamma = \frac{1-\gamma}{\gamma}$, and $q = \frac{1}{1+\Gamma p}$. Then, the HJB equation (3.1) with an initial condition $V(0,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} h(y)^q$, where $h > 0$, has a classical solution in separable power form, $V(t,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} g(t,y)$, with $g > 0$ if and only if there exists a positive solution to the linear PDE problem*

$$\partial_t u + \mathscr{L}u = 0 \;\; on \;\; [0,\infty)\times D, \quad u(0,\cdot) = h \tag{3.4}$$

posed in the "wrong" time direction. Hereby, $\mathscr{L}$ is the linear elliptic operator of the second order with the coefficients of (2.11). In that case, the two solutions are related through

$$V(t,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} u(t,y)^q.$$

Proof Since we are looking for solutions of the HJB equation (3.1) in separable power form, we plug in the ansatz $V(t,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} g(t,y)$ to arrive at

$$\partial_t g + \mathscr{L}_y g + \frac{\Gamma}{2}\lambda^\top\lambda g + \Gamma\lambda^\top\rho\kappa\nabla_y g + \Gamma\frac{(\nabla_y g)\kappa^\top\rho^\top\rho\kappa\nabla_y g}{2g} = 0, \quad g(0,\cdot) = h^q. \tag{3.5}$$

Next, we employ the distortion transformation $g(t,y) = u(t,y)^q$ and get the PDE

$$qu^{q-1}\partial_t u + \frac{1}{2}\sum_{i,j=1}^{k}(\kappa^\top\kappa)_{ij}\Big(qu^{q-1}\partial_{y_iy_j}u + q(q-1)u^{q-2}(\partial_{y_i}u)(\partial_{y_j}u)\Big)$$
$$+\frac{\Gamma}{2}q^2u^{q-2}(\nabla_y u)^\top\kappa^\top\rho^\top\rho\kappa\nabla_y u + q\Big(\alpha+\Gamma\kappa^\top\rho^\top\lambda\Big)^\top u^{q-1}\nabla_y u + \frac{\Gamma}{2}\lambda^\top\lambda u^q = 0, \tag{3.6}$$

equipped with the initial condition $u(0,\cdot) = h$. Moreover, the assumed positivity of g translates to $u > 0$, so that we can divide both sides of (3.6) by u^{q-1}. In addition, we insert the identity $\rho^\top\rho = pI_{d_B}$ of Definition 2.3 to end up with

$$\begin{aligned}&\partial_t u + \frac{1}{2}\sum_{i,j=1}^{k}(\kappa^\top\kappa)_{ij}\partial_{y_iy_j}u + \Big(\alpha+\Gamma\kappa^\top\rho^\top\lambda\Big)^\top\nabla_y u + \frac{\Gamma}{2q}\lambda^\top\lambda u\\ &+\frac{1}{2u}(q+\Gamma pq-1)(\nabla_y u)^\top\kappa^\top\kappa\nabla_y u = 0.\end{aligned} \tag{3.7}$$

The crucial observation is now that the non-linear term in the PDE (3.7) drops out thanks to $q = \frac{1}{1+\Gamma p}$. Hence, u is a positive solution of (3.4). The converse follows by carrying out the transformations we have used in the reverse order.

Proposition 3.3 reduces the task of finding solutions of the HJB equation (3.1) in separable power form to solving the linear PDE problem (3.4) set in the "wrong" time direction. The latter has been studied in Widder [31] with $\mathscr{L}$ being the Laplace operator on $\mathbb{R}^k$ and in Nadtochiy and Tehranchi [27] for more general linear second-order elliptic operators on $\mathbb{R}^k$. We establish subsequently a variant of [27, Theorem 3.12] that allows for linear second-order elliptic operators on domains $D \subseteq \mathbb{R}^k$.

Theorem 3.4 *Under Assumption 2.2, a function* $u : \{(0,y_0)\}\cup((0,\infty)\times D) \to (0,\infty)$ *is a classical solution of* $\partial_t u + \mathscr{L}u = 0$ *with* $u(0,y_0) = 1$ *if and only if it admits the representation*

$$u(t,y) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta}\,\Psi(\zeta,y)\,\nu(\mathrm{d}\zeta), \tag{3.8}$$

where ν *is a Borel probability measure on* $\mathbb{S}_{\mathscr{L}}(D)$ *and* $\Psi : \mathbb{S}_{\mathscr{L}}(D)\times D \to (0,\infty)$ *is a* ν*-Bochner integrable selection of positive eigenfunctions. In this case, the pairing* (Ψ,ν) *is uniquely determined by the function u.*

Proof Let $u : \{(0,y_0)\}\cup((0,\infty)\times D)\to(0,\infty)$ be a classical solution of $\partial_t u+\mathscr{L}u = 0$ with $u(0,y_0) = 1$. Recalling the C^3-diffeomorphism $\Xi : D \to \mathbb{R}^k$ from Assumption 2.2 and taking without loss of generality $\Xi(y_0) = 0$ (otherwise we compose Ξ with the translation by $-\Xi(y_0)$) we define $\overline{u} : \{(0,0)\}\cup((0,\infty)\times\mathbb{R}^k) \to (0,\infty)$, $(t,z)\mapsto u(t,\Xi^{-1}(z))$. Then, $\partial_t u(t,y) = \partial_t\overline{u}(t,\Xi(y))$, $\partial_{y_i}u(t,y) = \sum_{j=1}^k \partial_{z_j}\overline{u}(t,\Xi(y))\,\partial_{y_i}\Xi_j(y)$, and

$$\partial_{y_iy_j}u(t,y) = \sum_{i',j'=1}^{k}\partial_{z_{i'}z_{j'}}\overline{u}\Big(t,\Xi(y)\Big)\,\partial_{y_i}\Xi_{i'}(y)\,\partial_{y_j}\Xi_{j'}(y) + \sum_{i'=1}^{k}\partial_{z_{i'}}\overline{u}\Big(t,\Xi(y)\Big)\,\partial_{y_iy_j}\Xi_{i'}(y).$$

Plugging these into the PDE for u we conclude that $\overline{u}$ is a classical solution of $\partial_t \overline{u} + \overline{\mathscr{L}}\overline{u} = 0$ with $\overline{u}(0,0) = 1$, where $\overline{\mathscr{L}}$ is the operator of (2.10), (2.9) on $\mathbb{R}^k$. From [27, Theorem 3.12] we infer that

$$\overline{u}(t,z) = \int_{\mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k)} e^{-t\zeta}\,\overline{\Psi}(\zeta,z)\,\nu(\mathrm{d}\zeta),$$

with a Borel probability measure ν on $\mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k)$ and a ν-Bochner integrable selection of positive eigenfunctions $\overline{\Psi} : \mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k) \times \mathbb{R}^k \to (0,\infty)$ for the operator $\overline{\mathscr{L}}$ (note that $\overline{\Psi}(\zeta,\cdot) \in C^{2,\eta}(\mathbb{R}^k)$ by the Schauder interior estimate, e.g., [11, inequality (6.23)]).

Next, we express $\overline{\mathscr{L}}\,\overline{\Psi}(\zeta,\cdot)$ as

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{k}\Big((\nabla\Xi_i)^\top a\nabla\Xi_j\Big)\Big(\Xi^{-1}(\cdot)\Big)\,\partial_{z_i z_j}\overline{\Psi}(\zeta,\cdot)\\
&+\sum_{i=1}^{k}\bigg(\Big((\nabla\Xi_i)^\top b\Big)\Big(\Xi^{-1}(\cdot)\Big)+\frac{1}{2}\mathrm{trace}\Big(\mathrm{Hess}(\Xi_i)a\Big)\Big(\Xi^{-1}(\cdot)\Big)\bigg)\partial_{z_i}\overline{\Psi}(\zeta,\cdot)+P\Big(\Xi^{-1}(\cdot)\Big)\,\overline{\Psi}(\zeta,\cdot)\\
&=\mathscr{L}\overline{\Psi}\Big(\zeta,\Xi(y)\Big)\Big|_{y=\Xi^{-1}(\cdot)}
\end{aligned}$$

and see that $\overline{\mathscr{L}}\,\overline{\Psi}(\zeta,\cdot) = \zeta\overline{\Psi}(\zeta,\cdot)$ is equivalent to $\mathscr{L}\overline{\Psi}(\zeta,\Xi(\cdot)) = \zeta\overline{\Psi}(\zeta,\Xi(\cdot))$. Thus, $\mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k) = \mathbb{S}_{\mathscr{L}}(D)$ and

$$u(t,y) = \overline{u}\Big(t,\Xi(y)\Big) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta}\,\overline{\Psi}\Big(\zeta,\Xi(y)\Big)\,\nu(d\zeta),$$

where $\Psi(\cdot,\cdot) := \overline{\Psi}(\cdot,\Xi(\cdot)) : \mathbb{S}_{\mathscr{L}}(D)\times D \to (0,\infty)$ is a ν-Bochner integrable selection of positive eigenfunctions for the operator $\mathscr{L}$ (observe that the images of compact sets under Ξ are compact).

Conversely, for a function u given by (3.8), it holds $u(0,y_0) = 1$. Moreover, defining the function $\overline{u}$ as before we find that

$$\overline{u}(t,z) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-t\zeta}\,\Psi\Big(\zeta,\Xi^{-1}(z)\Big)\,\nu(\mathrm{d}\zeta).$$

As above, we have that $\mathbb{S}_{\mathscr{L}}(D) = \mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k)$ and that $\overline{\Psi}(\cdot,\cdot) := \Psi(\cdot,\Xi^{-1}(\cdot)) : \mathbb{S}_{\overline{\mathscr{L}}}(\mathbb{R}^k)\times\mathbb{R}^k \to (0,\infty)$ provides a ν-Bochner integrable selection of positive eigenfunctions for the operator $\overline{\mathscr{L}}$. By [27, Theorem 3.12], $\overline{u}$ is a classical solution of $\partial_t\overline{u} + \overline{\mathscr{L}}\overline{u} = 0$ with $\overline{u}(0,0) = 1$ (here, we again assume without loss of generality that $\Xi(y_0) = 0$). Since $(\partial_t\overline{u} + \overline{\mathscr{L}}\overline{u})(t,\Xi(y)) = (\partial_t u + \mathscr{L}u)(t,y)$, the function u is a classical solution of $\partial_t u + \mathscr{L}u = 0$. Lastly, according to [27, Theorem 3.12] the pairing $(\overline{\Psi}(\cdot,\cdot),\nu) = (\Psi(\cdot,\Xi^{-1}(\cdot)),\nu)$ is uniquely determined by the function $\overline{u}(\cdot,\cdot) = u(\cdot,\Xi^{-1}(\cdot))$, so the pairing (Ψ,ν) is uniquely determined by the function u.

We now have all the ingredients needed to prove Theorem 2.11.

Proof (Proof of Theorem 2.11) Take a function $h : D \to (0,\infty)$, and consider

$$V(t,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} g(t,y), \quad \text{where} \quad g > 0 \quad \text{and} \quad g(0,y) = h(y)^q.$$

First, we will show that $V(t,x,Y_t)$ is a separable power factor form local FPP if and only if $V(t,x,y)$ is a classical solution to the HJB equation (3.1). Sufficiency follows trivially from Proposition 3.1. To prove necessity, consider a portfolio allocation $\pi \in \mathscr{A}$. We apply Itô's formula to $\gamma^\gamma \frac{(X_t^\pi)^{1-\gamma}}{1-\gamma} g(t,Y_t)$ and infer from the conditions (ii) and (iii) in Definition 2.6 that the resulting drift coefficient must be non-positive for all $\pi \in \mathscr{A}$ and equal to 0 for any maximizer $\pi^* \in \mathscr{A}$. Equating the maximum of the drift coefficient over all $\pi \in \mathscr{A}$ to 0 we end up with the PDE in (3.5) for g. Thus, V is a classical solution to the HJB equation (3.1).

It follows by Proposition 3.3 that $V(t,x,Y_t)$ is a separable power factor form local FPP if and only if $g(t,y) = u(t,y)^q$, where

$$\partial_t u + \mathscr{L} u = 0 \;\text{ on }\; [0,\infty) \times D \;\text{ with }\; u(0,\cdot) = h(\cdot). \tag{3.9}$$

By Theorem 3.4 each solution u of (3.9) is uniquely identified with a pairing (Ψ,ν), and is given by the right-hand side of (3.8). This yields the necessity and sufficiency of the representation (2.12), as well as the identity (2.13). Finally, the characterization (2.14) of the optimal portfolios is a direct consequence of (3.2) and (2.13).

4 Proof of Theorem 2.14 and further ramifications

4.1 Proof of Theorem 2.14

We start our analysis of the pairing (Ψ,ν) by establishing Theorem 2.14.

Proof (Proof of Theorem 2.14) Let $D' \subset D$ be a bounded subdomain with a C^3 boundary $\partial D' \subset D$ and $\psi : D' \to [0,1]$ be a thrice continuously differentiable function with compact support in D'. Then, Assumption 2.2 and the formulas

$$\begin{aligned} a_{ij}(y) &= \Big((\nabla \Xi_i^{-1})^\top \,\overline{a}\, \nabla \Xi_j^{-1} \Big) \Big(\Xi(y) \Big), \\ b_i(y) &= \Big((\nabla \Xi_i^{-1})^\top \,\overline{b} \Big) \Big(\Xi(y) \Big) + \frac{1}{2} \text{trace} \Big(\text{Hess}(\Xi_i^{-1})\, \overline{a} \Big) \Big(\Xi(y) \Big), \\ P(y) &= \overline{P}\Big(\Xi(y) \Big) \end{aligned} \tag{4.1}$$

render Ladyženskaja et al. [17, Chapter IV, Theorem 5.2] applicable to the problem

$$\partial_t u_{D'} + \mathscr{L} u_{D'} = 0 \;\text{ pointwise on }\; [-\varepsilon,0] \times D', \quad u_{D'}|_{[-\varepsilon,0]\times\partial D'} = 0, \quad u_{D'}(0,\cdot) = h\psi$$

(posed in the "right" time direction), yielding a unique classical solution with η-Hölder continuous $\partial_t u_{D'}$, $\partial_{y_i y_j} u_{D'}$ in the y variable, $\frac{\eta}{2}$-Hölder continuous $\partial_t u_{D'}$, $\partial_{y_i y_j} u_{D'}$

in the t variable, and $\frac{1+\eta}{2}$-Hölder continuous $\partial_{y_i} u_{D'}$ in the t variable. In particular, $u_{D'}$ obeys the Feynman-Kac formula

$$u_{D'}(-t,y) = \mathbb{E}\left[e^{\int_0^t P(Z_s)\,\mathrm{d}s}\,(h\psi)(Z_t)\,\mathbf{1}_{\{\tau_{D'}>t\}}\,\Big|\,Z_0 = y\right], \quad (t,y) \in [0,\varepsilon]\times D',$$

where $\tau_{D'}$ is the first exit time of Z from D'.

Using the described construction for a sequence of subdomains D' and functions ψ increasing to D and $\mathbf{1}_D$, respectively, we arrive at the monotone limit

$$u(-t,y) = \mathbb{E}\left[e^{\int_0^t P(Z_s)\,\mathrm{d}s}\,h(Z_t)\,\mathbf{1}_{\{\tau_D>t\}}\,\Big|\,Z_0 = y\right], \quad (t,y) \in [0,\varepsilon]\times D$$

of $u_{D'}$, which is locally bounded on $[0,\varepsilon]\times D$ by assumption. Thanks to this and the local regularity estimate [17, Chapter IV, inequality (10.5)] on every fixed set $(-\varepsilon,0)\times D'$ (and, hence, on its closure $[-\varepsilon,0]\times\overline{D'}$) we can extract a subsequence of $u_{D'}$ converging uniformly together with $\partial_t u_{D'}$, $\partial_{y_i} u_{D'}$, and $\partial_{y_i y_j} u_{D'}$ on every fixed set $[-\varepsilon,0]\times\overline{D'}$. Thus, u is a positive classical solution of the problem (2.17).

Now, consider an arbitrary positive classical solution u of the problem (2.17), and suppose that there exist pairings $(\Psi^{(1)},\nu^{(1)})$ and $(\Psi^{(2)},\nu^{(2)})$ such that for all $y\in D$

$$h(y) = \int_{\mathbb{S}_{\mathscr{L}}(D)} \Psi^{(1)}(\zeta,y)\,\nu^{(1)}(\mathrm{d}\zeta) = \int_{\mathbb{S}_{\mathscr{L}}(D)} \Psi^{(2)}(\zeta,y)\,\nu^{(2)}(\mathrm{d}\zeta).$$

In view of [29, Chapter 4, Theorem 3.2 and Exercise 4.16] (see also Section 4.2 for more details), the elements of $\mathbb{S}_{\mathscr{L}}(D)$ are bounded below, so that the functions $\widetilde{u}^{(i)}(t,y) := \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\,\Psi^{(i)}(\zeta,y)\,\nu^{(i)}(\mathrm{d}\zeta)$, $i=1,2$ are finite on $[0,\infty)\times D$. By Theorem 3.4, each $\widetilde{u}^{(i)}$ is a classical solution of

$$\partial_t \widetilde{u}^{(i)} + \mathscr{L}\widetilde{u}^{(i)} = 0 \;\text{ on }\; \{(0,y_0)\}\cup((0,\infty)\times D). \tag{4.2}$$

Moreover, each

$$v^{(i)}(t,y) := \begin{cases} u(t,y) & \text{for } (t,y)\in[-\varepsilon,0]\times D, \\ \widetilde{u}^{(i)}(t,y) & \text{for } (t,y)\in(0,\infty)\times D \end{cases}$$

is a positive classical solution of the PDE $\partial_t v^{(i)} + \mathscr{L} v^{(i)} = 0$ on $[-\varepsilon,\infty)\times D$. Indeed, on the sets $[-\varepsilon,0]\times D$ and $(0,\infty)\times D$ this PDE holds by construction, whereas

$$\partial_t \widetilde{u}^{(i)}(0,y) = \lim_{t\downarrow 0} \partial_t \widetilde{u}^{(i)}(t,y) = -\lim_{t\downarrow 0} \mathscr{L}\widetilde{u}^{(i)}(t,y) = -\mathscr{L}\widetilde{u}^{(i)}(0,y), \quad y\in D$$

by the interior Schauder estimate of [27, Theorem 6.2].

Shifting the time by ε and renormalizing $v^{(i)}$, $i=1,2$ we get $\widetilde{v}^{(i)}(t,y) := \frac{v^{(i)}(t-\varepsilon,y)}{v^{(i)}(-\varepsilon,y_0)}$, $i=1,2$, which solve the PDE (4.2) on $[0,\infty)\times D$. By Theorem 3.4, there exist pairings

$(\widetilde{\Psi}^{(i)}, \widetilde{\nu}^{(i)})$, $i = 1, 2$ such that $\widetilde{v}^{(i)}(t,y) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \widetilde{\Psi}^{(i)}(\zeta,y)\, \widetilde{\nu}^{(i)}(\mathrm{d}\zeta)$, $i = 1, 2$. In particular, for $(t,y) \in (0,\infty) \times D$ and $i = 1, 2$,

$$\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta(t+\varepsilon)}\, \widetilde{\Psi}^{(i)}(\zeta,y)\, \widetilde{\nu}^{(i)}(\mathrm{d}\zeta) = \widetilde{v}^{(i)}(t+\varepsilon,y) = \frac{v^{(i)}(t,y)}{v^{(i)}(-\varepsilon,y_0)} = \frac{\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \Psi^{(i)}(\zeta,y)\, \nu^{(i)}(\mathrm{d}\zeta)}{v^{(i)}(-\varepsilon,y_0)}. \tag{4.3}$$

Plugging in first $y = y_0$, then $y \in D\backslash\{y_0\}$, and relying on the uniqueness of the Laplace transform ([3, Section 30]) we read off $\widetilde{\nu}^{(i)}(\mathrm{d}\zeta) = \frac{e^{\zeta\varepsilon}}{v^{(i)}(-\varepsilon,y_0)}\, \nu^{(i)}(\mathrm{d}\zeta)$ and $\widetilde{\Psi}^{(i)} = \Psi^{(i)}$, $i = 1, 2$, from (4.3). Hence, for $(t,y) \in (-\varepsilon, 0] \times D$ and $i = 1, 2$,

$$\begin{aligned} u(t,y) = v^{(i)}(t,y) &= v^{(i)}(-\varepsilon,y_0)\, \widetilde{v}^{(i)}(t+\varepsilon,y) \\ &= v^{(i)}(-\varepsilon,y_0) \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta(t+\varepsilon)}\, \widetilde{\Psi}^{(i)}(\zeta,y)\, \widetilde{\nu}^{(i)}(\mathrm{d}\zeta) \\ &= \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \Psi^{(i)}(\zeta,y)\, \nu^{(i)}(\mathrm{d}\zeta). \end{aligned} \tag{4.4}$$

In particular, we get from the latter equation:

$$\int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \Psi^{(1)}(\zeta,y)\, \nu^{(1)}(\mathrm{d}\zeta) = \int_{\mathbb{S}_{\mathscr{L}}(D)} e^{-\zeta t}\, \Psi^{(2)}(\zeta,y)\, \nu^{(2)}(\mathrm{d}\zeta).$$

Just like above, plugging in $y = y_0$, then $y \in D\backslash\{y_0\}$, and utilizing the uniqueness of the Laplace transform we obtain $\nu^{(1)}(\mathrm{d}\zeta) = \nu^{(2)}(\mathrm{d}\zeta) =: \nu(\mathrm{d}\zeta)$ and $\Psi^{(1)} = \Psi^{(2)} =: \Psi$. Combining this with equation (4.4) we get that any positive classical solution to the problem (2.17) must be as given in (2.18). This yields uniqueness as desired, and in the special case of $y = y_0$, we obtain (2.19).

4.2 Preliminaries on positive eigenfunctions

As a preparation for the proofs of Propositions 2.17, 2.19, 2.21 and 2.22, we recall some facts about the sets $\mathbb{S}_{\mathscr{L}}(D)$ and $C_{\mathscr{L}-\zeta}(D)$, $\zeta \in \mathbb{S}_{\mathscr{L}}(D)$ from the positive harmonic function theory. Throughout the subsection we let $\mathscr{L}$ satisfy Assumption 2.2 and infer from (4.1) that $\mathscr{L}$ is then locally uniformly elliptic with locally bounded and locally η-Hölder continuous coefficients.

Definition 4.1 (Green's measure) Consider the solution Z of the generalized martingale problem on D associated with $\mathscr{L}_0 = \mathscr{L} - P(y)$ (see [29, Section 1.13]). If

$$D' \mapsto \mathbb{E}\left[\int_0^\infty e^{\int_0^t P(Z_s)\,\mathrm{d}s}\, \mathbf{1}_{D'}(Z_t)\,\mathrm{d}t \,\middle|\, Z_0 = y \right] < \infty \tag{4.5}$$

for all bounded subdomains $D' \subset D$ with $\overline{D'} \subset D$ and $y \in D$, then the positive Borel measure defined by (4.5) is called the Green's measure for $\mathscr{L}$ on D. The density $G(y,z)$ of the Green's measure, if it exists, is referred to as the Green's function.

By [29, Chapter 4, Theorem 3.1 and Exercise 4.16] for the operators $\mathscr{L}-\zeta$, $\zeta \in \mathbb{R}$, we have the next proposition.

Proposition 4.2 *If $\zeta \in \mathbb{R}$ is such that the Green's function exists for $\mathscr{L}-\zeta$, then $C_{\mathscr{L}-\zeta}(D) \neq \emptyset$.*

We proceed to the corresponding classification of the operators $\mathscr{L}-\zeta$, $\zeta \in \mathbb{R}$.

Definition 4.3 An operator $\mathscr{L}-\zeta$ on D is described as

(i) subcritical if it possesses a Green's function,
(ii) critical if it is not subcritical, but $C_{\mathscr{L}-\zeta}(D) \neq \emptyset$,
(iii) and supercritical if it is neither critical nor subcritical.

Thus, we are interested in the values of ζ for which $\mathscr{L}-\zeta$ is subcritical or critical, that is, $\zeta \in \mathbb{S}_{\mathscr{L}}(D)$. As it turns out, $\mathbb{S}_{\mathscr{L}}(D)$ is a half-line under Assumption 2.2.

Proposition 4.4 (Pinsky [29], Chapter 4, Theorem 3.2 and Exercise 4.16) *There exists a critical eigenvalue $\zeta_c = \zeta_c(D) \in \mathbb{R}$ such that $\mathscr{L}-\zeta$ is subcritical for $\zeta > \zeta_c$, supercritical for $\zeta < \zeta_c$, and either critical or subcritical for $\zeta = \zeta_c$.*

When the potential P is non-positive, more information about the classification of the operator $\mathscr{L}$ is available.

Proposition 4.5 (Pinsky [29], Chapter 4, Theorem 3.3 and Exercise 4.16) *For an operator $\mathscr{L}$ with $P \leq 0$ one of the following holds:*

(i) $P \leq 0$, $P \not\equiv 0$, and $\mathscr{L}$ is subcritical,
(ii) $P \equiv 0$, the solution of the generalized martingale problem on D associated with $\mathscr{L}$ is transient, and $\mathscr{L}$ is subcritical,
(iii) $P \equiv 0$, the solution of the generalized martingale problem on D associated with $\mathscr{L}$ is recurrent, and $\mathscr{L}$ is critical.

Remark 4.6 When $\gamma > 1$, the potential term in (2.11) is non-positive. This, put together with Proposition 4.5, yields $0 \in \mathbb{S}_{\mathscr{L}}$. Thus, $[0,\infty) \subset \mathbb{S}_{\mathscr{L}}$ by Proposition 4.4.

4.3 Proofs of Propositions 2.17, 2.19, 2.21 and 2.22

At this point, we can read off Propositions 2.17, 2.19 and 2.21 from appropriate results in [21] and [29].

Proof (Proof of Proposition 2.17) By Propositions 4.2 and 4.4,

$$\inf\Big\{\zeta \in \mathbb{R} : \zeta \in \mathbb{S}_{\mathscr{L}}(D)\Big\} = \zeta_c(D) \in \mathbb{S}_{\mathscr{L}}(D).$$

If P is constant and the solution of the generalized martingale problem on D for $\mathscr{L}-P$ is recurrent, then $\mathscr{L}-P$ is critical by Proposition 4.5, and hence, $\zeta_c(D) = -P$. In this case, [29, Chapter 4, Theorem 3.4 and Exercise 4.16] yield $|C_{\mathscr{L}-\zeta_c(D)}(D)| = 1$.

Proof (Proof of Proposition 2.19) It suffices to put together Proposition 4.4 with [29, Chapter 4, Remark 2 on p. 149, Theorem 3.4 and Exercise 4.16].

Proof (Proof of Proposition 2.21) Note that, for any $\zeta \geq \zeta_c(\mathbb{R}^k)$ and $f \in C_{\mathscr{L}-\zeta}$, one has $\phi f \in C_{\widetilde{\mathscr{L}}-\zeta}$. Therefore, it is enough to prove $|C_{\widetilde{\mathscr{L}}-\zeta}| = 1$, $\zeta \geq \zeta_c(\mathbb{R}^k)$, which is readily obtained by combining Proposition 4.4 with Murata [21, Theorem 5.3].

Remark 4.7 The condition (2.22), on $\mathscr{L}$ and ζ, needs to be verified on a case-by-case basis. For example, consider a locally η-Hölder continuous non-positive bounded radially symmetric potential P_0 with $\widetilde{P}_0(r) = cr^{-2}$, $r \geq 1$ for some $c < 0$. Take $\mathscr{L} = \frac{1}{\phi}(\Delta + P_0)\phi$ for some ϕ as in Proposition 2.21 and $\zeta = 0$. Then,

$$g_0(r) = c_1 r^{\frac{2-k+\sqrt{(k-2)^2-4c}}{2}} + c_2 r^{\frac{2-k-\sqrt{(k-2)^2-4c}}{2}}, \quad r \geq 1$$

for some $c_1, c_2 \in \mathbb{R}$. By [21, Theorem 4.6(iii) and Theorem 2.4(ii)] the operator $\Delta + P_0$ is subcritical, so that $c_1 \neq 0$ by [21, Theorem 3.1(ii)]. Thus, (2.22) holds.

In the context of Proposition 2.22, the structure of the sets $C_{\mathscr{L}-\zeta}(D)$, $\zeta > \zeta_c(D)$ has been described in [1, Theorems 6.1 and 6.3], which we briefly recall for the convenience of the reader.

Definition 4.8 (Minimal eigenfunction) A function $f \in C_{\mathscr{L}-\zeta}(D)$ is referred to as minimal if $\widetilde{f} \leq f$ implies $\widetilde{f} = f$ for all $\widetilde{f} \in C_{\mathscr{L}-\zeta}(D)$.

Proposition 4.9 (Ancona [1], Theorems 6.1 and 6.3) *In the setting of Proposition 2.22, every minimal element $f \in C_{\mathscr{L}-\zeta}(D)$ has the property $\lim_{z\to y} f(z) > 0$ for exactly one point $y \in \partial D$ and is uniquely determined by y. In addition, for every $f \in C_{\mathscr{L}-\zeta}(D)$, there exists a unique Borel probability measure ξ on ∂D such that*

$$f(\cdot) = \int_{\partial D} f_y(\cdot)\, \xi(\mathrm{d}y), \tag{4.6}$$

where f_y is the minimal eigenfunction associated with y.

Proposition 2.22 is a direct consequence of Proposition 4.9.

Proof (Proof of Proposition 2.22) The uniqueness of the Borel probability measure ξ in the representation (4.6) shows that the extreme points of $C_{\mathscr{L}-\zeta}(D)$ are precisely the minimal eigenfunctions f_y, $y \in \partial D$. Clearly, $|\{f_y : y \in \partial D\}| = |\partial D| = \infty$.

5 Merton problem in stochastic factor models

In this section, we consider the framework of the Merton problem, in which an investor aims to maximize her expected terminal utility from the wealth acquired through investment:

$$\sup_{\pi \in \mathscr{A}} \mathbb{E}\Big[\upsilon_T(X_T^{\pi}, Y_T)\Big].$$

Thereby, the time horizon T and the utility function υ_T are chosen once and for all at time zero. It is well-known (see e.g. [10, Section IV.3]) that the dynamic programming equation for the Merton problem within the Markovian diffusion model (2.1)-(2.3) takes the shape of the HJB equation

$$\partial_t V + \mathscr{L}_y V - \frac{1}{2}\frac{|\lambda\,\partial_x V + \rho\kappa\,\partial_x \nabla_y V|^2}{\partial_{xx} V} = 0. \tag{5.1}$$

In contrast to the preceding discussion, here the HJB equation is equipped with a terminal condition $V(T,\cdot,\cdot) = \upsilon_T$ and, hence, posed in the backward ("right") time direction. It turns out that, under Definition 2.3, we can reduce the backward problem to a linear second-order parabolic PDE posed in the "right" time direction, provided that the terminal utility function is of separable power form: $\upsilon_T(x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} g_T(y)$, and that appropriate technical assumptions hold.

Theorem 5.1 *Let $\gamma \in (0,1)$. Suppose the market model (2.1)-(2.3), the correlation matrix ρ, and the linear elliptic operator of the second order $\mathscr{L}$ with the coefficients*

$$a(\cdot) = \kappa(\cdot)^\top \kappa(\cdot), \quad b(\cdot) = \alpha(\cdot) + \Gamma\kappa(\cdot)^\top \rho^\top \lambda(\cdot), \quad P(\cdot) = \frac{\Gamma}{2q}\lambda(\cdot)^\top \lambda(\cdot)$$

satisfy Assumptions 2.1, 2.3, and 2.2, respectively, where $\Gamma = \frac{1-\gamma}{\gamma}$ and $q = \frac{1}{1+\Gamma p}$. Suppose further that the volatility matrix $\kappa(\cdot)$ of the factor process is bounded, the weak solution Z of the SDE associated with $\mathscr{L}_0 = \mathscr{L} - P(y)$ remains in D, and the terminal utility function is of separable power form $\upsilon_T(x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} h(y)^q$, with an $h \in C^{2,\eta}(D)$ bounded above and below by positive constants and such that

$$(t,y) \mapsto \nabla_y \mathbb{E}\Big[e^{\int_0^t P(Z_s)\,\mathrm{d}s} h(Z_t)\,\Big|\,Z_0 = y\Big]$$

is bounded on $[0,T]\times D$. Then, the value function for the corresponding Merton problem, $V(t,x,y) = \sup_{\pi\in\mathscr{A}} \mathbb{E}[\upsilon_T(X_T^\pi, Y_T)\,|\,X_t^\pi = x, Y_t = y]$, can be written as

$$V(t,x,y) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} u(t,y)^q. \tag{5.2}$$

Hereby, u is a classical solution of the linear PDE problem

$$\partial_t u + \mathscr{L} u = 0 \ \text{ on } \ [0,T]\times D, \quad u(T,\cdot) = h.$$

Moreover, every portfolio allocation π^* fulfilling

$$\sigma(Y_t)\pi_t^* = \frac{1}{\gamma}\left(\lambda(Y_t) + q\rho\,\kappa(Y_t)\frac{\nabla_y u(t,Y_t)}{u(t,Y_t)}\right) \tag{5.3}$$

is optimal.

Proof By the classical verification paradigm (see e.g. [10, Chapter IV, proof of Theorem 3.1]), it is enough to show that for every portfolio allocation $\pi \in \mathscr{A}$ the process $V(t,X_t^\pi,Y_t)$, $t\in[0,T]$ is a supermartingale, and that for every solution π^* of (5.3) the process $V(t,X_t^{\pi^*},Y_t)$, $t\in[0,T]$ is a martingale.

We follow the proof of Proposition 3.3 in the reverse direction and find that $g(t,y) := u(t,y)^q$ is a classical solution of the problem (3.5), whereas the function V defined by (5.2) is a classical solution of the HJB equation (5.1) with $V(T,\cdot,\cdot)=\upsilon_T$. For any $\pi\in\mathscr{A}$, we may now apply Itô's formula to $V(t,X_t^\pi,Y_t)$ and replace $\partial_t V + \mathscr{L}_y V$ by $\frac{1}{2}\frac{|\lambda\,\partial_x V+\rho\kappa\,\partial_x\nabla_y V|^2}{\partial_{xx}V}$ to see that the drift coefficient of $V(t,X_t^\pi,Y_t)$ is the product of $\frac{1}{2}\partial_{xx}V(t,X_t^\pi,Y_t)$ with the expression in (3.3) and, in particular, non-positive. Hence, the local martingale part of $V(t,X_t^\pi,Y_t)$ is bounded below by $-V(0,x,y)$ and, consequently, a supermartingale. Thus, $V(t,X_t^\pi,Y_t)$ is a supermartingale as well.

Next, we deduce from the proof of Theorem 2.14 that $u(t,y)$ admits the stochastic representation

$$u(t,y)=\mathbb{E}\left[e^{\int_0^{T-t}P(Z_s)\,\mathrm{d}s}\,h(Z_{T-t})\,\middle|\,Z_0=y\right]$$

(recall that Z remains in D by assumption). In addition, our further assumptions imply that $\nabla_y u$ is bounded on $[0,T]\times D$, and that u is bounded above and below by positive constants on $[0,T]\times D$. Together with the boundedness of the volatility matrix $\kappa(\cdot)$ of the factor process and the Sharpe ratio $\lambda(\cdot)$ (see Assumption 2.2(i)) this yields the boundedness of $\sigma(Y_t)\pi_t^*$ via (5.3). Finally, the drift coefficient of $V(t,X_t^{\pi^*},Y_t)$ vanishes and the quadratic variation of its local martingale part computes to

$$\begin{aligned}\int_0^t \gamma^{2\gamma}(X_s^{\pi^*})^{2-2\gamma}\big|\sigma(Y_s)\pi_s^*\big|^2 + \frac{\gamma^{2\gamma}q^2}{(1-\gamma)^2}(X_s^{\pi^*})^{2-2\gamma}u(s,Y_s)^{2q-2}\big|\kappa(Y_s)\nabla_y u(s,Y_s)\big|^2 \\ + \frac{2\gamma^{2\gamma}q}{1-\gamma}(X_s^{\pi^*})^{2-2\gamma}u(s,Y_s)^{q-1}\Big(\sigma(Y_s)\pi_s^*\Big)^\top\rho\,\kappa(Y_s)\nabla_y u(s,Y_s)\,\mathrm{d}s.\end{aligned} \tag{5.4}$$

The expectation of the latter integral is finite for all $t\in[0,T]$, since $\sigma(Y_s)\pi_s^*$ and $u(s,Y_s)^{q-1}\kappa(Y_s)\nabla_y u(s,Y_s)$ are bounded, while $\sup_{t\in[0,T]}\mathbb{E}[(X_t^{\pi^*})^{2-2\gamma}]<\infty$ thanks to the boundedness of $\sigma(Y_s)\pi_s^*$ and $\lambda(Y_s)$ in

$$X_t^{\pi^*}=x\exp\left(\int_0^t\Big(\sigma(Y_s)\pi_s^*\Big)^\top\lambda(Y_s)\,\mathrm{d}s+\int_0^t\Big(\sigma(Y_s)\pi_s^*\Big)^\top\mathrm{d}W_s-\frac{1}{2}\int_0^t\big|\sigma(Y_s)\pi_s^*\big|^2\,\mathrm{d}s\right).$$

We conclude that $V(t,X_t^{\pi^*},Y_t)$ is a true martingale.

6 Discussion of EVE assumption

This last section is devoted to a thorough investigation of Definition 2.3 that plays a key role in the proof of Theorem 2.11. It is instructive to start with the two extreme cases corresponding to taking $p=1$ and $p=0$ therein, respectively. Suppose first that $A=0$ in (2.3), in other words, the components of the Brownian motion B driving the factors are given by linear combinations of the components of the Brownian motion W driving the stock prices. We can then reparametrize the model such that $B = W$, $\rho = I_{d_W}$, and $\rho^\top \rho = I_{d_W}$. Consequently, Definition 2.3 holds with $p = 1$. The resulting market is complete, and we find ourselves in the framework of [27, Section 2.3]. It is therefore not surprising that the HJB equation (3.1) can be reduced to a linear PDE, even though the linearization in Proposition 3.3 differs from the one in [27, Section 2.3]. On the other hand, when $\rho = 0$ in (2.3), the Brownian motions B and W become independent, leading to an incomplete market. Nonetheless, Definition 2.3 is still satisfied with $p = 0$. Thus, the linearization in Proposition 3.3 goes far beyond the complete market setup.

More generally, Definition 2.3 can be put to use as follows. In practice, the correlation matrix ρ can have hundreds or thousands of entries and, hence, might be difficult to estimate accurately in its entirety. However, one can attempt to obtain a less noisy estimate by projecting an estimate for ρ onto the submanifold of $d_W \times d_B$ matrices fulfilling Definition 2.3. Restricting the attention to the non-trivial case $d_W \geq d_B$ (see Remark 2.4), with the exception of the zero matrix, the latter matrices can be written uniquely as rQ, where $r \in (0,1]$ and Q has orthonormal columns, thereby forming a $(1 + d_W d_B - d_B(d_B+1)/2)$-dimensional submanifold of $\mathbb{R}^{d_W \times d_B}$. As it turns out, the most tractable projection onto this submanifold is that with respect to the Frobenius norm (also known as the Hilbert-Schmidt norm) on $\mathbb{R}^{d_W \times d_B}$.

6.1 Choice of r and Q

Let us equip the space $\mathbb{R}^{d_W \times d_B}$ with the Frobenius norm

$$|A|_F = \left(\sum_{i=1}^{d_W} \sum_{j=1}^{d_B} a_{ij}^2 \right)^{1/2} = \Big(\operatorname{trace} A^\top A \Big)^{1/2}.$$

For an estimate $\widehat{\rho}$ of ρ, we are able to find a constant $r \in [0,1]$ and a matrix with orthonormal columns Q that minimize the distance induced by the Frobenius norm.

Proposition 6.1 *Consider the minimization problem*

$$\min |\widehat{\rho} - rQ|_F \ \text{ such that } \ r \in [0,1], \ \ Q^\top Q = I_{d_B}.$$

Then, $r^* = \frac{\operatorname{trace}(\widehat{\rho}^\top \widehat{\rho})^{1/2}}{d_B}$ *and* $Q^* = \widehat{\rho}(\widehat{\rho}^\top \widehat{\rho})^{-1/2}$ *are the minimizers.*

Proof Equivalently, consider the problem

$$\min |\widehat{\rho} - \widetilde{Q}|_F^2 \ \text{ such that } \ \widetilde{Q}^\top \widetilde{Q} = r^2 I_{d_B}$$

for fixed $r \in [0,1]$ and minimize over $r \in [0,1]$ subsequently. Applying the method of Lagrange multipliers with a $d_B \times d_B$ Lagrange multiplier matrix Λ we get

$$2(\widetilde{Q} - \widehat{\rho}) = \widetilde{Q}(\Lambda + \Lambda^\top) \quad \Longleftrightarrow \quad \widetilde{Q}(2I_{d_B} - \Lambda - \Lambda^\top) = 2\widehat{\rho}. \tag{6.1}$$

Passing to the transpose on both sides of the last equation, taking the product of the resulting equation with the original equation, and recalling the constraint we see

$$r^2(2I_{d_B}-\Lambda-\Lambda^\top)^2=4\widehat{\rho}^\top\widehat{\rho} \quad\iff\quad r(2I_{d_B}-\Lambda-\Lambda^\top)=2(\widehat{\rho}^\top\widehat{\rho})^{1/2},$$

where $(\widehat{\rho}^\top\widehat{\rho})^{1/2}$ is the $d_B\times d_B$ square root of the matrix $\widehat{\rho}^\top\widehat{\rho}$. Together with (6.1) and the notation $(\widehat{\rho}^\top\widehat{\rho})^{-1/2}$ for the inverse of $(\widehat{\rho}^\top\widehat{\rho})^{1/2}$ this yields

$$\widetilde{Q}=r\widehat{\rho}(\widehat{\rho}^\top\widehat{\rho})^{-1/2}. \tag{6.2}$$

Plugging the formula for $\widetilde{Q}$ back into the objective function we are left with the minimization problem

$$\min_{r\in[0,1]}\big|\widehat{\rho}-r\widehat{\rho}(\widehat{\rho}^\top\widehat{\rho})^{-1/2}\big|_F^2 \quad\iff\quad \min_{r\in[0,1]}\Big(\mathrm{trace}(\widehat{\rho}^\top\widehat{\rho})-2r\,\mathrm{trace}(\widehat{\rho}^\top\widehat{\rho})^{1/2}+r^2d_B\Big).$$

Consequently, the optimal r is $\frac{\mathrm{trace}(\widehat{\rho}^\top\widehat{\rho})^{1/2}}{d_B}$, that is, the average of the singular values of $\widehat{\rho}$, whereas $\widetilde{Q}$ should be picked according to (6.2).

6.2 Choice of p

If one is only interested in the parameter p from Definition 2.3, then it is most natural to minimize $|\widehat{\rho}^\top\widehat{\rho}-pI_{d_B}|$ for a selection of a norm $|\cdot|$ on $\mathbb{R}^{d_B\times d_B}$. When $|\cdot|$ is the operator norm (also known as the spectral radius or the Ky Fan 1-norm),

$$|\widehat{\rho}^\top\widehat{\rho}-pI_{d_B}|=\max_{1\le i\le d_B}|\theta_i-p|,$$

where $\theta_1\le\theta_2\le\cdots\le\theta_{d_B}$ are the ordered eigenvalues of $\widehat{\rho}^\top\widehat{\rho}$ (or, equivalently, the ordered squared singular values of $\widehat{\rho}$). In this case, $|\widehat{\rho}^\top\widehat{\rho}-pI_{d_B}|$ is minimized by $p=(\theta_1+\theta_{d_B})/2$. When $|\cdot|$ is the Frobenius norm,

$$|\widehat{\rho}^\top\widehat{\rho}-pI_{d_B}|=\left(\sum_{i=1}^{d_B}|\theta_i-p|^2\right)^{1/2}.$$

The minimizer for the latter is $p=(\theta_1+\theta_2+\cdots+\theta_{d_B})/d_B$. When $|\cdot|$ is the trace norm (also known as the nuclear norm or the Ky Fan d_B-norm),

$$|\widehat{\rho}^\top\widehat{\rho}-pI_{d_B}|=\sum_{i=1}^{d_B}|\theta_i-p|,$$

which is smallest for the median of $\{\theta_1,\theta_2,\ldots,\theta_{d_B}\}$.

6.3 Example: affine factor models

We conclude by illustrating the use of the EVE assumption in the framework of affine market models with non-negative factors. In that situation, both the forward investment problem and the Merton problem can be reduced to the solution of a system of Riccati ordinary differential equations (ODEs). Consider the affine specialization of the factor model (2.1)-(2.3):

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_i(Y_t)\,\mathrm{d}t + \sum_{j=1}^{d_W} \sigma_{ji}(Y_t)\,\mathrm{d}W_t^j, \quad i = 1, 2, \ldots, n,$$
$$\mathrm{d}Y_t = (M^\top Y_t + w)\,\mathrm{d}t + \kappa(Y_t)^\top\,\mathrm{d}B_t,$$
$$B_t = \rho^\top W_t + A^\top W_t^\perp,$$

where M has non-negative off-diagonal entries, $w \in [0,\infty)^k$, and $\mu(\cdot)$, $\sigma(\cdot)$, $\kappa(\cdot)$, ρ are such that

$$\lambda(y)^\top \lambda(y) = \mu(y)^\top \sigma(y)^{-1}\Big(\sigma(y)^\top\Big)^{-1}\mu(y) = \Lambda^\top y,$$
$$\kappa(y)^\top \kappa(y) = \mathrm{diag}(L_1 y_1, L_2 y_2, \ldots, L_k y_k) \ \text{ with } L_1, L_2, \ldots, L_k \geq 0,$$
$$\Gamma \kappa(y)^\top \rho^\top \lambda(y) = N^\top y.$$

Remark 6.2 The condition (6.7) is necessary for the process Y of (6.4) to be $[0,\infty)^k$-valued and affine (see [9, Theorem 3.2]). Conversely, the SDE (6.4) with volatility coefficients satisfying (6.7) has a unique weak solution, which is affine and takes values in $[0,\infty)^k$ (see [9, Theorem 8.1]).

Suppose now that the initial utility function for the forward investment problem or the terminal utility function for the Merton problem is of separable power form with $h(y) = \exp(H^\top y + h_0)$. Under the EVE assumption, the HJB equation (3.1) arising in the two problems can be transformed into the linear second-order parabolic PDE of (3.4) (see the proof of Proposition 3.3), which in the setting of (6.3)-(6.8) amounts to

$$\partial_t u + \frac{1}{2}\sum_{i=1}^k L_i y_i \partial_{y_i y_i} u + y^\top (M+N)\nabla_y u + w^\top \nabla_y u + \frac{\Gamma}{2q} y^\top \Lambda u = 0.$$

Inserting the exponential-affine ansatz $u(t,y) = \exp(\Phi_t^\top y + \Theta_t)$ we obtain

$$y^\top \dot{\Phi}_t + \dot{\Theta}_t + \frac{1}{2}\sum_{i=1}^k L_i y_i (\Phi_t^i)^2 + y^\top (M+N)\Phi_t + w^\top \Phi_t + \frac{\Gamma}{2q} y^\top \Lambda = 0.$$

Equating the linear and the constant terms in y to 0 leads to the following system of Riccati ODEs

$$\dot{\Phi}_t^i + \frac{1}{2} L_i (\Phi_t^i)^2 + \sum_{j=1}^k (M+N)_{ij} \Phi_t^j + \frac{\Gamma}{2q}\Lambda_i = 0, \quad i = 1, 2, \ldots, k,$$
$$\dot{\Theta}_t + w^\top \Phi_t = 0.$$

We note that Θ is completely determined by the solution Φ of the system (6.9). The latter can be solved numerically in general and, for special kinds of M and N, even explicitly. For example, when M and N are diagonal the system (6.9) splits into k one-dimensional Riccati ODEs

$$\dot{\Phi}_t^i + \frac{1}{2} L_i (\Phi_t^i)^2 + (M_{ii} + N_{ii}) \Phi_t^i + \frac{\Gamma}{2q} \Lambda_i = 0, \quad i = 1, 2, \ldots, k. \tag{6.11}$$

These ODEs can be solved by a separation of variables and subsequent integration. For instance, when $\gamma \in (0,1)$ and the discriminants $D_i := (M_{ii} + N_{ii})^2 - L_i \frac{\Gamma}{q} \Lambda_i$ associated with the quadratic equations $\frac{1}{2} L_i z^2 + (M_{ii} + N_{ii}) z + \frac{\Gamma}{2q} \Lambda_i = 0$ are positive for all i, we obtain the (real) roots

$$z_{+,i} = \frac{-M_{ii} - N_{ii} + \sqrt{D_i}}{L_i}, \quad z_{-,i} = \frac{-M_{ii} - N_{ii} - \sqrt{D_i}}{L_i}.$$

The general solution of (6.11) then becomes

$$\Phi_t^i = \frac{z_{+,i} - \chi_i z_{-,i} e^{-\sqrt{D_i} t}}{1 - \chi_i e^{-\sqrt{D_i} t}}, \quad i = 1, 2, \ldots, k, \tag{6.12}$$

and one can find the constants χ_i by setting Φ to H at the terminal time (for the Merton problem) or at time 0 (for the forward investment problem).

We conclude by discussing, in the latter setting and with $M_{ii} + N_{ii} \geq 0$ for all i, the true FPP property of the process

$$V(t,x,Y_t) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} u(t,Y_t)^q = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} \exp\Big(q \Phi_t^\top Y_t + q \Theta_t\Big). \tag{6.13}$$

By arguing as in the second paragraph of the proof of Theorem 5.1 we conclude that $V(t, X_t^\pi, Y_t)$ is a true supermartingale for each $\pi \in \mathscr{A}$. It remains to see if $V(t, X_t^{\pi^*}, Y_t)$ is a true martingale for some $\pi^* \in \mathscr{A}$ as in (3.2). To this end, we consider the expectation of the integral in (5.4). In view of the Cauchy-Schwarz inequality and Fubini's theorem, it suffices to control the expectations of the two summands in the first line of (5.4) uniformly over $s \in [0,t]$. The random variables entering the two summands read in the case at hand as follows

$$X_s^{\pi^*} = x \exp\left(\int_0^s \left(\frac{\Lambda^\top}{\gamma} + \frac{q \Phi_r^\top N^\top}{1-\gamma}\right) Y_r \,\mathrm{d}r + \int_0^s \Big(\sigma(Y_r)\pi_r^*\Big)^\top \mathrm{d}W_r - \frac{1}{2}\int_0^s \big|\sigma(Y_r)\pi_r^*\big|^2 \mathrm{d}r\right),$$

$$\big|\sigma(Y_s)\pi_s^*\big|^2 = \frac{1}{\gamma^2}\left(\Lambda^\top Y_s + \frac{2q}{\Gamma}\Phi_s^\top N^\top Y_s + pq^2 \sum_{i=1}^k L_i (\Phi_s^i)^2 Y_s^i\right),$$

$$\big|\kappa(Y_s)\nabla_y u(s,Y_s)\big|^2 = u(s,Y_s)^2 \sum_{i=1}^k L_i (\Phi_s^i)^2 Y_s^i.$$

With $p_1, p_2 > 1$ satisfying $p_1^{-1} + p_2^{-1} = 1$, $\widetilde{\gamma} := 1 - \gamma$ and $p_3 := 1 - 2\widetilde{\gamma} p_2 < 1$, we bound the expectation of the first summand in the first line of (5.4) using Hölder's inequality and the supermartingale property of stochastic exponentials (see e.g. [15, Chapter 3, discussion before Proposition 5.12]) by $\gamma^{2\gamma}x^{2\widetilde{\gamma}}$ times

$$\mathbb{E}\left[\left|\sigma(Y_s)\pi_s^*\right|^{2p_1}\exp\left(2p_1\widetilde{\gamma}\int_0^s\left(\sigma(Y_r)\pi_r^*\right)^\top\lambda(Y_r)\,\mathrm{d}r-p_1\widetilde{\gamma}p_3\int_0^s\left|\sigma(Y_r)\pi_r^*\right|^2\mathrm{d}r\right)\right]^{\frac{1}{p_1}}$$
$$\cdot\,\mathbb{E}\left[\exp\left(2p_2\widetilde{\gamma}\int_0^s\left(\sigma(Y_r)\pi_r^*\right)^\top\mathrm{d}W_r-p_2\widetilde{\gamma}(1-p_3)\int_0^s\left|\sigma(Y_r)\pi_r^*\right|^2\mathrm{d}r\right)\right]^{\frac{1}{p_2}}$$
$$\le\mathbb{E}\left[\left|\sigma(Y_s)\pi_s^*\right|^{2p_1}\exp\left(2p_1\widetilde{\gamma}\int_0^s\left(\frac{\Lambda^\top}{\gamma}+\frac{q\Phi_r^\top N^\top}{1-\gamma}\right)Y_r\,\mathrm{d}r-p_1\widetilde{\gamma}p_3\int_0^s\left|\sigma(Y_r)\pi_r^*\right|^2\mathrm{d}r\right)\right]^{\frac{1}{p_1}}.$$

For every i and r, the coefficient of Y_r^i in the latter exponential admits the estimate

$$p_1\widetilde{\gamma}\left(\frac{2\Lambda_i}{\gamma}+\frac{2qN_{ii}\Phi_r^i}{1-\gamma}-\frac{p_3}{\gamma^2}\left(\Lambda_i+\frac{2q}{\Gamma}N_{ii}\Phi_r^i+pq^2L_i(\Phi_r^i)^2\right)\right)$$
$$\le p_1\widetilde{\gamma}\left(\Lambda_i\,\frac{2\gamma-p_3}{\gamma^2}+N_{ii}c_{i,1}^\Phi\,\frac{2q(\gamma-p_3)}{\gamma(1-\gamma)}-L_i(c_{i,2}^\Phi)^2\,\frac{pq^2p_3}{\gamma^2}\right)=:\beta_i,$$

where

$$c_{i,1}^\Phi=\begin{cases}z_{+,i} & \text{if}\quad N_{ii}\ge 0,\\ z_{-,i} & \text{if}\quad N_{ii}<0\end{cases}\qquad\text{and}\qquad c_{i,2}^\Phi=\begin{cases}z_{+,i} & \text{if}\quad p_3\ge 0,\\ z_{-,i} & \text{if}\quad p_3<0\end{cases}$$

(note that (6.12) and $M_{ii}+N_{ii}\ge 0$ imply $z_{-,i}\le\Phi_s^i\le z_{+,i}\le 0$).

In view of the uniform boundedness of any given moment of Y_s over $s\in[0,t]$ (see Filipović and Mayerhofer [9, Lemma A.1, Lemma 2.3(iv) and Theorem 3.2]) and Hölder's inequality, it suffices to control the exponential moment of $\int_0^s Y_r\,\mathrm{d}r$ of an order slightly larger (componentwise) than $\beta:=(\beta_1,\beta_2,\ldots,\beta_k)$ uniformly over $s\in[0,t]$. With the explicit solution

$$-M_{ii}+\sqrt{\Delta_i}\tan\left(\arctan\frac{M_{ii}}{\sqrt{\Delta_i}}+\frac{\sqrt{\Delta_i}}{2}\,t\right),\quad i=1,2,\ldots,k$$

to the system of Riccati ODEs in [9, Theorem 4.1(ii), third line of display (4.5)], where $\Delta_i=2L_{ii}\beta_i-M_{ii}^2$, we find that the exponential moment in consideration is bounded uniformly over $s\in[0,t]$ as long as

$$t<\min_{i=1,2,\ldots,k}\frac{\pi-2\arctan(M_{ii}/\sqrt{\Delta_i})}{\sqrt{\Delta_i}}.\tag{6.14}$$

Similarly, the expectation of the second summand in the first line of (5.4) is less or equal to $\frac{\gamma^{2\gamma}q^2}{(1-\gamma)^2}e^{2q\Theta_s}$ times

$$\mathbb{E}\left[\left(\sum_{i=1}^k L_i(\Phi_s^i)^2Y_s^i\right)^{p_1}\exp\left(2p_1\widetilde{\gamma}\int_0^s\left(\frac{\Lambda^\top}{\gamma}+\frac{q\Phi_r^\top N^\top}{1-\gamma}\right)Y_r\,\mathrm{d}r-p_1\widetilde{\gamma}p_3\int_0^s\left|\sigma(Y_r)\pi_r^*\right|^2\mathrm{d}r\right)\right]^{\frac{1}{p_1}},$$

which is also bounded uniformly over $s \in [0,t]$ as long as (6.14) holds. All in all, the process in (6.13) is a true FPP at least until (but possibly not including) the time on the right-hand side of (6.14).

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