Variable Costs in Dynamic Cournot Energy Markets

Anirudh Dasarathy* Ronnie Sircar†

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Abstract

We study a game-theoretic model for energy markets. Our framework is an N-player stochastic dynamic Cournot game where one producer has a reserve (or stock) that depletes over time, while the others can produce indefinitely with no such quantity restriction. We think of the first player as producing energy from a fossil fuel such as oil, which is an exhaustible resource, while the others are producing from renewables. All players have costs of production that evolve over time, and the exhaustible player can choose to invest in R&D (research and development, including exploration) which may yield increases in stock probabilistically over time. The assumption that the players have heterogeneous and time-varying costs requires a reexamination and extension of previous literature which has typically considered homogeneous costs. We also study how this model may be applied to energy policy, comparing when it is optimal to consider taxing oil producers, opposed to subsidizing green energy, as a matter of public policy.

1 Introduction

The motivation behind the work presented in this paper is to present a model for energy markets which may be considered as oligopolies, where a small number of different producers compete against each other to maximize profits. The initial work in the economics literature on oligopolistic competition was by Cournot [2] in 1838, who introduced the idea of competition through production output. This work was re-envisioned by Bertrand [1] in 1883, who framed competitions in terms of prices. More recently, however, energy markets have been modeled through dynamic, as opposed to the static games that Cournot and Bertrand considered. For more modern interpretations of oligopolistic competition, we recommend Friedman [4], Vives [10], or for dynamic models, Dockner *et al.* [3].

Additionally, energy markets often have two distinct types of players: some, like oil, depend on a fixed reserve, while others, like renewable players such as wind and solar, have effectively infinite or inexhaustible resources. Study of the impact of exhaustibility of resources was initiated by Hotelling [6], within a monopoly. In [5], energy production is modeled as a dynamic Cournot game, where certain players depend on stock remaining to produce, while others have a higher extraction cost but can produce indefinitely.

Previous work to model the behavior we consider in energy market production has made certain assumptions that we relax here. For example, [8] assumes that there is no R&D or exploration, while [9] assumes that the only cost oil producers incur are their research costs. We expand such work in two ways. First, we relax the assumption that the oil producer has zero cost of extraction. Then we allow for evolving costs of energy production: costs over time are realistically not constant. In particular, as stock begins to run out, costs for oil producers often increase (deeper drilling, more expensive extraction technology required), whereas costs for green energy often decrease due to external investment and financing that leads to more

*ORFE Department, Princeton University, Sherrerd Hall, Princeton NJ 08544; *anirudhd@princeton.edu*. Work partially supported by NSF grant DMS-0739195.

†ORFE Department, Princeton University, Sherrerd Hall, Princeton NJ 08544; *sircar@princeton.edu*. Work partially supported by NSF grant DMS-1211906.

efficient technology. For instance "the price of solar panels has fallen more than 75 percent just since 2008" [7].

The outline of this paper is as follows. In Section 2, we analytically solve a special case of constant costs, which requires no numerical calculations. We define the notion of "blockading" and find the Markov perfect equilibrium for the case where all costs are positive, yet constant. In Section 3, we provide a partially analytical solution for the case without exploration. Then, in Section 4, we solve the full model which incorporates exploration. This extends the stochastic model in [9] to allow for varying costs. Finally, in Section 5, we demonstrate how our model may be applied to the setting of energy policy. We provide two examples, the latter of which compares taxing finite resource producers and subsidizing green energy. We then conclude and suggest methods by which this work may be extended.

2 Dynamic Game Model with Constant Costs

We begin with the case of constant production costs and no exploration where we can establish analytical results.

2.1 Preliminary Notation

We consider an N-player oligopoly game that models energy markets. The first $k < N$ of the energy players have exhaustible stocks (or reserves) $\{x_1(t), \ldots, x_k(t)\}$. They are active whenever their stock $x_i(t) > 0$ and are "eliminated" or exit production when their reserve $x_i(t)$ hits 0. The remaining players can be considered "renewable," and have infinite (or inexhaustible) reserves. Each of the N players has a marginal cost function s_i that is associated with the ith player (so it costs the ith player s_i units of money to produce one unit of output), and they compete through Cournot (quantity-setting) competition. We will see that all players will participate provided that their cost is low enough compared to the other active players.

The case where $k = N \geq 2$ (that is, there are only exhaustible players who compete against each other), both with zero extraction costs for all time, was examined in [5], and involved the analysis of a coupled system of nonlinear partial differential equations (PDEs) which are typically difficult to solve even numerically. We will thus consider the case where $k = 1$, so there is only one exhaustible player. Since the motivation behind this paper is to provide a framework within which energy policy comparisons can be made, we will constrain ourselves to this case.

Notationally, when we consider an N-player game, we will let player 0 be the exhaustible player. Given the evident link to energy markets, we will interchangeably refer to player 0 as the "exhaustible" player, the "stock" player, and the "oil" player. Then, players $1, \ldots, N-1$ shall be our renewable players. Since there is only one exhaustible player in our consideration, we denote by $x(t) = x_0(t)$, the remaining stock of player 0. We let the costs also vary in the amount $x(t)$ remaining: $s_i(x)$. In general, costs evolve as stock begins to run out and this generalization allows us to encapsulate the meaning of such changes. Further, to have a meaningful model that can be applied to policy, we must be able to capture evolving costs, as most elements of energy policy affect the perceived price of various players in the market. The case where $s_0(x) = 0$ and $s_i(x) = s_i$, that is, the oil player has zero extraction costs and the renewable players have constant cost has been evaluated analytically in [8].

As assumed there, we will have a representative market for the energy that is produced by these players. In particular, we assume that the utility function represented by market demand makes no differentiation between the energy produced by all N players; that is, the individual consumer cares only about prices and quantities produced, but not the actual source fuel (or technology) itself. We can, in general, assume that the market demand function follows a constant prudence price curve; that is, if we let Q denote the total market output, then the inverse demand curve is

$$P(Q) = \begin{cases} \dfrac{1 - Q^{1-\rho}}{1-\rho} & \text{if } \rho \neq 1 \\ -\log Q & \text{if } \rho = 1, \end{cases} \tag{1}$$

where we restrict $0 \leq Q \leq 1$. Here, we will consider the case where $\rho = 0$, so the inverse market demand curve reduces to $P(Q) = 1 - Q$.

We first review the static (one-period) Cournot game with constant costs.

2.2 Static Game

In the static (or one-period or stage) Cournot game with N players who have constant costs $s_0, \dots, s_{N-1}$, each player i chooses a production quantity $q_i \geq 0$ to maximize revenue, that is price minus cost multiplied by quantity produced:

$$\pi(q_i, q_{-i}) = q_i \left(1 - q_i - \sum_{j=0, j\neq i}^{N-1} q_j - s_i\right), \quad \text{where} \quad q_{-i} = (q_0, \cdots, q_{i-1}, q_{i+1}, \cdots, q_{N-1}).$$

Here there is no distinction between player 0 and the others as there is no dynamic component whereby things change over time.

We say that $(q_0^*, \cdots, q_{N-1}^*)$ is a Nash equilibrium if $\pi(q_i^*, q_{-i}^*) \geq \pi(q_i, q_{-i}^*)$ for any $q_i \geq 0$ and all $i = 0, 1, \cdots, N-1$. That is q_i^* maximizes revenue for player i when all the other players play their Nash equilibrium strategies. For this problem, the Nash equilibrium is as follows.

Proposition 2.1. *In an N-player static game with constant costs $0 \leq s_0 < s_1 < \cdots < s_{N-1} < 1$, we let*

$$P_i = \frac{1}{i+1}\left(1 + \sum_{j=0}^{i-1} s_j\right), \qquad i = 1, \dots, N,$$

and $\bar{P} = \min\{P_i \mid i = 1, \dots, N\}$. The number of active players is given by

$$n = \min\{i \mid P_i = \bar{P}, i = 1, \dots, N\}.$$

Then players $\{0, \dots, n-1\}$ are active, and players $\{n, \dots, N-1\}$ do not produce. The unique Nash equilibrium is given by

$$q_i^* = \frac{1}{n+1}\left(1 - ns_i + \sum_{j=0; j\neq i}^{n-1} s_j\right), \qquad i \in \{0, \dots, n-1\},$$

and $q_i^* = 0$ for $i \in \{n, \dots, N-1\}$.

Proof. See [5, Proposition 2.9]. □

Remark. *We chose to have the costs to be strictly increasing; if costs of players are allowed to be the same, then the blockading points for the dynamic game (defined in section 2.3) might coincide. For purposes of consistency, we use the same assumption here. The result in Proposition 2.1 would hold even if the sequence of costs is only weakly increasing and the calculations presented in section 2.3 and beyond can be reproduced for weakly increasing costs. For simplicity and to highlight the focus of our results, we use a strictly increasing costs assumption.*

When the oil producer is inactive (as will occur in the dynamic game when his reserves are exhausted) and only players $1 \leq i \leq N-1$ are in the market, the static game Nash equilibrium is

$$q_i^* = \begin{cases} \frac{1}{n+1}\left(1 - ns_i + \sum_{j=1; j\neq i}^{n} s_j\right), & i \in \{1, \dots, n\} \\ 0 & i \in \{n+1, \dots, N-1\}, \end{cases} \tag{2}$$

where $n = \min\{i \mid P_i = \bar{P}, i = 1, \dots, N-1\}$ and

$$P_i = \frac{1}{i+1}\left(1 + \sum_{j=1}^{i} s_j\right), \quad i = 1, \dots, N-1, \qquad \bar{P} = \min\{P_i \mid i = 1, \dots, N-1\}. \tag{3}$$

We denote by $S^{(k)} = \sum_{i=1}^{k} s_i$, the cumulative cost of the first k renewable players, and we also define

$$\rho_i = \frac{1 + S^{(i-1)}}{i}, \qquad i \in 2, \dots, N, \tag{4}$$

with $\rho_1 = 1$. Further, we make the following assumption.

Assumption 2.2. *We assume that $s_{N-1} < \rho_{N-1}$.*

Remark. *This assumption is necessary to guarantee that when the oil producer is not present, the costs of the renewable producers are low enough that all participate in equilibrium. This follows from the fact that $P_{i-1} = \rho_i$, where P_i is given in* (3)*. Then a calculation shows that*

$$\rho_N < \rho_{N-1} \quad \Longleftrightarrow \quad s_{N-1} < \rho_{N-1},$$

and so under Assumption 2.2, $P_{N-1} < P_{N-2}$. Therefore $\bar{P} = P_{N-1}$ and $n = N-1$.

We also have:

Lemma 2.3. *Under Assumption 2.2, we have that $s_j < \rho_j$ for all $j \in 1, \dots, N-1$.*

Proof. For player $N-2$:

$$s_{N-2} - \rho_{N-2} = \frac{(N-1)s_{N-2} - (1 + S^{(N-2)})}{N-2} = \frac{N-1}{N-2}(s_{N-2} - \rho_{N-1}) \leq \frac{N-1}{N-2}(s_{N-1} - \rho_{N-1}) < 0,$$

where in the second to last step, we used $s_j < s_{j+1}$ for all $j \in \{1, \dots, N-2\}$. The implication $s_j < \rho_j$ can be shown inductively from here. □

Lemma 2.4. *When s_i is increasing in i and Assumption 2.2 holds, we have that ρ_i is decreasing in i.*

Proof. This follows from

$$\rho_i - \rho_{i-1} = \frac{1 + S^{(i-1)}}{i} - \frac{1 + S^{(i-2)}}{i-1} = \frac{i s_{i-1} - S^{(i-1)} - 1}{i(i-1)} < 0,$$

since

$$s_{i-1} < \rho_{i-1} \implies s_{i-1} < \frac{1 + S^{(i-1)}}{i} \implies i s_{i-1} < 1 + S^{(i-1)}.$$

□

Next, we introduce the dynamic N player Cournot game, where player 0 is our "oil" producer with exhaustible resources, and the remaining players $1, \dots, N-1$ are renewable energy producers.

2.3 Dynamic Game

Energy is produced from different sources by players $0, \ldots, N-1$, who have constant costs of production $(s_0, \ldots, s_{N-1})$. The case where $s_0 = 0$ has been solved analytically in [8]. The general case where s_0 is allowed to be a constant that is in $[0,1)$ can also be solved completely analytically and will be presented below. We will study the game when costs vary as time goes on and oil runs out in Section 3.

Player 0 is our oil producer who plays when his stock $x(t) > 0$ and has an extraction cost $s_0 > 0$. The stock $x(t)$ evolves according to the flow equation

$$\frac{dx(t)}{dt} = -q_0(x(t))\mathbb{1}_{\{x(t)>0\}},$$

where q_0 is the extraction strategy of player 0, and his initial reserve is $x(0)$. Players $i = 1, \ldots, N-1$ are renewable producers and have a fixed marginal cost of production $s_i > 0$. We order the players such that $s_1 \leq s_2 \leq \cdots \leq s_{N-1}$ and further require that Assumption 2.2 holds. They produce energy at the rates q_i.

Each player has an infinite time horizon objective value function that is determined by future profits discounted at rate $r > 0$. In particular, the Nash equilibrium $(q_0^*(\cdot), q_1^*(\cdot), \ldots, q_{N-1}^*(\cdot))$ are given by the arguments of the following suprema:

$$v(x) = \sup_{q_0} \int_0^\tau e^{-rt} q_0(x(t)) \left(1 - q_0(x(t)) - \sum_{j=1}^{N-1} q_j^*(x(t)) - s_0\right) dt \tag{5}$$

$$w_i(x) = \sup_{q_i} \int_0^\tau e^{-rt} q_i(x(t)) \left(1 - q_0^*(x(t)) - \sum_{j=1, j\neq i}^{N-1} q_j^*(x(t)) - q_i(x(t)) - s_i\right) dt + \frac{1}{r} e^{-r\tau} G_i, \tag{6}$$

where $i = 1, \cdots, N-1$. Here τ is the exhaustion time $\tau = \inf\{t \mid x(t) = 0\}$, and G_i is the equilibrium profit of player i in the static game with only players $1, \ldots, N-1$ (who all participate under Assumption 2.2):

$$G_i = q_i^* \left(1 - \sum_{j=1}^{N-1} q_j^* - s_i\right) = \left(\frac{1}{N}(1 - Ns_i + S^{(N-1)})\right)^2,$$

where we have used (2) for q_i^* with $n = N-1$. The admissible Markov strategies $q_i(x)$ are such that $q_i \geq 0$ and the $q_i(x)$ are Lipschitz continuous.

2.4 Blockading of Renewable Producers

Under some conditions, some subset of the players are blockaded from production because their costs are too high to generate a profit given the competition from players with lower costs. In the context of the renewable players $i \in \{1, \ldots, N-1\}$, this will be denoted by a point x_b^i such that

$$x_b^i = \inf\{x > 0 : q_i^*(x) = 0\}.$$

That is, for all points $x < x_b^i$, player i produces and participates in the game, but for $x \geq x_b^i$, the supply of cheap oil makes the market energy price too low for him to participate. We define this to be the blockading point for player i. In the case where ${q_i}^*(x) > 0$ for all x, we set $x_b^i = \infty$. In such an instance, we say that player i is never blockaded. We also identify $x_b^N = 0$ and $x_b^0 = \infty$.

Additionally, unlike the $s_0 = 0$ case, since s_0 is now positive, there is also the chance that the oil player may be blocked from playing if his extraction cost s_0 is too high compared to the renewable players. In other words, it may be possible that the oil player is inactive even if $x(t) > 0$. We will defer this case to Section 2.5. The intuition behind the case where s_0 is sufficiently low so that the oil player plays whenever he has remaining stock is presented in [8] and reproduced in Figure 1, where the blocking times t_b^i are defined by $x(t_b^i) = x_b^i$.

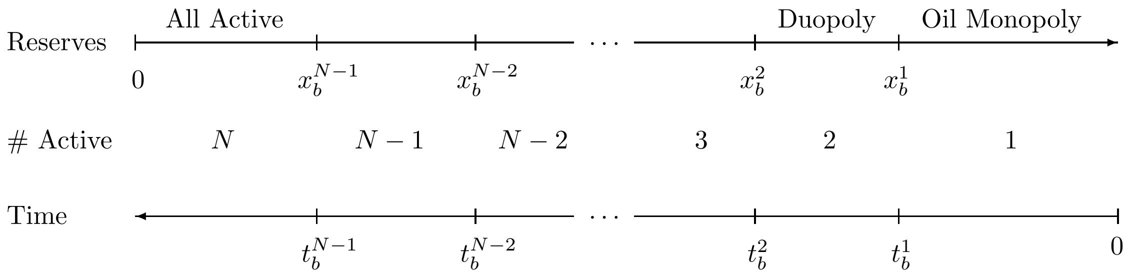

Figure 1: *Blockading intuition*

In the region $x \in [x_b^n, x_b^{n-1})$, there are $n \leq N$ active players *including the oil player*. Shifting the variable x, we write $v(x_b^n + x) = v^{(n)}(x)$ for $x \in (0, x_b^{n-1} - x_b^n)$. A straightforward extension of [8, Proposition 5.3] to incorporate the cost s_0 shows that $v^{(n)}$ solves the Hamilton-Jacobi-Bellman (HJB) equation

$$rv^{(n)} = \frac{1}{(n+1)^2}\left(1 - n(v^{(n)'} + s_0) + \sum_{j=1}^{n-1} s_j\right)^2, \qquad n = 1, \dots, N. \tag{7}$$

These are solved with the boundary conditions $v^{(N)} = 0$ and $v^{(n)}(0) = v^{(n+1)}(x_b^n - x_b^{n+1})$ for continuity of the value function $v(x)$. The equilibrium strategies in the region $x \in [x_b^n, x_b^{n-1})$ are given by the formula in Proposition 2.1 with the replacement $s_0 \mapsto s_0 + v^{(n)'}(x - x_b^n)$:

$$q_0^*(x) = \frac{1}{n+1}\left(1 - n(s_0 + v^{(n)'}(x - x_b^n)) + \sum_{j=1}^{n-1} s_j,\right) \tag{8}$$

$$q_i^*(x) = \frac{1}{n+1}\left(1 - ns_i + (s_0 + v^{(n)'}(x - x_b^n)) + \sum_{j=1; j\neq i}^{n-1} s_j\right), \quad i = 1, \dots, n-1. \tag{9}$$

Further, since at any given point x, $q_j^*(x) > q_{j+1}^*(x)$ (that is, higher cost renewable players produce less), and since costs are constant in x, we have that $x_b^j > x_b^{j+1}$; so, it takes more oil to run out before player j enters compared to player $j+1$, as indicated in Figure 1. To solve (7) analytically, the following Lemma shall be useful:

Lemma 2.5. *The solution to the ODE*

$$(\alpha - v')^2 = \kappa v, \tag{10}$$

where $v_0 = v(0) \geq 0$ *and* $\alpha, \kappa > 0$ *is*

$$v(x) = \frac{\alpha^2}{\kappa}\left(1 + \mathbf{W}\left(\theta(x)\right)\right)^2,$$

where $\mathbf{W}(\cdot)$ *is the Lambert-W function, satisfying* $Z = \mathbf{W}(Z)\, e^{\mathbf{W}(Z)}$ *restricted to* $Z \geq -e^{-1}$*. Further,* $\theta(x) = \beta e^{\beta} e^{-\kappa x/(2\alpha)}$ *and* $\beta = -1 + \dfrac{\kappa v_0}{\alpha}$.

Proof. The result follows by direct evaluation, which is explained in detail in [8]. ∎

Writing (7) in the form (10), we see that $\alpha = \dfrac{1 - ns_0 + S^{(n-1)}}{n+1} > 0$ by Assumption 2.2. Consequently, the closed form solution to (7) is

$$v^{(n)}(x) = \frac{1}{r}\left(\frac{1 - ns_0 + S^{(n-1)}}{n+1}\right)^2 (1 + \mathbf{W}(\theta(x)))^2, \tag{11}$$

where $\theta(x) = \beta e^{\beta} e^{-\kappa x/(2\alpha)}$ with $\beta = -1 + \dfrac{\kappa v^{(n)}(0)}{\alpha}$ and $\kappa = r$. Taking the derivative, we have

$$v^{(n)'}(x) = -\frac{1}{n}\mathbf{W}(\theta(x))(1 - ns_0 + S^{(n-1)}) = -\mathbf{W}(\theta(x))(\rho_n - s_0), \tag{12}$$

where ρ_n was defined in (4). Assumption 2.2 and Lemma 2.3 guarantee that $v^{(n)'}(x) > 0$.

We define:

$$\hat{\rho}_n = \frac{1 + S^{(n-1)} - ns_0}{n} = \rho_n - s_0 \tag{13}$$

for $n = 2, \ldots, N$. When $\hat{\rho}_n < 0$ for all n, the oil producer does not play in our game, and we have a perpetually repeated static game with $N-1$ players, meaning v and w_i ($i = 1, \ldots, N-1$) are given by (5)-(6) with $\tau = 0$.

It is straightforward to show from the concavity of $v(x)$ that whenever the stock for the oil producer is greater than the blockading point x_b^n, player n produces nothing. The blockading point is thus a threshold, below which the player enters the market, and above which, the player does not enter the market.

2.5 Blockading of the Oil Producer

When we allow s_0 to be greater than zero, we cannot assume as in [8] that the oil producer will always participate should $x(t) > 0$. Indeed, should s_0 be sufficiently high, it is possible that the oil producer does not play because his extraction cost is too high.

Heuristically, we can view the "cost" for the oil producer as $s_0 + v'(x)$ at any given stock $x(t)$ (the sum of extraction and shadow costs). This is bounded below by s_0, so if s_0 is sufficiently high, it may be too expensive for him to produce and he may be forced to exit the market. This shall be referred to as blockading of the oil player.

We define a blockading point for our oil producer as the point

$$x_b^* = \sup\{x > 0 : q_0^*(x) = 0\}.$$

For all $x > x_b^*$, the oil producer will produce, whereas for all $x < x_b^*$, the oil producer does not participate. If there is no such $x > 0$ such that $q_0^*(x) = 0$, then we say the oil producer is never blockaded and let $x_b^* = 0$.

We will first demonstrate that provided that the oil producer always plays when $x(t) > 0$ provided s_0 is small enough.

Proposition 2.6. *If $s_0 < \rho_n$ for all $n \in \{2, \ldots, N\}$, then the oil producer is never blockaded.*

By Lemma 2.4, this condition is equivalent to $s_0 < \rho_N$, as ρ_n is decreasing in n.

Proof. In the interval $[0, x_b^{N-1})$, the candidate value function is $v(x) = v^{(N)}(x)$ which solves the ODE (7) with boundary condition $v^{(N)}(0) = 0$, and the corresponding equilibrium oil production is

$$q_0^*(x) = \frac{N}{N+1}\left(\rho_N - (s_0 + v^{(N)'}(x))\right),$$

following from formula (8). Then $q_0^*(x) \geq 0$ as long as $v^{(N)'}(x) \leq \rho_N - s_0$, where the bound is positive by hypothesis. It follows from the ODE (7) that $v^{(N)'}(0) = \rho_N - s_0$, and it is easily verified from the formula

(11) for $v^{(N)}$ that $v^{(N)}(x)$ is strictly concave and so $v^{(N)'}(x) \leq \rho_N - s_0$ for all $x \in [0, x_b^{N-1})$. Therefore the candidate solution in which player 0 is not blockaded hold in the first interval as the unique Markov perfect equilibrium. A similar argument hold in the other intervals (in which the shadow cost v' of the oil producer is even lower). □

This is consistent for the $s_0 = 0$ limiting case, since then it is always true that $s_0 \leq \rho_N$ and thus there are no blockading points for player 0. In this case, we can evaluate the blockading points explicitly. We first define

$$\delta_n = (n+1)s_n - (1 + s_0 + S^{(n-1)}).$$

Then, the following proposition holds:

Proposition 2.7. *Provided that $s_0 < \rho_N$, the blockading point for the kth player is finite if $\delta_k > 0$. Let $i = \min\{k : \delta_k > 0\}$, or the lowest cost player who is blockaded at some point. Players $\{i, \dots, N-1\}$ are blockaded to the right of their blockading points which are determined recursively by the equations:*

$$x_b^{N-1} = \frac{1}{\mu_N}\left[-1 + \frac{\delta_{N-1}}{\hat{\rho}_N} - \log\left(\frac{\delta_{N-1}}{\hat{\rho}_N}\right)\right] \tag{14}$$

$$x_b^{n-1} = x_b^n + \frac{1}{\mu_n}\left[\log\left(\frac{\delta_n}{\delta_{n-1}}\right) - \frac{(n+1)(s_n - s_{n-1})}{\hat{\rho}_n}\right], \tag{15}$$

where, for $n \in \{i, \dots, N-2\}$,

$$\mu_n = \frac{2r}{\hat{\rho}_n}\left(\frac{1}{2} + \frac{1}{2n}\right)^2,$$

and $\hat{\rho}_n$ was defined in (13).

Proof. To find x_b^{N-1}, we look for a solution of $q^*_{N-1}(x_b^{N-1}) = 0$, and thus, from (9), we require that

$$v^{(N)'}(x_b^{N-1}) = Ns_{N-1} - (1 + s_0 + S^{(N-2)}) = \delta_{N-1}.$$

Since, from (11), $v^{(N)'} = -\hat{\rho}_N \mathbf{W}(\theta(x))$, we have that

$$\mathbf{W}\left(\theta(x_b^{N-1})\right) = -\frac{\delta_{N-1}}{\hat{\rho}_N}.$$

This equation only has a positive solution for x_b^{N-1} for $\delta_{N-1} > 0$, which is given by (14). The recursion formula (15) follows from shifting the axes left by x_b^{N-1} and proceeding with similar analysis to that for the $s_0 = 0$ case in the proofs of Propositions 5.2 and 5.3 in [8]. □

The following proposition solves the case where $s_0 \geq \rho_N$:

Proposition 2.8. *If $s_0 \geq \rho_N$, then the Markov perfect equilibrium is that the oil producer does not play and the renewables play an $N-1$ player Cournot game as given by* (2) *with $n = N-1$:*

$$q_i^*(x(t)) = \frac{1}{N}\left(1 - Ns_i + S^{(N-1)}\right), \qquad i = 1, \dots, N-1. \tag{16}$$

Proof. In the candidate $N-1$ player equilibrium (16), the total output is

$$Q = \sum_{i=1}^{N-1} q_i^* = 1 - \rho_N,$$

and so the market price is $P = 1 - Q = \rho_N$. Since $s_0 \geq \rho_N$, player 0 will not enter the market and his best response is $q_0^* = 0$. Therefore (16) gives the Nash equilibrium in this case. □

Finally, in Figure 2, we plot $v(x)$ for two constant costs: $s_0 = 0.2$ and $s_0 = 0.4$. For the lower cost, note that the value function is strictly higher, as expected, because profits are greater. On the right, however, we note that the oil producer lasts longer in time t with a higher cost than with a lower cost. The additional cost produces an amplified incentive to save until tomorrow and hence the oil producer lasts longer. This is reflected in the market price as well, as there is a greater market price with a higher cost, but the jump in market price is not as significant when oil runs out. In other words, the higher fixed cost of extraction results in price stability over time at the cost of higher market prices even when the oil player produces.

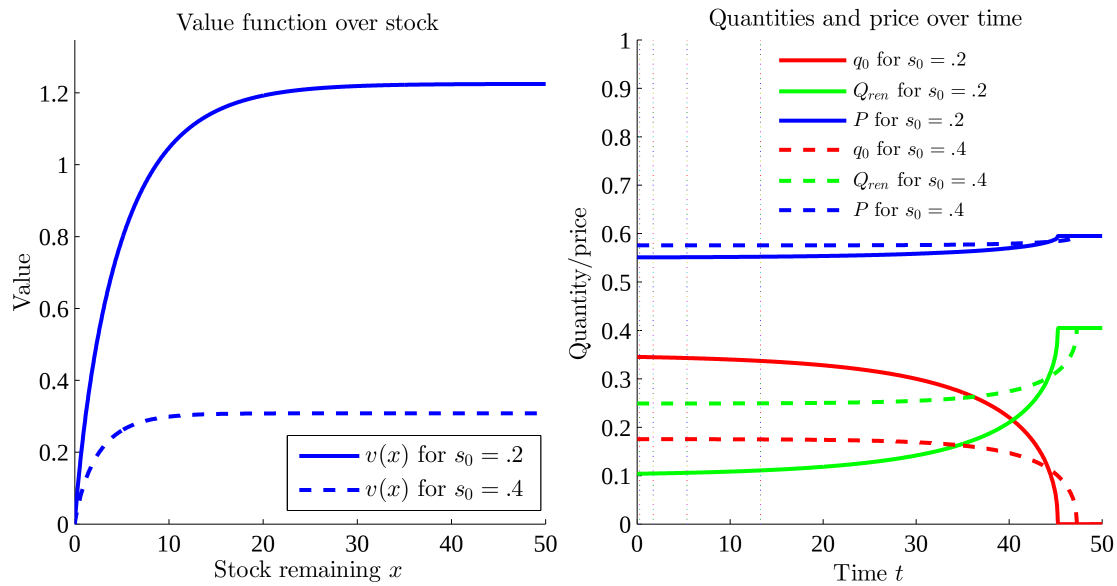

Figure 2: *Constant cost dynamic game based on analytical solution for* $s_0 = 0.2$ *and* $s_0 = 0.4$ *for* $s_1 = 0.51, s_2 = 0.52, \dots, s_9 = 0.59$. *Notationally,* $Q_{ren} = \sum_{i=1}^{9} q_i$ *(total renewable production). In this case* $\rho_N = 0.595$*, and so, as* $s_0 < 0.595$*, the oil producer always produces.*

3 Varying Costs

Now that we have considered the one subcase in which we can derive a fully analytical Markov strategy, we consider the general case, where costs vary for each of the players. We associate to each player i in our N player game a cost function $s_i(x)$, where x is the remaining stock of player 0, the only exhaustible player. To understand how this generalizes the earlier result, we first consider a base case, where $s_1, \dots, s_{N-1}$ are constant.

In this section, we do not yet allow discovery of new reserves. Since disabling discovery of new reserves enables us to have partially analytical solutions, we present these calculations here and resort to a numerical approach to the full discovery problem in Section 4.

3.1 Holding renewable costs constant

The array of costs for this base case is such that $s_i(x) = s_i \in [0, 1]$ for each $i \in \{1, \dots, N-1\}$, while $s_0(x)$ is a decreasing function. The following assumptions are made:

Assumption 3.1. *We retain the assumption that* $s_i < \rho_i$ *for the renewable players* $i = 1, \dots, N-1$*. This implies that should the oil producer exit, the appropriate Cournot solution among the renewable players, given in formula* (2)*, entails that all players are active.*

Assumption 3.2. *We assume that the cost function of the oil producer* $s_0(x) \in \mathcal{C}^1$ *and* $s_0'(x) < 0$*, so cost increases as remaining reserves are depleted. This assumption can be justified, as oil producers typically delay extraction from their most expensive ores, so as reserves begin to run out, costs are increased.*

When there are $n-1$ active renewable players, and hence n total players, should the oil producer play, the Hamilton-Jacobi equation for his value function $v(x)$ is

$$rv = \frac{1}{(n+1)^2}\left(1 - n(v'(x) + s_0(x)) + S^{(n-1)}\right)^2, \qquad x \in [x_b^n, x_b^{n-1}), \tag{17}$$

analogous to (7) with the constant s_0 replaced by $s_0(x)$. The oil producer's strategy is

$$q_0^*(x) = \frac{1}{(n+1)}\left(1 - n(v'(x) + s_0(x)) + S^{(n-1)}\right) = \frac{n}{(n+1)}\left(\rho_n - (v'(x) + s_0(x))\right), \tag{18}$$

where ρ_n was defined in (4). We will also see that, while the effective cost for the oil player is always decreasing in x, even when Assumption 3.2 does not hold, the shadow cost $v'(x)$ may be increasing or decreasing in x, depending on the properties of $s_0(x)$.

Proposition 3.3. *The sum* $v'(x) + s_0(x)$ *is strictly decreasing, but* $v'(x)$ *is not necessarily decreasing. In particular, if we define*

$$T(x,n) = -\frac{(n+1)^2 r v'(x)}{2n q_0^*(x)},$$

then if $|s_0'(x)| \geq |T(x,n)|$*, when there are* n *players total including the oil producer, then* $v(x)$ *is convex (*$v'(x)$ *increasing), while* $|s_0'(x)| \leq |T(x,n)|$ *implies that* $v(x)$ *is concave, with* $v'(x)$ *decreasing.*

Proof. The result follows immediately from taking the derivative of equation (17) with respect to x:

$$v''(x) + s_0'(x) = -T(x,n),$$

so if $|s_0'(x)| \geq |T(x,n)|$, then $v''(x) > 0$ and hence $v'(x)$ is increasing, and similarly for the other case. □

Note that $q_0^*(x)$ is increasing in x since the effective cost $v'(x) + s_0(x)$ is decreasing; further, as $x \to 0$, $q_0^*(x) \to 0$, but v' is bounded above ($v'(0) + s_0(0) = \rho_N$). Assuming that $s_0(x)$ is bounded above as well, then in a neighborhood around 0, $v(x)$ is concave. If $s_0(x)$ decreases sufficiently quickly in some neighborhood of a point $x > 0$, then $v(x)$ becomes convex. Figure 3 illustrates with two different cost functions $s_0(x)$.

We now let $x_b^* = \sup\{x > 0 : q_0^*(x) = 0\}$. Then, it is evident that since $v'(x) + s_0(x)$ is decreasing that for all $x > x_b^*$, the oil producer participates and for all $x < x_b^*$, the oil producer does not participate. In the region the oil producer does not participate, the assumption $s_n < \rho_n$ for renewable players implies the quantities and market price is determined by an $N-1$ player Cournot static game among the renewable players. The following establishes that the condition upon which the oil producer plays is solely dependent on extracting costs and not remaining stock:

Proposition 3.4. *The blockading point* x_b^* *for player* 0 *is given by*

$$x_b^* = \sup\{x > 0 : q_0^*(x) = 0\} = \inf\{x > 0 : s_0(x) \leq \rho_N\}.$$

Proof. Let $I = \inf\{x > 0 : s_0(x) \leq \rho_N\}$. It suffices to show that when $x < I$, the oil producer does not play and when $x > I$, the oil producer does. When $x < I$, we have that $s_0(x) > \rho_N$; assume the oil producer plays. Prior to this assumption, the equilibrium in this region would have been an $N-1$ player Markov game, so for the assumption to hold, the quantity produced in an N player game once the oil producer enters should be positive. That is, from (18) with $n = N$, this requires $v'(x) + s_0(x) < \rho_N$, but this forces $v' < 0$, a contradiction. So, the oil producer does not play on $x < I$.

Similarly, the oil producer must play when $x > I$. Assume he doesn't; then it must not be profitable for him to play. Hence, if $q_0 > 0$ when $x > I$, the oil player plays. This is evident, since if $x > I$, $s_0(x) \leq \rho_N$, so $v'(I) \leq 0$ and the Hamilton-Jacobi equation is well-defined and evolves to give positive q_0 and hence positive profit. Since $s(x)$ is decreasing and continuous, the result follows. □ □

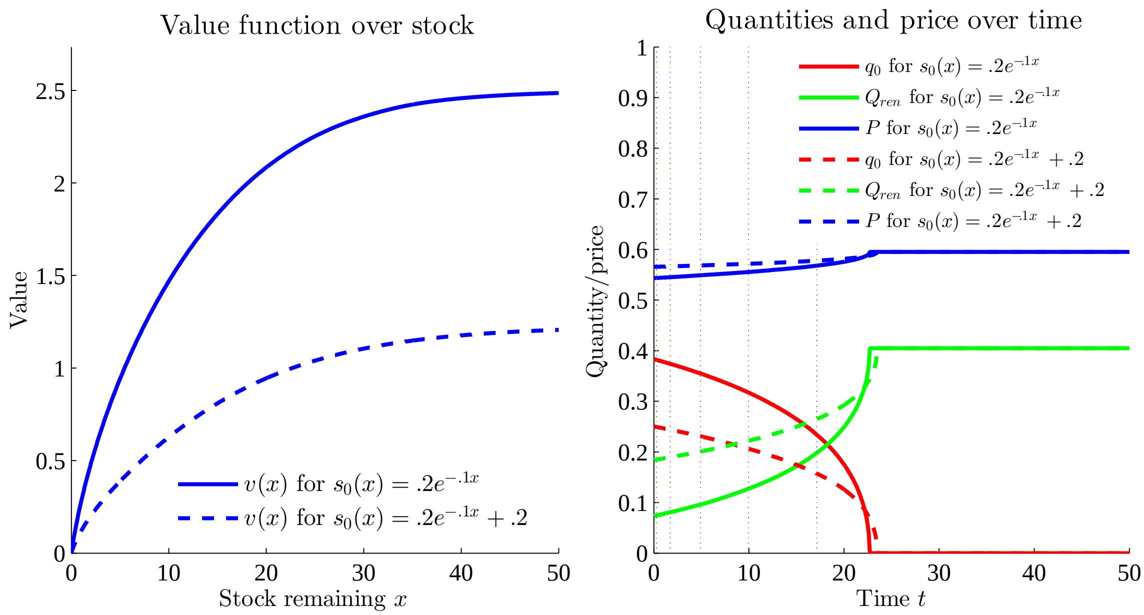

Figure 3: *Compares value functions as well as quantity and price evolution over time for 9 renewable players with costs 0.51, ..., 0.59, while we adjust the cost of extraction for the oil producer from* $s_0(x) = .2e^{-.1x}$ *(solid) to* $s_0(x) = .2e^{-.1x} + .2$ *(dashed).*

Remark: The first point coming from the left at which $s(x) = \rho_N$ (and also the only point since $s(x)$ is decreasing) is the threshold point at which the oil producer enters. The mechanism here is not that the oil producer's shadow cost is too high. Rather, the oil producer's cost of extraction becomes too prohibitive in comparison to the other alternatives. To the left of this point, since the assumption $s_n < \rho_n$ for the renewable players binds, only the renewable players remain and they play indefinitely in a static game. To the right of this point, blockading for the renewable players once again can occur; that is, there may be associated threshold values for each of the renewable players to the left of which they do play and to the right of which they do not. In particular, the following proposition highlights when the renewable players are blockaded.

3.2 Numerical examples with renewable costs held constant

Figures 3 and 4 depict numerically evaluated solutions for the case of decreasing $s_0(x)$. In particular, in Figure 3, we note that if $s_0'(x) < s_0(x)$ for all x, then the $v(x)$ that corresponds to $s_0'(x)$ is less than or equal to than the $v(x)$ for $s_0(x)$. In particular, we note that similar to earlier, increasing $s_0(x)$ by a fixed cost leads to the oil player staying in for a longer period of time. In Figure 3, the vertical dotted lines correspond to blockading points x_b^i for the oil producers for $s_0(x) = .2e^{-.1x}$. That is, to the left of these lines, additional players enter since the reserves have depleted sufficiently. When we increase $s_0(x)$, the blockading points are shifted to the left, since the relative cost of oil is higher, so the threshold for the renewables entering is lower. Hence, when simulating such a game over time, the renewable players come in earlier.

Figure 4 depicts the case where for some $x < x^*$, we have that $s_0(x) > \rho_N$. In particular, note that until $x = 11.6$, the oil producer does not produce. This is reflected in the graph on the left. At the time when the oil producer leaves, there are still reserves left that have yet to be drilled (specifically, 11.6 units). Further, Figure 4 depicts a situation that is consistent with that explained in Proposition 8. That is, $s_0'(x)$ is greater than the threshold value $T(x, n)$ for a subset of $x \in [0, 50]$, resulting in a shift from concavity to convexity. This is due to the high Lipschitz constant that bounds the value of $s_0'(x)$ from the above and continuity, which ensures concavity in a neighborhood where the derivative is maximized.

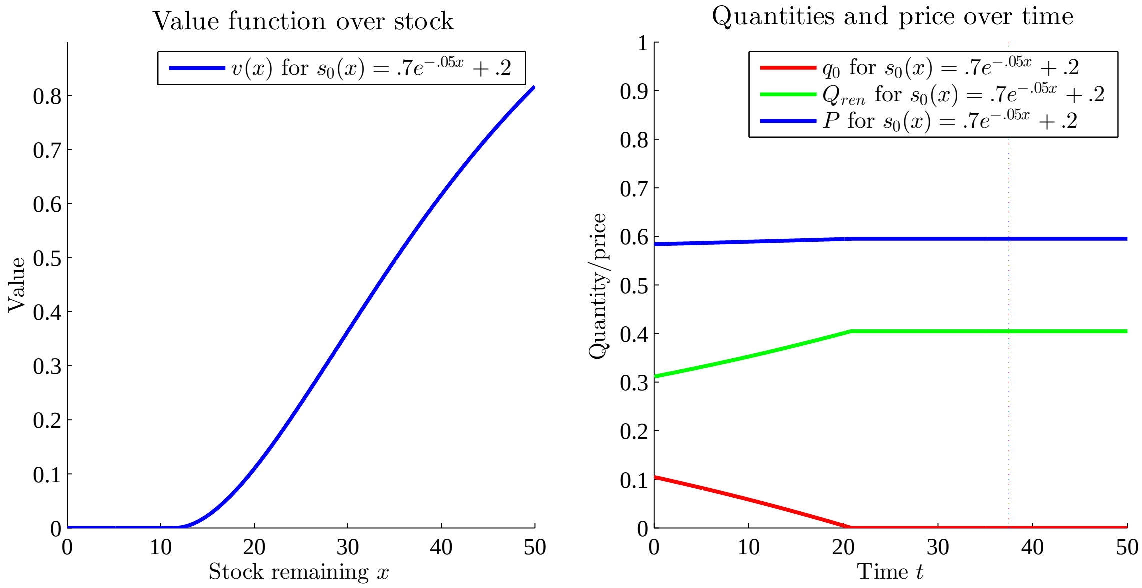

Figure 4: *Demonstrates evolution of game when $s_0(x)$ is high enough for small enough x so that the oil player is effectively blockaded for low x. Notationally, $Q_{ren} = \sum_{i=1}^{9} q_i$. Also demonstrates case where Lipschitz constant of $s_0(x)$ is so large to cause concavity of $v(x)$ on certain regions of x. The dotted vertical line reflects blockading of the 9th renewable player.*

3.3 Varying renewable costs: Analytic Results

The above problem can be solved partially analytically for N players, all of whom have varying costs that obey the following assumption:

Assumption 3.5. *In an N player game, let player 0 be the exhaustible player and players $1, \dots, N-1$ be renewable players. Let $s_i(x)$ be the cost of player i. We assume that $s_0'(x) < 0$ and $s_i'(x) > 0$ for all $i \in \{1, \dots, N-1\}$.*

This assumption is justified because, as oil tends to deplete, governments tend to subsidize green energy, thus reducing costs for the renewable players as x approaches 0.

The following assumption is made so that players are not forced out of the game due solely to their absolute extraction cost, but rather only choose not to play because of relative extraction cost:

Assumption 3.6. *For all $i \in \{0, 1, \dots, N-1\}$, we require $s_i(x) \in [0,1]$ for all $x \geq 0$.*

We now let $N = 2$ for a couple of reasons: first, the structure of the game is not substantively different with $N > 2$, and second, $N = 2$ simplifies numerical computation significantly.

Proposition 3.7. *We define*

$$\rho_1(x) = \frac{1 + s_1(x)}{2}.$$

The dominant strategy for the oil producer is not to produce when $s_0(x) > \rho_1(x)$.

Proof. Assume that the oil producer will want to produce; then there must be some neighborhood around the point x by continuity of $s_0(x)$ and $s_1(x)$ such that both players will participate with positive quantities. However, in such a case,

$$q_0^*(x) = \frac{1}{3}\left(1 - 2v' - 2s_0(x) + s_1(x)\right) > 0,$$

implying that $v' < \rho_1(x) - s_1(x) < 0$, a contradiction to the obvious fact that giving a player extra resources does not make him worse off. □

Remark. *When $s_1(x)$ is held constant, then $\rho_1(x)$ is also constant, so this result is identical with that of the fixed cost case discussed previously.*

In particular, since $s_1(x)$ is decreasing, so must $\rho_1(x)$. However, at the same time, $s_0(x)$ is decreasing, leading to the following corollary:

Corollary 3.8. *It cannot be the case that at some point x_1, the oil producer produces, but for some $x_2 > x_1$ the oil producer does not produce.*

Thus, we define the point

$$x_b^* = \inf\{x > 0 : q_0^*(x) \geq 0\}.$$

For all points $x < x_b^*$, the oil producer will not produce. In particular, it is easy to see that

$$\sup\{x > 0 : q_0^*(x) = 0\} = \inf\{x > 0 : q_0^*(x) \geq 0\}.$$

Proposition 3.9. *If there exists a point x such that $s_0(x) < \rho_1(x)$, then the oil producer will participate.*

Proof. Assume for sake of contradiction that the oil player does not play. Then, should he play, the cost must be too high. In other words, $v'(x) + s_0(x) > \rho_1(x)$ must hold, requiring $v'(x) > \rho_1(x) - s_0(x) > 0$, so the value function is increasing at the point x. However, then, there must be a neighborhood to the left of x on which the value function is also rising; since the value function measures objective utility, the player must produce nonzero quantity. If the player were to produce zero quantity, then his utility could not increase. Thus, there must be a neighborhood to the left of x on which the oil player produces. However, Corollary 3.8 then implies that at x, the oil player must participate, contradicting the assumption. □

It thus follows immediately that there are three cases for the form of $s_0(x)$

- CI: It could be that $s_0(0) < \rho_1(0)$, which implies that $s_0(x) < \rho_1(x)$ for all x, in which case the oil producer always plays.
- CII: It could be that $s_0(x) \geq \rho_1(x)$ for all x, in which case the oil producer never plays.
- CIII: It could be that $s_0(0) \geq \rho_1(0)$, but there is some later $x' > 0$ such that $s_0(x') < \rho_1(x')$. By continuity, it is evident that the point x_b^* can be expressed as

$$x_b^* = \inf\{s_0(x) < \rho_1(x)\}.$$

In the third case we were examining, continuity and closure of $[0, x']$ requires that $x_b^* \in [0, x']$. In particular, continuity of $s_0(x)$ and $s_0(x)$ decreasing implies that in this case, x_b^* is the unique point that satisfies $s_0(x_b^*) = \rho_1(x_b^*)$.

The following Proposition then establishes the Markov perfect equilibrium:

Proposition 3.10. *The Markov perfect equilibrium for the game is that for all points $x < x_b^*$, the renewable player is the only player and the strategies are*

$$(q_0^*, q_1^*) = \left(0, \frac{1 - s_0(x)}{2}\right).$$

For all $x > x_b^*$,

$$(q_0^*, q_1^*) = \left(1 - 2v'(x) - 2s_0(x) + s_1(x), 1 - 2s_1(x) + v'(x) + s_0(x)\right).$$

Proof. Should we show that this holds for Case III, we can merely take the limits $x_b^* \to 0$ for Case I and and $x_b^* \to \infty$ for Case II, so proving that this is the relevant equilibrium for Case III suffices. Assume for contradiction that this is not a Markov equilibrium. Then, either there is some strategy q_0 such that profit is larger for the oil player holding q_1^* constant (Case A), or there is some strategy q_1 such that profit is larger for the renewable player holding q_0^* constant (Case B). Assume Case A holds; then, Proposition 12 implies optimality of q_0^*. Assume Case B holds; then, when $x < x_b^*$, the renewable player has a monopoly on the market, which implies q_1^* is optimal. For $x > x_b^*$, the optimal Cournot equilibrium is described by q_1^* and hence is already optimal. Thus, the assumption fails to hold and we have a contradiction, implying the result. □

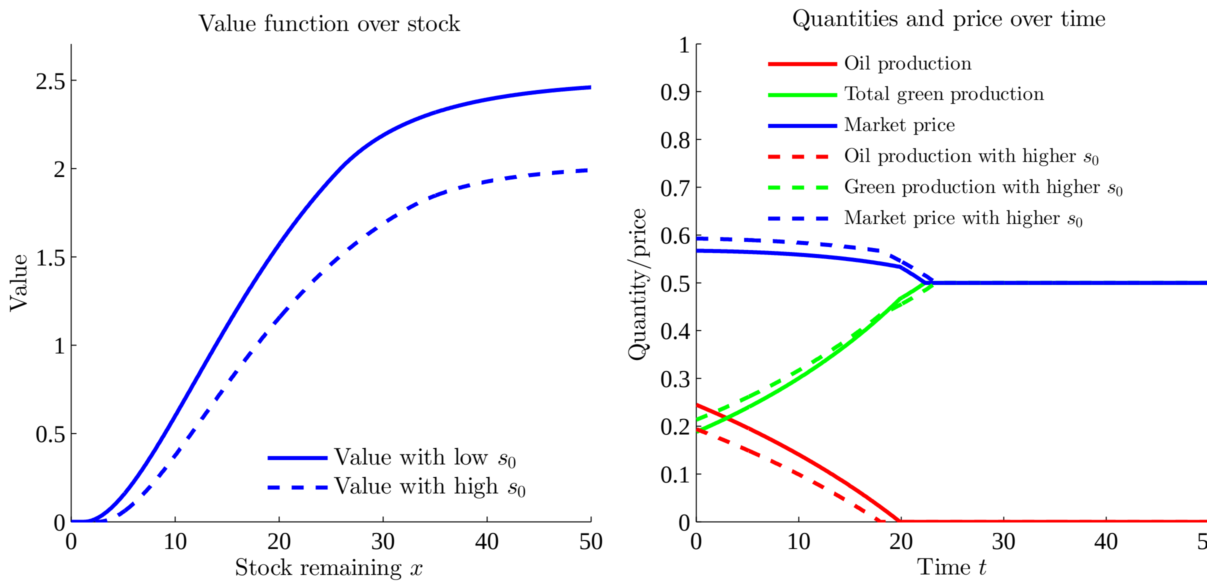

Figure 5: *The "low" cost corresponds to $s_0(x) = .6e^{-.1x}$ and $s_1(x) = .6(1 - e^{-.1x})$, while the "high" cost case corresponds to $s_0(x) = .6e^{-.1x} + .1$ and $s_1(x) = .6(1 - e^{-.1x})$.*

3.4 Varying costs: Numerical calculations

We now investigate the problem numerically. We first consider a simple case, shown in Figure 5; the oil producer shall have an exponentially decreasing cost in stock, that is $s_0(x) = .6e^{-x}$ and player 1, our renewable player, shall have an exponentially increasing cost in stock, that is, $s_1(x) = .6(1 - e^{-x})$. The left plot suggests a result that follows directly from the analytical propositions above. That is, for sufficiently low stock, that is for all stock $x < x^*$, where x^* satisfies $s_0(x^*) = \rho_1(x^*)$, the oil producer has a cost too high for him to ever produce. For all $x > x^*$, his cost is lower than the necessary threshold. This is reflected in the value functions $v(x)$ which satisfy $v(x) = 0$ for all $x \in [0, x^*]$.

Although not explicitly depicted, for large reserves of oil, the renewable players' cost increases to the point where he must drop out, leading to an oil monopoly for sufficiently large x. In particular, we note that the total quantity produced is not smooth at these two points. This is as expected; if we let x' be the point at which the oil player has a monopoly for all $x > x'$, the intervals $[0, x^*]$, (x^*, x'), and (x', ∞) correspond to fundamentally different games.

Finally, Figure 6 plots two value functions; this time, we once again fix $s_1(x) = .6(1 - e^{-.1x})$. However, the low $s_0(x)$ case corresponds to $s_0(x) = .6e^{-.1x}$, while the high $s_0(x)$ case corresponds to $s_0(x) = .8e^{.1x}$. While the right plot exhibits similar features to those discussed earlier, the left plot is of particular interest.

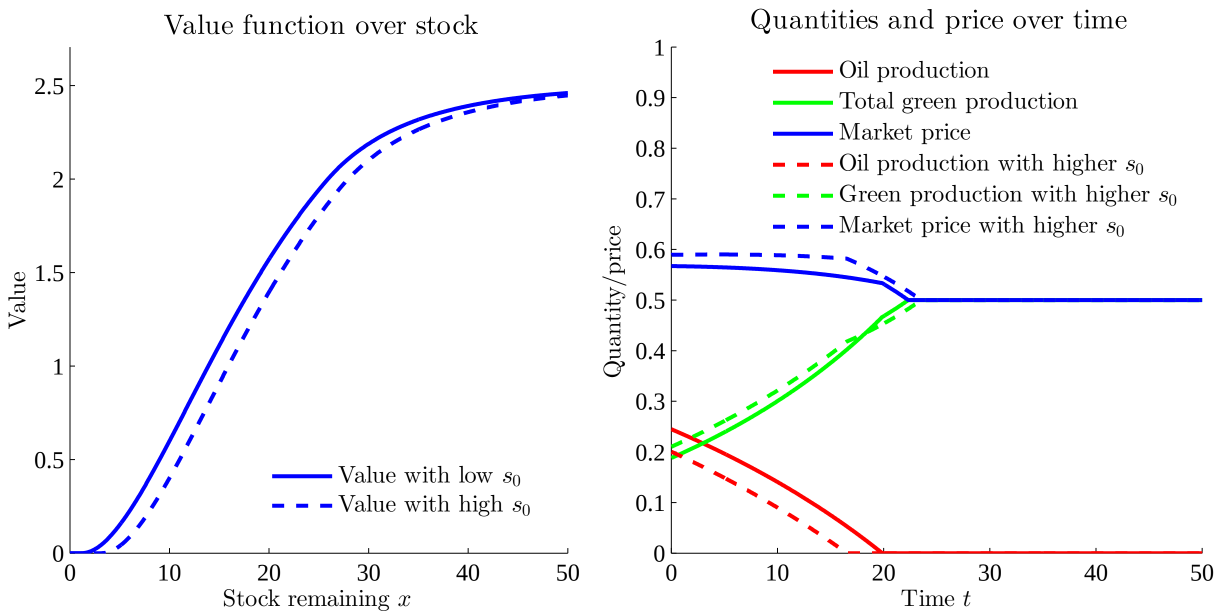

Figure 6: *The "low" cost corresponds to* $s_0(x) = .6e^{-.1x}$ *and* $s_1(x) = .6(1 - e^{-.1x})$*, while the "high" cost case corresponds to* $s_0(x) = .8e^{-.1x}$ *and* $s_1(x) = .6(1 - e^{-.1x})$.

If we denote $v_L(x)$ to correspond to the low cost case and $v_H(x)$ to correspond to the high cost, we note that as $x \to \infty$, $\|v_H(x) - v_L(x)\| \to 0$. This is a consequence of the lower and higher costs both converging to 0, which reduces the difference in oil value over time. This suggests that as stock is increased to arbitrarily large amounts, the differences in discounted profits for the oil producer becomes negligible as long as the costs over time converge to the same value.

4 Resource Discovery

We now allow that the oil producer may invest in research and development, choosing to invest in exploration that yields a possibility of discovering new reserves. In particular, the oil producer may, at any given point, explore with intensity a at a cost $\mathcal{C}(a)$. We choose $\mathcal{C}(a)$ to be increasing in a. As intensity increases, the probability of finding additional reserves also increases. In particular, we let reserves at any time t evolve according to

$$dX_t = -q_0(X_t)\mathbb{1}_{\{X_t>0\}}\, dt + \delta\, dN_t.$$

Here, N_t is a point process with intensity λa_t, where λ is determined exogenously. The probability of increasing one's reserves by a fixed quantity δ at a given time t is thus $\lambda a_t dt$. This is the situation considered in [9] for constant costs.

4.1 Two Players

We first consider a scenario similar to above where there are two players, an oil player (denoted player 0) and a renewable player (denoted player 1). The oil producer's strategy will depend on his stock at any given time and produces at $q_0(X_t)$, where q_0 is part of a Markov strategy with player 1.

We let both players have a cost of extraction $s_0(x)$ and $s_1(x)$, where $s_0'(x) < 0$. This reflects that as oil runs out, the oil producer's cost of extraction also goes up. Meanwhile, we let $s_1'(x) > 0$, since as x decreases, investment and the like in renewable resources increasing.

We then can write the value functions for both players, where (q_0^*, q_1^*) is a Markov equilibrium:

$$v(x) = \sup_{q_0,a} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(q_0(X_t)(1-q_0(X_t)-q_1^*(X_t)-s_0(X_t)) - \mathcal{C}(a_t)\right)dt \Big| X_0 = x\right]$$

$$w(x) = \sup_{q_1} \mathbb{E}\left[\int_0^\infty e^{-rt} q_1(X_t)(1-q_0^*(X_t)-q_1(X_t)-s_1(X_t))\mathbb{1}_{\{X_t>0\}}dt + \int_0^\infty e^{-rt}\frac{1}{4}(1-s_1(X_t))^2\mathbb{1}_{\{X_t=0\}}dt \Big| X_0 = x\right].$$

As earlier, we will note that $v_0(x)$ will be the only value function necessary for a description of the strategies. From just the value functions, we may note that the oil player has to optimize over two variables, q_0 and a. In other words, in certain intervals it may be optimal for player 1 to defer production by investing in exploration (R&D) while at other times, he may choose to not invest in R&D at all.

The corresponding Hamilton-Jacobi equation for the oil producer is

$$rv(x) = \sup_{q_0,a}\left[(1-q_0-q_1^*)q_0 - q_0(v'(x)+s_0(x)) - \mathcal{C}(a) + a\lambda\Delta v(x)\right].$$

Here, we have that $\mathcal{C}(a)$ is the cost of exploring with intensity a and $\Delta v(x)$ is a jump term defined by

$$\Delta v(x) = v(x+\delta) - v(x).$$

Since q_1 and a are additively separable, we can simplify the above Hamilton-Jacobi equation to

$$rv(x) = \sup_{q_0}\left\{(1-q_0-q_1^*)q_0 - q_0(v'(x)+s_0(x))\right\} + \sup_a\left\{-\mathcal{C}(a) + a\lambda\Delta v(x)\right\}.$$

In particular, the optimum exploration intensity at any given x is given by

$$a^* = \operatorname*{argsup}_{a\geq 0} -\mathcal{C}(a) + a\lambda\Delta v(x),$$

the Legendre transform of the exploration cost function evaluated at $\lambda\Delta v(x)$.

As is done in [9], we will take the cost of exploration to be

$$\mathcal{C}(a) = \frac{1}{\beta}a^\beta + \kappa a, \tag{19}$$

with $\beta > 1$ and $\kappa \geq 0$. We define a saturation point x_{sat} to be the point where the oil producer stops exploring. In particular,

$$x_{sat} = \inf\{a^*(x) = 0 : x > 0\}.$$

Since we require $\mathcal{C}(a)$ to be increasing in a, it is evident from $\lambda > 0$ and $\Delta v(x) > 0$ ($v(x)$ non-decreasing is guaranteed by Lemma 5), we have that for all $x > x_{sat}$, $a^*(x) = 0$ and for all $x < x_{sat}$, $a^*(x) > 0$. It is immediately seen that

$$a^*(x) = \left[\max(\lambda\Delta v(x) - \kappa, 0)\right]^{\gamma-1}, \qquad \text{where} \qquad \gamma = \frac{\beta}{\beta-1}. \tag{20}$$

4.1.1 Structure of Solution

In particular, there are three possibilities. In regions where $s_0(x) > \dfrac{1+s_1(x)}{2}$, the oil producer is blockaded and does not produce. On this interval, the above Hamilton-Jacobi equation reduces to

$$v(x) = -\mathcal{C}(a^*) + a^*\lambda\Delta v(x).$$

When both the oil and renewable players produce, then the Hamilton-Jacobi equation reduces to

$$rv = \frac{1}{9}\left(1 - 2s_0(x) - 2v'(x) + s_1(x)\right)^2 + \frac{1}{\gamma}\left[\max(\lambda\Delta v(x) - \kappa, 0)\right]^\gamma.$$

If the renewable player is blockaded, in the sense that there exists a finite x_b^1 such that

$$x_b^1 = \inf\{q_1^*(x) = 0 : x > 0\},$$

then for all $x \geq x_b^1$, the Hamilton-Jacobi equation is

$$rv = \frac{1}{4}(1 - s_0(x) - v'(x))^2 + \frac{1}{\gamma}\left[\max(\lambda\Delta v(x) - \kappa, 0)\right]^\gamma.$$

The case where $s_0(x) = 0$ has been asymptotically evaluated when $\lambda < \epsilon$ for small ϵ in [9].

Finally, to compute the boundary condition, we note that at $x = 0$, the oil producer cannot produce; that is, we have $q_0(0) = 0$. This implies that

$$v(0) = \sup_{a\geq 0}\mathbb{E}\left[e^{-rT}v(\delta) - \int_0^T e^{-rt}\mathcal{C}(a)dt\right],$$

where T is the time until when the next discovery is made.

In the case where $s_0(x) > \dfrac{1+s_1(x)}{2}$ for some x, we let

$$x^* = \sup\left\{x : s_0(x) > \frac{1+s_1(x)}{2}\right\}.$$

If no such x^* exists (that is $s_0(x) < (1+s_1(x))/2$ for all $x \in \mathbb{R}_+$), we let $x^* \to 0$. In particular, the constraint

$$v(x^*) = \sup_{a\geq 0}\mathbb{E}\left[e^{-rT}v(x^*+\delta) - \int_0^T e^{-rt}\mathcal{C}(a)dt\right]$$

must also hold. The above thus implies that

$$v(0) = \sup_{a\geq 0}\frac{\lambda a v(\delta) - \mathcal{C}(a)}{\lambda a + r}. \tag{21}$$

4.1.2 Numerical Discretization

As explained in [9], we can reduce the above to an iterative ODE that can be solved with Runge-Kutta methods by defining $v^0(x) = v(x)$, which is the no-exploration case solved in Section 3.3. For all $n \geq 1$, we recursively define

$$\begin{aligned}
rv^n &= (q_0^n(x))^2 + \frac{1}{\gamma}\left(\max(\lambda(v^{n-1}(x+\delta) - v^n(x)) - \kappa, 0)\right)^\gamma \\
q_0^n &= \begin{cases} \frac{1}{2}(1 - s_0(x) - v'(x)) & \text{if } x \geq x_b^1 \\ \frac{1}{3}(1 - 2s_0(x) - 2v'(x) + s_1(x)) & \text{if } x \in (x^*, x_b^1) \\ 0 & \text{if } x \in [0, x^*] \end{cases} \\
v^n(0) &= \sup_{a\geq 0}\frac{\lambda a v^{n-1}(\delta) - \mathcal{C}(a)}{\lambda a + r}. \\
v^n(x \in (0, x^*)) &= \sup_{a\geq 0}\frac{\lambda a v^{n-1}(x+\delta) - \mathcal{C}(a)}{\lambda a + r},
\end{aligned}$$

where $\mathcal{C}(a)$ is defined in (19). The above can then be solved using standard Runge-Kutta methods. In particular, it is shown in [9] that for a monopoly with zero costs, the above iterative scheme does converge uniformly to the value function with exploration.

4.2 Numerical Solution for Two Players

A numerical solution for the case where $s_0(x) = .15e^{-.05x}$ and $s_1(x) = .15(1 - e^{-.1x}) + 0.5$ is evaluated graphically using the iterative approach above in Figures 7 and 8. We note in particular that as $x \to \infty$, $\|v(x) - v^0(x)\| < \epsilon$ for $\epsilon \to 0$. This follows from $v'(x) \to 0$ as $x \to \infty$ for all iterations n; hence, we also have from the Mean Value Theorem that $\Delta v(x) \to 0$ as $x \to \infty$, and $\kappa > 0$ forces $a^*(x)$ to be realized at 0 as $x \to \infty$. Applying this limit to the Hamilton-Jacobi equation, we see that as $x \to \infty$, we recover the original equation for the non-exploration case, implying that $\|v(x) - v^0(x)\| \to 0$ as $x \to \infty$. Further, we can see that for all x, $v(x) \geq v^0(x)$. This follows immediately from a revealed preference argument, since when we allow exploration, the oil producer can never be strictly worse off, as he can always choose to never explore.

From Figure 8, we have that for all $x < x_{sat}$, the oil producer (player 0), does indeed explore and has a value $a^*(x) > 0$. We also have analytically that $a^*(x)$ is strictly decreasing in x. This follows from Figure 7 which indicates that $v(x)$ is concave everywhere. This is a result of our chosen cost function which has a low enough Lipschitz constant to obey the condition outlined in Proposition 8. Hence, $\Delta v(x)$ is decreasing in x so $a^*(x)$ must also be strictly decreasing. Since the process that governs evolution of reserves δdN_t has increased probability of identifying with δ as $x \to \infty$, we can see that there are more "jumps" or discoveries as x decreases.

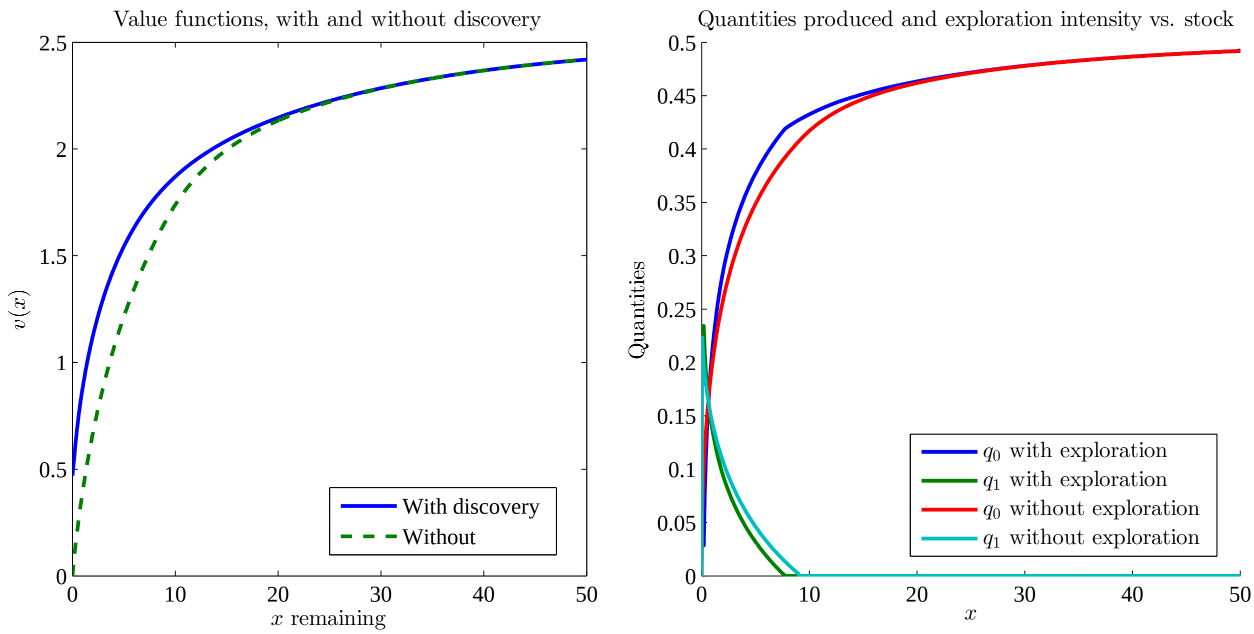

Figure 7: *Oil producer has cost function* $s_0(x) = .15e^{-0.05x}$ *and the renewable player has cost function* $s_1(x) = .15(1 - e^{-.1x}) + 0.5$.

Intuitively, as oil runs out, the oil producer needs to amp up discovery in order to stay in the game and is thus willing to pay additional cost, since the opportunity cost of not exploring is to leave the game. In the right panel of Figure 8, we can also see such jumps. In particular, in the beginning, the oil producer does fairly well, and for each discovery, the oil producer does produce additional quantities; even though the cost for the renewable player approaches a cost significantly higher than that of the oil producer (0.5 for player 1 vs. .15 for player 0), the renewable producer does indeed outproduce when the oil effectively runs out. Finally, we can note a trend of total quantity $q_0 + q_1$ decreasing in x, which corresponds to a higher market price over time.

If we now increase the relative cost of $s_0(x)$ by both increasing $s_0(x)$ and by decreasing $s_1(x)$, we obtain

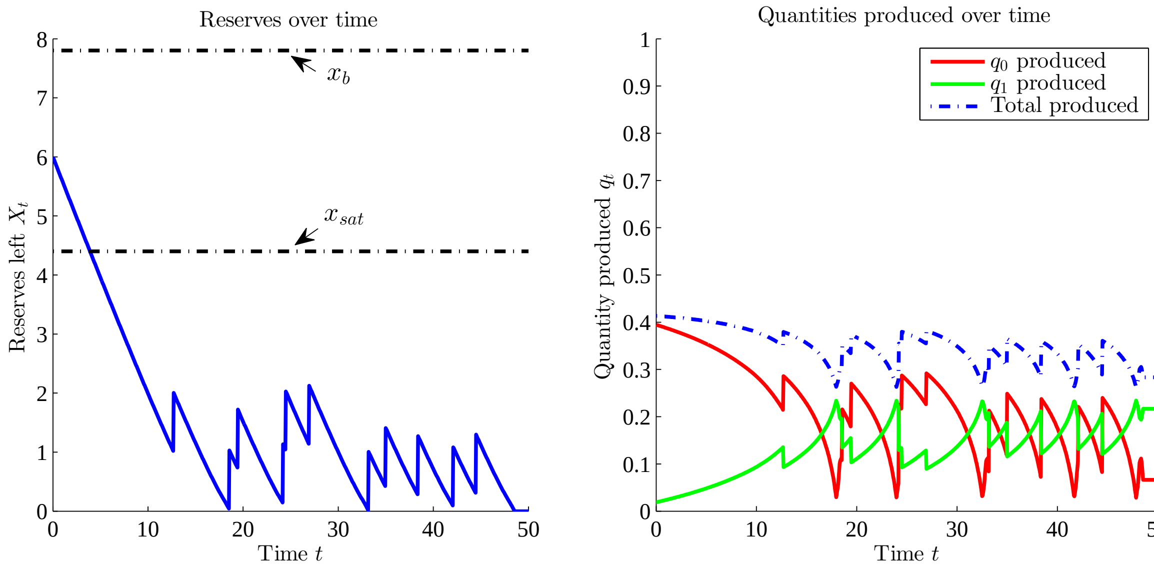

Figure 8: *Simulation vs. time of an exhaustible resource producer (player 0) and renewable producer (player 1) with costs* $s_0 = .15e^{-.05x}$ *and* $s_1(x) = .15(1 - e^{-.1x}) + 0.5$.

the simulation presented in Figure 9. We can note a few differences. First, $x_b \to \infty$, so the renewable producer is never blockaded. Further, the saturation point x_{sat} is now strictly lower. That is, the oil producer stops researching for new reserves at a lower threshold. This is a mathematical triviality from the expression for $a^*(x)$; as we increase the relative cost of $s_0(x)$, the value function $v(x)$ is decreased since profits are lower. This in turn causes $\Delta v(x)$ to go down since the marginal utility of an additional δ amount of oil at any given x is also lower. Hence, the threshold for which the cost becomes too prohibitive to research is also lower.

In particular, $a^*(x)$ is also strictly lower in this case due to an argument similar to that in the previous paragraph. This means the the probability of successful discovery of reserves is also lower. Hence, there is a dual effect to increasing relative cost. Not only does the oil producer produce less and hence have less discounted profit overall, but he also spares less for researching, which harms him when $x \to 0$.

Finally, we can note that in the long term, the market price is quite similar since the total quantity produced, plotted in the right panel of Figure 9 is about the same as before. However, now, the green producer almost always overtakes the red producer because the probability of discovery is now lower.

4.3 N-player case

Finally, we consider a multi-player oligopoly differential game with exploration. Consider an N player game where player 0 is an exhaustible producer, whom we will term the "oil" producer, while players $1, \dots, N-1$ are renewable players with fixed costs $s_1, \dots, s_{N-1}$. For simplicity, we set each of these costs to be constant for all x. Further, let the "oil" producer have a decreasing cost of extraction $s_0(x)$ such that $s_0'(x) < 0$. Unlike previously, we now allow the oil producer to invest in R&D which may result in discovery of new oil sources. Borrowing notation from previously, we let

$$S^{(N-1)} = \sum_{i=1}^{N-1} s_i, \qquad \text{and} \qquad \rho_N = \frac{1 + S^{(N-1)}}{N}.$$

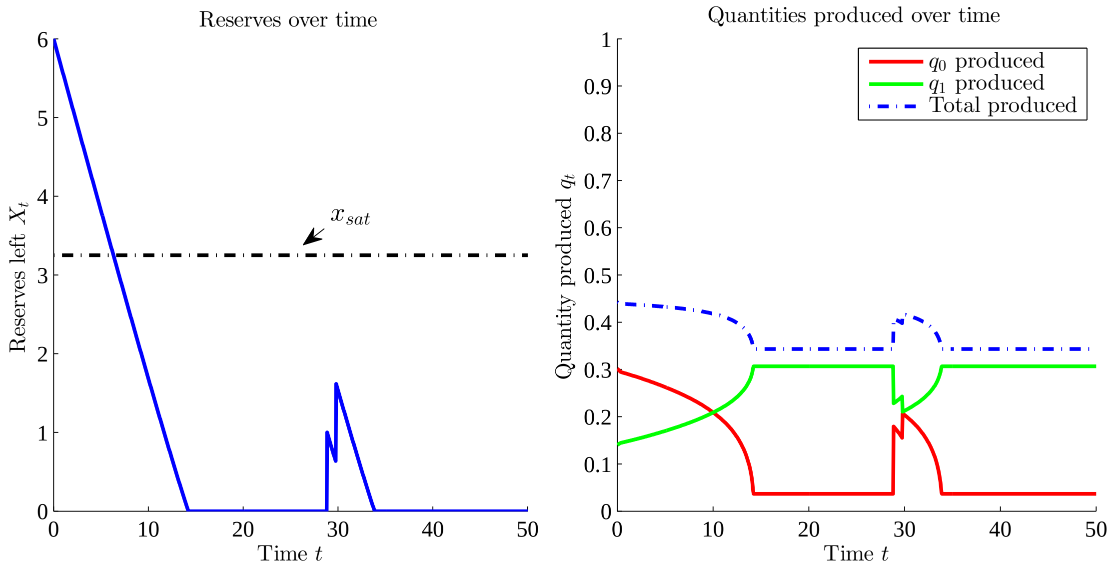

Figure 9: *Simulation vs. time with increased relative cost for player 0 (exhaustible producer). In particular, we now let* $s_0(x) = .25e^{-.05x}$ *and* $s_1(x) = .15(1 - e^{-.1x}) + .35$.

Letting $(q_0, \ldots, q_{N-1})$ be the Markov strategies of production given any remaining stock $x \in \mathbb{R}_+$, the reserves evolve according to

$$dX_t = -q_0(X_t)\mathbb{1}_{\{X_t>0\}}\, dt + \delta dN_t.$$

Now, there are N players, but player 0 has a cost of extraction $s_0(x)$ which depends on the amount of stock left. Denoting the cost of exploration with intensity a as $\mathcal{C}(a)$, the appropriate value function for the oil producer is, given that we denote $(q_0^*, \ldots, q_{N-1}^*)$ as a Markov perfect equilibrium,

$$v(x) = \sup_{q_0,a} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(q_0(X_t)\left(1 - q_0(X_t) - \sum_{i=1}^{N-1} q_i^*(X_t) - s_0(X_t)\right) - \mathcal{C}(a_t)\right) dt\big|X_0 = x\right].$$

As throughout the other games that we have considered, the value functions for the other players are not needed for analysis of blockading and saturation points. However, for completeness, we have that

$$w_i(x) = \sup_{q_i} \mathbb{E}\left[\int_0^\infty e^{-rt} q_i(X_t)\left(1 - q_0^*(X_t) - \sum_{j=1, j\neq i}^{N-1} q_j^*(X_t) - q_i(X_t) - s_i\right)\mathbb{1}_{\{X_t>0\}}dt \right.$$
$$\left. + \int_0^\infty e^{-rt}\frac{1}{(N+1)^2}\left(1 - (N+1)s_i + \sum_{i=1}^{N-1} s_i\right)^2 \mathbb{1}_{\{X_t=0\}}\, dt\big|X_0 = x\right],$$

for $i \in \{1, \ldots, N-1\}$.

The corresponding Hamilton-Jacobi equation is

$$rv(x) = \sup_{q_0}\left\{\left(1 - q_0 - \sum_{i=1}^{N-1} q_i^*\right) q_0 - q_0 v'(x)\right\} + \sup_a \left\{-\mathcal{C}(a) + a\lambda\Delta v(x)\right\}.$$

It is easy to verify that assuming the same cost function for exploration, the expression for $a^*(x)$ is given by (20) as before, with the difference being internalized in the jump term $\Delta v(x)$. Further, we can note that the boundary condition (21) from before also holds. However, now that $s_0(x)$ is decreasing, we have that for all x such that $s_0(x) > \rho_N$, the oil producer does not play and the appropriate Hamilton-Jacobi equation in that instance is simply

$$rv(x) = \frac{1}{\gamma}\left[\max(\lambda\Delta v(x) - \kappa, 0)\right]^\gamma .$$

As we did in Section 4.1.2, we recast the problem into a set of iterative ODEs which uniformly converge to $v(x)$. In particular, letting $v^0(x)$ be the solution where $\lambda = 0$ (the no-exploration case), which was analyzed in Section 3.3, we define

$$\begin{aligned}
rv^n &= (q_0^n(x))^2 + \frac{1}{\gamma}\left(\max\left(\lambda(v^{n-1}(x+\delta) - v^n(x)) - \kappa, 0\right)\right)^\gamma \\
q_0^n(x) &= \begin{cases} \dfrac{1}{R+2}\left(1 - s_0(x) - v'(x) + \displaystyle\sum_{i=1}^{R} s_i\right) & \text{if } s_0(x) < \rho_N \\ 0 & \text{if } s_0(x) \geq \rho_N \end{cases} \\
v^n(0) &= \sup_{a\geq 0} \frac{\lambda a v^{n-1}(\delta) - \mathcal{C}(a)}{\lambda a + r} \\
v^n(x \in (0, x^*)) &= \sup_{a\geq 0} \frac{\lambda a v^{n-1}(x+\delta) - \mathcal{C}(a)}{\lambda a + r}
\end{aligned}$$

Here, x^* is the value of x such that $x^* = \inf\{x \in \mathbb{R}_+ : s(x) < \rho_N\}$ and R is the number of renewable players who play at x. We can determine R by assuming that all players play, and then checking the quantity produced by the highest cost player. We denote

$$\begin{aligned}
q_n &= \frac{1}{n+2}\left(1 - (n+2)s_n + s_0(x) + v'(x) + S^{(n)}\right) \\
R &= \min\{n : q_n < 0; n \in \{1, \dots, N-1\}\}.
\end{aligned}$$

4.4 N-player simulations

We numerically evaluate the game using the expressions above. We choose $s_0(x) = .2e^{-x}$ and $s_1, \dots, s_9$ to be $0.51, 0.52, \dots, 0.59$ and depict the appropriate results in Figures 10 and 11. In particular, we note that once again $v(x) > v^0(x)$ for all x, as expected. We note that as x increases, the market price decreases, since a low cost option (oil) is available. When comparing the blockading points (denoted by vertical bars) to the case where $\lambda = 0$ (no exploration), the blockading points are to the right. This follows intuitively, since when the oil producer can explore, he is better off and hence is capable of driving the other players out faster. Mathematically, the shadow cost v' is lower in x with exploration, so at any given x, the oil producer can produce at a lower effective cost when exploration is an option, driving the blockading points to the right for the renewables.

In Figure 11, we simulate the game over time. We note that in this case, x_{sat} is higher than the initial quantity of reserves, so the oil producer always puts in a finite amount of exploration effort, a^*. Overall production goes down over time, reflecting increased market price over time. Further, as the oil producer runs out, the renewables produce more, but each discovery marks a resurgence of oil into the market while the renewables temporarily cut back production.

Further, if we increase $s_0(x)$, then x_{sat} is decreased. That is, when $s_0(x)$ increases, the value function $v(x)$ is lower everywhere than before because profits are lower. In particular, $\Delta v(x)$ is also lower since over the interval $(x, x+\delta)$, the oil producer is not as well off with a higher cost of extraction, leading to lower discounted profits and in turn a lower value for $\Delta v(x)$. Since $a^*(x)$ is proportional to $\Delta v(x)$ when $a^*(x) > 0$ and since the threshold value x_{sat} is also dependent on the magnitude of $\Delta v(x)$, an increased $s_0(x)$ marks a lower x_{sat}.

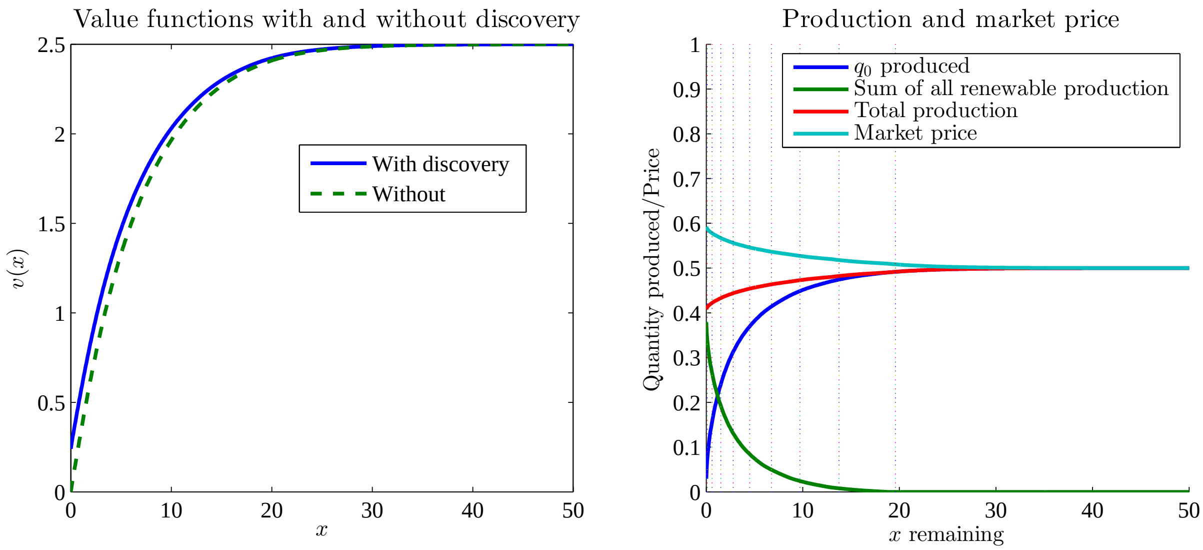

Figure 10: *We let $N = 10$ and initially start off with 9 renewable players and one exhaustible player. In particular, $s_0(x) = .2e^{-x}$ and $s_1, \ldots, s_9$ to be $0.51, \ldots, 0.59$ respectively. The vertical dotted lines in the right figure denote blockading points for each of the renewable players.*

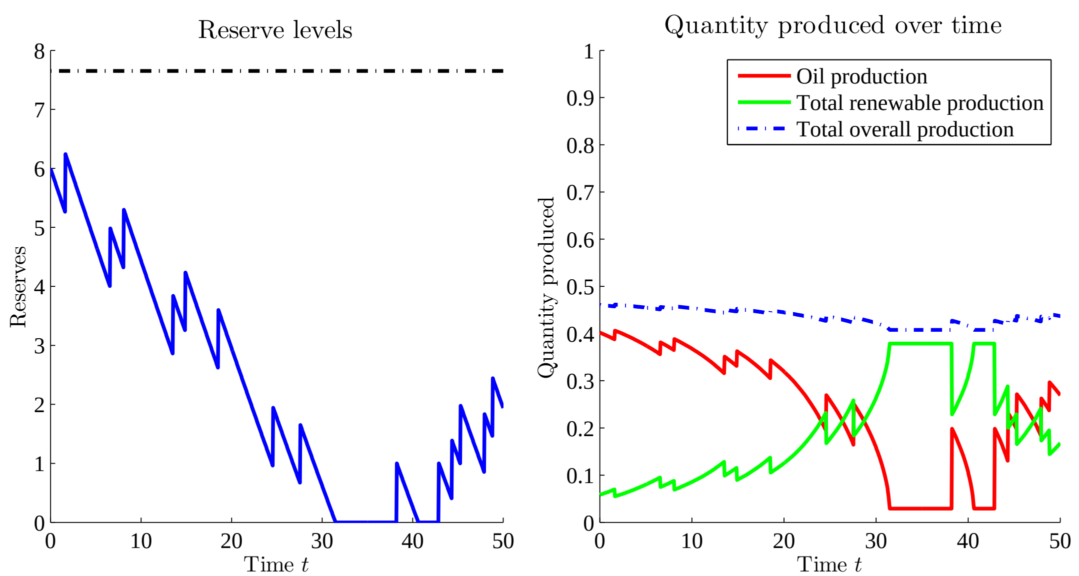

Figure 11: *We simulate the game over time $t \in [0, 50]$ for the same structure as described previously in Figure 10.*

Intuitively, increasing $s_0(x)$ means that there is a relative disincentive to explore, since even if exploration is successful, the marginal benefits are not as great because profit margins are lower. This suggests that in terms of policy implications, increasing $s_0(x)$ must be offset by a decrease in κ, which is reflective of the cost of exploration.

5 Application to Energy Policy

The above calculations demonstrate that the model that we have set forth is directly applicable to energy policy. We consider two such examples below.

5.1 Taxing oil production but subsidizing research

We now apply the above stochastic differential game models to energy policy. As alluded to throughout this paper, energy markets are similar to oligopoly models with varying costs and exploration. For simplicity, we will consider the two player model most recently explored whose costs both vary in time. We consolidate all finite stock based suppliers into one player (player 0) and all renewable players into a second (player 1).

In Figure 12, we model a policy option of taxing oil production but subsidizing their exploration costs. That is, we lower $\mathcal{C}(a)$ but increase $s_0(x)$. Such a policy has the potential of being revenue neutral and we lower κ significantly to model this. We note as a result that the oil producer is active on average for a longer time, since $a^*(x)$ is greater for all x in this situation. This follows immediately from $\mathcal{C}(a)$ being lower. Further, we note that x_{sat} is much higher. Specifically, x_{sat} increases from 7.65 to $x_{sat} = 21.25$. In addition, since the oil producer has a lower cost of exploration, he stays in for a longer period of time, leading to a lower market cost.

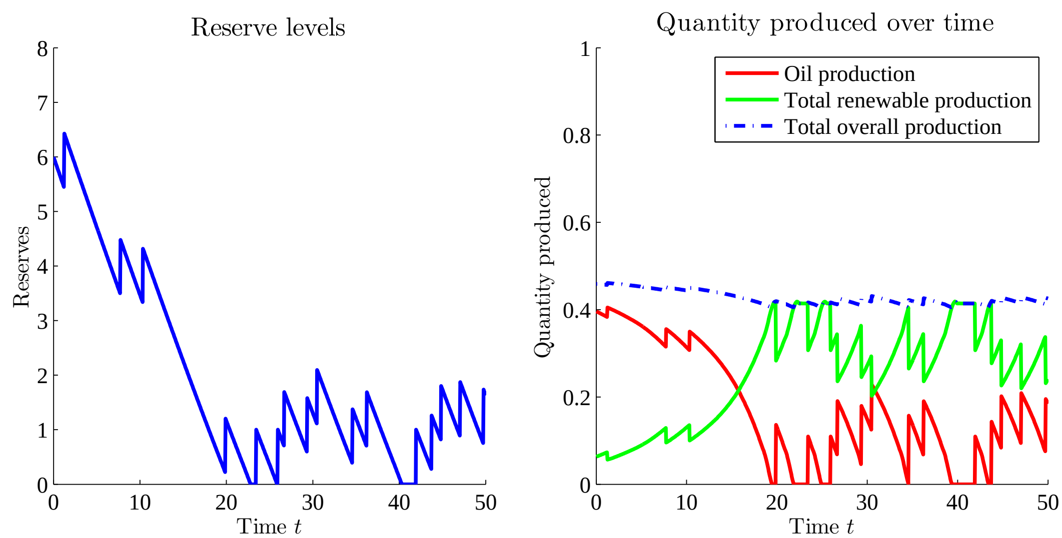

Figure 12: *We now increase $s_0(x)$ to $s_0(x) = .5e^{-x}$ but we lower $\kappa \to 0.01$ from $\kappa = 0.1$. This models taxing production but incentivizing R&D.*

However, such policy is limited at best since this model assumes that the probability of finding additional reserves is independent of the amount of discoveries already made, when in reality, such probability is not independent. As we find additional reserves, the marginal probability of finding another one is lower in the number of discoveries. However, it is of particular interest to note that lowering κ may have short term benefits in terms of price stability but has long term harms in that the total number of possible reserves is exhausted at a faster rate even if corrective action is taken in terms of increasing $s_0(x)$.

5.2 Taxing oil vs. subsidizing green energy

We will now compare the policy options of taxing our oil player and subsidizing our green player. These are both policy options that are currently on the table and have been implemented to some extent. However, to truly compare welfare overall, we must also consider the third player, the consumer, who drives the inverse demand function that we have assumed throughout. Assuming that demand is positive for both goods (as reflected in our function), we can apply the Gorman Aggregation Theorem to note that there exists an aggregate utility function that consolidates the preferences of the representative household. In particular, the appropriate utility function for our inverse demand function can be verified to be

$$u(Q, m, x) = Q(x)\left(1 - \frac{Q(x)}{2}\right) + m(x),$$

where $Q(x) = q_0(x) + q_1(x)$, the total quantity produced when x stock is remaining, and $m(x)$ is the total money the consumer has at time x. In particular, the consumer will seek to maximize the time discounted integral of utility:

$$\sup_{Q(x(t))} \mathbb{E}\left[\int_0^\infty \left(e^{-\rho t} Q(x(t))\left(1 - \frac{Q(x(t))}{2}\right) + m(x(t))\right) dt\right],$$

where the evolution of stock is given by

$$dX_t = -q_0(X_t)\mathbb{1}_{\{X_t>0\}}\, dt + \delta\, dN_t,$$

and ρ is the discounting factor of our consumer.

We will consider three states of the world. Notationally, we denote the set of costs and utilities by the array $(s_0^i, s_1^i, u^i(Q_i, m_i, x))$. Here, s_0^i refers to the cost function for the oil producer in state i and s_1^i is the cost function for the renewable player in state i. Additionally, $u^i(Q_i, m_i, x)$ refers to the utility of the representative consumer in state i, with Q_i the total production and m_i the amount of disposable money. Finally, we will let q_0^i and q_1^i be the quantities produced by the oil and renewable players respectively in state i.

State 1 will be our control case. For our numerical simulations, we will let $s_0^1(x) = .3e^{-.05x}$ and $s_1^1(x) = .6(1 - e^{-.1x})$. The appropriate utility function in this case is simply $u^1(Q_1, m_1, x)$. State 2 will be the case where we impose a tax on the oil producer. This is not to be conflated with a Pigouvian tax, which is typically introduced to correct externalities. The utility function $u(\cdot)$ does not contain any direct disutility from consuming oil and the inverse demand function has no differentiation from energy derived from oil or renewable sources. Hence, this is merely a tax on finite resources, aimed at prolonging the time duration for which the oil producer will play and also inducing further smoothing over time of oil production. For numerical purposes, we will introduce a 33% tax, so $s_0^2 = .4e^{-.05x}$, $s_1^1(x) = .6(1 - e^{-.1x})$ and $u^2(Q_2, m_2, x) = u^1(Q_2, m_2, x) + .1e^{-.1x} q_0^2$. We assume that the "government," or agent taxing the oil producer redistributes 100% of the tax revenue to the consumer.

Finally, State 3 will be the case where subsidize green energy. We assume a 33% subsidy for sake of consistency, so $s_0^3(x) = s_0^1(x) = .3e^{-.05x}$, $s_1^3(x) = s_1^1(x) - .2(1 - e^{-.1x}) = .4(1 - e^{-.1x})$, and $u^3(Q_3, m_3, x) = u^1(Q_3, m_3, x) - .2e^{-.1x} q_1^3$.

We define the aggregate welfare $W(x)$ to be $W(x) = u(Q, m, x) + \Pi_0(x) + \Pi_1(x)$, where $\Pi_i(x)$ is the profit of the ith firm at any point x in stock. We note that if at any given point x, if for two states i and j, $W_i(x) > W_j(x)$, then, state i is potentially Pareto improving over state j and hence preferable. In particular, since aggregate welfare is higher, it is possible to make every player better off up to some redistribution of profits.

In Figure 13, the top left panel plots oil profits over remaining stock. The control is clearly the best case scenario for the oil producer, as either taxing the oil player or subsidizing green energy raises the relative cost of oil, lowering profits. In particular, for low oil reserves, taxing oil is worse for oil profits than subsidizing green, but the reverse is true for high reserve levels. The top right panel measures renewable profits; as expected, the renewable producer is best off when his cost is subsidized and does marginally better when oil is taxed, as doing so reduces the relative cost of green energy.

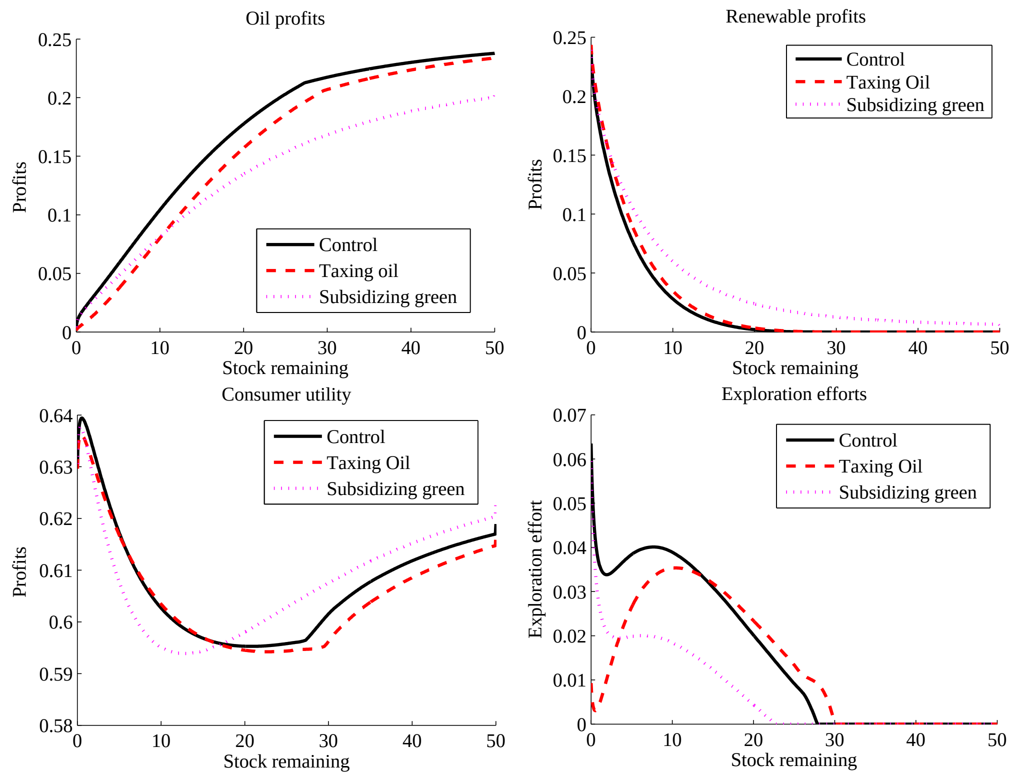

Figure 13: Top left: *plots oil (player 0's) profits over remaining stock.* Top right: *computes renewable energy (player 1's) profits for remaining stock* X_t*.* Bottom left: *plots consumer utility over* X_t*.* Bottom right: *plots exploration intensity* a^* *over* X_t *for all three policy options.*

Of particular interest is the bottom left panel of Figure 13, which measures consumer utility over remaining stock. The consumer is best off by subsidizing green energy at high reserve levels and at such high reserve levels, taxing oil worsens the consumer's situation. However, as oil begins to run out, the utility derived from subsidizing green energy begins to diminish. The bottom right panel accounts for exploration efforts. In particular, at high oil reserves, taxing oil incentivizes the oil player to conduct more research, but as oil begins to run out, the additional taxation reduces the discounted profits for the oil player should he discover, lowering discovery efforts.

Figure 14 consolidates the above welfare analysis, accounting for the welfare of both players and the representative consumer. For high oil reserves, the best policy seems to be to do nothing, but as oil begins to run out, subsidizing green energy is an effective policy, and as oil continues to deplete, taxation of oil might result in a marginal increase in aggregate utility.

Finally, since the utility function that corresponds to the inverse demand function does not factor beneficial macroeconomic effects such as price stability, we compare these in Figure 15, which depicts evolution of production levels. The left panel suggests that taxing oil reduces oil production a bit, more so than subsidizing oil for low reserve levels, though subsidizing oil reduces oil production when reserves are large. However, subsidizing oil is far better at stimulating green energy for all energy reserves. The right panel demonstrates that subsidizing green energy also results in greatest price stability, while there is not as significant of a difference between taxing oil and the control.

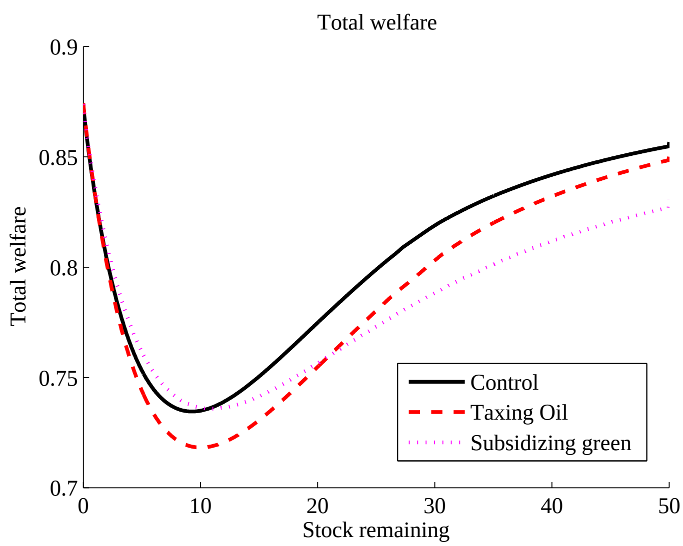

Figure 14: *Considers the aggregate welfare index W for three policy options: doing nothing (control), taxing oil, and subsidizing green energy.*

6 Conclusion

We developed a mathematical model for oligopoly markets with exhaustible resources, evolving extraction costs, and discovery. For the case where all costs are constant and where one player plays when his resource stock is non-negative and all others are renewable players, we presented a completely analytical solution, extending the work presented in [8]. We then presented partially analytical results with numerical solutions for the subcases where costs evolve over time but discovery is disabled, and finally when discovery of new reserves is permitted.

We next demonstrated how this model can be applied to energy policy. Of particular interest is our comparison of taxing oil and subsidizing green energy, two leading policy options that are currently being

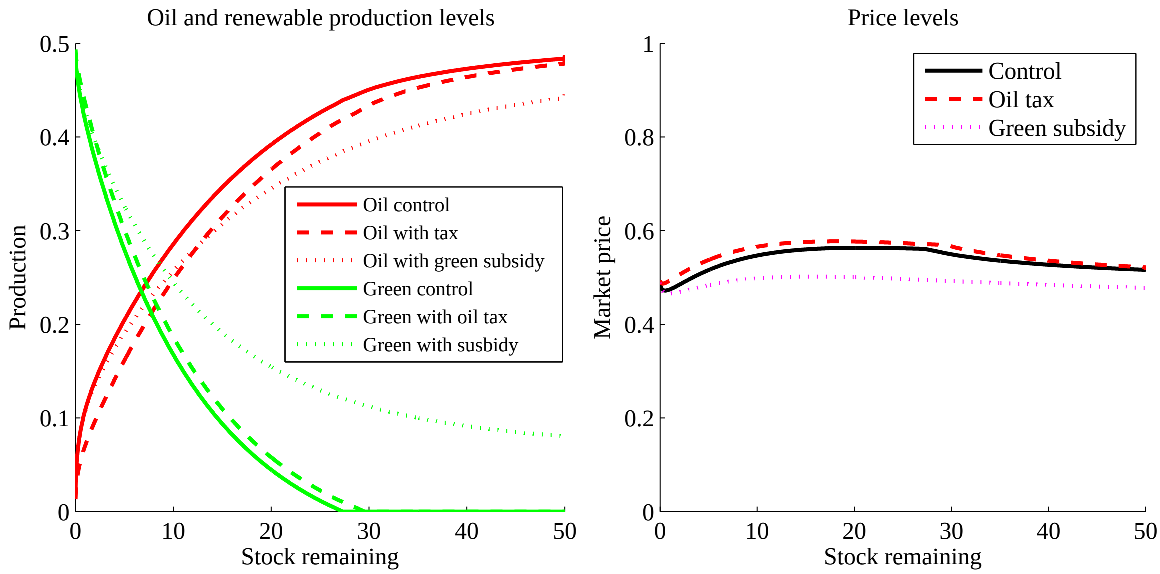

Figure 15: *Considers evolution of market price and quantities produced by both players over stock.*

considered. We compared the two in terms of aggregate welfare, which measures whether a policy option is potentially Pareto improving. We found that for high reserve levels, taxation of exhaustible players should be done only to correct for environmental externalities, and that subsidizing green energy later in our game, when $x(t)$ decreases beyond a threshold value, is potentially Pareto improving.

Future work may be considered in two directions. First, additional work may be done to solve this problem in general for a constant prudence demand curve, as in equation (1), which may alter the results slightly. We feel, however, that this should not significantly alter the policy implications that are demonstrated in Section 5. Secondly, as evidenced by the two examples in Section 5, this model itself may be applied to derive further policy implications.

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