SINGULAR PERTURBATIONS FOR BOUNDARY VALUE PROBLEMS ARISING FROM EXOTIC OPTIONS *

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Abstract. We study the pricing of three exotic derivative securities (barrier, lookback and passport options) which can be characterized by boundary value PDE problems in the context of popular Markovian stochastic volatility models of stock prices. By extending the fast mean-reverting asymptotic analysis in [6], the usual "Greek" correction to the Black-Scholes prices of these contracts is further corrected by a boundary integral term that is rapidly computed numerically. In the case of the passport option, the asymptotic method is effective in accounting for stochastic volatility effects in a simple and robust fashion even in the presence of a highly nonlinear embedded stochastic control problem.

1. Introduction. In this paper, we describe a framework for approximating the prices of certain path-dependent derivative securities to take into account the observed "implied volatility skew", which contains information about the market's view of the asymmetry and leptokurtosis in stock price returns. The pricing problems for these exotic options are characterized by boundary value problems for partial differential equations, under the class of stochastic volatility diffusion models we consider here. Our examples are a barrier option, a lookback option and a passport option, whose prices solve Dirichlet, mixed and Neumann boundary value problems respectively. From the point of view of the practical application, there is a need for a quick calculation from which a trader can quote a price to a client. The approximation method used here is computationally fast and robust to specific modeling of the unobserved stochastic volatility process.

The analysis extends the singular perturbation approximations for stochastic volatility models studied in [6]. The basis of the approximations is a rapid time-scale of fluctuation in the stock price volatility, relative to the time horizon of the options contract. Such a fast scale has been identified in market data in [9, 1, 2] for example, and is convenient for constructing approximations over times when other, slower factors in the volatility can be considered relatively benign. Extension of the approach in [6] to incorporate a slower scale is begun in [7]. Asymptotic analysis of a different type of exotic path-dependent contract, Asian options, is studied in [5].

The three options studied here are called exotic (and are listed in increasing order of 'exotic-ness') because they are less heavily traded than standard (or vanilla) call and put options. Lookbacks and passport options are usually sold as over-the-counter products. However, market vanilla option prices contain valuable information about the market's perception of future risks. This is typically expressed in units of implied volatility. Given the observed price C_{obs} of a European call option, which gives the holder the right but not the obligation to buy one unit of stock for strike price K on date T, the implied volatility I is defined as that volatility which equates the Black-Scholes option pricing formula $C^{BS}(t, X_t; K, T; I)$ to this price:

$$C^{BS}(t, X_t; K, T; I) = C_{\text{obs}}.$$

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Here, t denotes the current time and X_t the current stock price.

A basic problem in financial engineering is, given the market's implied volatilities, find prices of exotic contracts that are consistent with the principle of *no arbitrage*. (In general, there is no unique solution without making further assumptions on an underlying model).

Under a large class of fast mean-reverting stochastic volatility models, it is shown in [6] that the implied volatility surface $I(K,T)$ (that is, I considered as a function of the option's strike price K and maturity date T for fixed t and X_t) is approximated by an affine function of the log-moneyness-to-maturity ratio (LMMR):

$$\mathrm{LMMR} = \frac{\log(K/X_t)}{T-t},$$
$$I \approx a \times \mathrm{LMMR} + b, \tag{1.1}$$

where a and b are some market constants to be estimated by fitting this formula to option implied volatility data. See Figure 1.1.

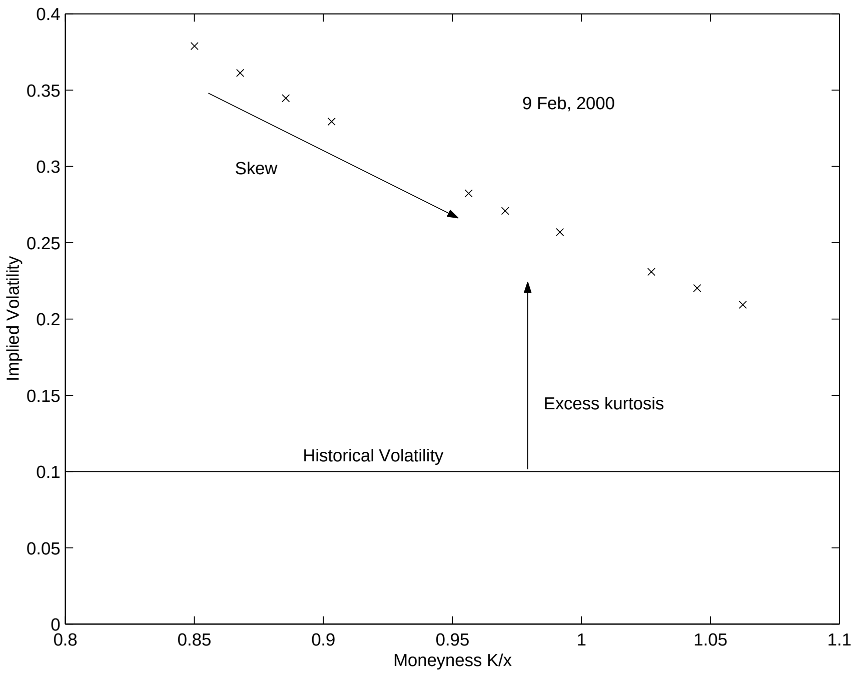

FIG. 1.1. Implied volatility as a function of moneyness for fixed maturity options. The skew represents asymmetry in the returns distribution of the stock, and the increase in level over historical volatility the excess kurtosis over lognormal models.

Then, given estimates of the slope a and the intercept b, we consider the problem of finding consistent approximations for various exotic options. The cases of American and Asian options were studied in [6], as well as barriers. The latter contained an error in the calculation, and we include it here, corrected, and in a somewhat different format from the subsequent erratum to [6], as our starting point.

When the formula (1.1) is fitted to certain regimes of S&P 500 implied volatility data, the estimated parameters a and b have good stability properties [6, 8]. This is particularly important for pricing path-dependent securities as considered here, because they depend not just on a one-time distribution of the stock price, but also on the evolution of the process. The stability and goodness-of-fit, particularly to short-dated options, can be improved by including time-dependent periodic factors, as is done in [8].

Given the condensation of the pricing measure contained in a and b, we show that the asymptotic correction term in these boundary value problems is explicit up to a one-dimensional integral, which can be computed very fast. It is then easy to gauge the impact of, for example, the slope of the implied volatility skew, measured by a, on the prices of these path-dependent contracts, as we illustrate numerically. The upshot of the analysis is that one obtains the usual asymptotic correction terms to the Black-Scholes prices of the exotics (which can be expressed in terms of the "Greeks", or partial derivatives of the Black-Scholes prices) plus an additional boundary integral term that corrects the Greek correction for skew effects.

2. Barrier Options. A *barrier option* is a path-dependent claim whose payoff depends on whether or not the underlying asset price hits a specified value before the maturity date. One example of a barrier option is the *down and out call option* which gives the holder the right to buy the underlying asset on expiration date T for strike price K unless the asset price has hit the barrier B at some time before T, in which case the contract expires worthless. The payoff at expiration T can be written as

$$h(X_T) = (X_T - K)^+ \, \mathbf{1}_{\left\{\min_{0\le t\le T} X_t \ge B\right\}},$$

where $\mathbf{1}$ denotes the indicator function.

2.1. Asymptotic Approximation. The fast mean-reverting stochastic volatility approximation for barrier options was studied in [6]. In this paper, we give a brief review, which derives the relevant PDE problems to solve for the terms in the asymptotic expansion. In this case, the boundary condition arises naturally due to the structure of the option.

We shall look at stochastic volatility models in which volatility (σ_t) is driven by an ergodic process (Y_t) that approaches its unique invariant distribution at an exponential rate $1/\varepsilon$. The size of this rate captures the volatility decorrelation speed, and in particular we shall be interested in asymptotic approximations when ε is small, which describes *fast* mean-reverting volatility.

As explained in [6], it is convenient for exposition to take a specific simple example for (Y_t) and allow the generality of the modeling to be in the unspecified relation between volatility and this process: $\sigma_t = f(Y_t)$, where f is some positive, (and sufficiently regular) function, bounded above and away from zero. Further, taking (Y_t) to be a Markovian Itô process allows us to simply model the asymmetry or fatter left-tails of returns distributions by incorporating a negative correlation between asset price and volatility shocks. We shall thus take (Y_t) to be a mean-reverting Ornstein-Uhlenbeck (OU) process, so that the stochastic volatility models we consider are

$$\begin{aligned} dX_t &= \mu X_t\,dt + \sigma_t X_t\,dW_t, \\ \sigma_t &= f(Y_t), \\ dY_t &= \frac{1}{\varepsilon}(m - Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho\,dW_t + \sqrt{1-\rho^2}\,dZ_t\right), \end{aligned} \tag{2.1}$$

where (X_t) is the stock price process. Here (W_t) and (Z_t) are independent standard Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and ρ is the instantaneous correlation between asset price and volatility shocks that captures the skew, asymmetry or leverage effect. The asymptotic results as they are used are *not* specific to the choice of the OU diffusion process, nor do they depend on specifying f. In this scaling, the invariant density of Y is Gaussian, $\mathcal{N}(m,\nu^2)$, which does not depend on ε.

The model (2.1) describes an *incomplete* market meaning that not all contingent claims can be replicated by trading only in the underlying stock, the volatility process being untradeable. This has profound consequences for pricing, hedging and calibration problems for derivative securities. By standard no-arbitrage pricing theory [4], there is more than one possible equivalent martingale (or risk-neutral pricing) measure $\mathbb{P}^{\star(\gamma)}$ because the volatility is not a traded asset; the nonuniqueness is denoted by the dependence on γ, which we identify as the market price of volatility risk.

By Girsanov's theorem, $(W_t^\star, Z_t^\star)$ defined by

$$W_t^\star = W_t + \int_0^t \frac{(\mu - r)}{f(Y_s)} ds,$$
$$Z_t^\star = Z_t + \int_0^t \gamma_s ds,$$

are independent Brownian motions under a measure $\mathbb{P}^{\star(\gamma)}$ defined by

$$\frac{d\mathbb{P}^{\star(\gamma)}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{(\mu-r)}{f(Y_s)} dW_s - \int_0^T \gamma_s dZ_s - \frac{1}{2}\int_0^T \left[\left(\frac{(\mu-r)}{f(Y_s)}\right)^2 + \gamma_s^2\right] ds\right),$$

assuming (γ_t) is a non-anticipating process with sufficient regularity.

In particular, γ_t is the risk premium factor from the *second* source of randomness Z that drives the volatility. We shall assume that the market price of volatility risk γ_t is a bounded function of the state Y_t: $\gamma_t = \gamma(Y_t)$. As explained in [6], we take the view that the market selects a pricing measure identified by a particular γ which is reflected in liquidly traded around-the-money European option prices. Other derivative securities must be priced with respect to this measure, if there are to be no arbitrage opportunities. Under our assumption about the volatility risk premium, the process (Y_t) remains autonomous and Markovian under the pricing measure.

Under $\mathbb{P}^{\star(\gamma)}$,

$$dX_t = rX_t dt + f(Y_t) X_t dW_t^\star, \tag{2.2}$$

$$dY_t = \left[\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho\, dW_t^\star + \sqrt{1-\rho^2}\, dZ_t^\star\right), \tag{2.3}$$

where

$$\Lambda(Y_t) = \rho\frac{(\mu - r)}{f(Y_t)} + \sqrt{1-\rho^2}\,\gamma(Y_t).$$

We also define the infinitesimal generator $\mathcal{L}^\varepsilon$ of (X, Y) under this measure, and write it grouped in powers of ε as:

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \tag{2.4}$$

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \tag{2.5}$$

$$\mathcal{L}_1 = \sqrt{2}\nu\rho f(y) x \frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \tag{2.6}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right). \tag{2.7}$$

Here $\mathcal{L}_0$ is the infinitesimal generator of the mean-reverting OU process, $\mathcal{L}_1$ contains the mixed derivative (from the correlation) and the market price of risk γ, and $\mathcal{L}_2$ is the Black-Scholes partial differential operator $\mathcal{L}_{BS}(f(y))$ at the volatility level $f(y)$.

The price $P(t,x,y)$ of the down-and-out barrier call option satisfies

$$\mathcal{L}^{\varepsilon} P = 0 \quad \text{in } x > B \text{ and } t < T,$$

with a terminal condition at $t = T$, $P(T,x,y) = (x-K)^+$, and a boundary condition at $x = B$, $P(t,B,y) = 0$. The latter expresses the knock-out condition on the barrier.

As shown in Appendix A, the fast mean-reverting approximation (in the limit $\varepsilon \downarrow 0$) for the barrier option is given by

$$P(t,x,y) \approx P^{(0)}(t,x) + \widetilde{P^{(1)}}(t,x).$$

Here, $P^0(t,x)$ is the Black-Scholes price of the option with constant volatility parameter $\bar{\sigma}$, which is related to the original volatility model by

$$\bar{\sigma}^2 = \langle f^2 \rangle,$$

where $\langle \cdot \rangle$ denotes averaging with respect to the invariant density of Y, $\mathcal{N}(m,\nu^2)$. Notice that, under the fast volatility scaling, the first two terms of the expansion do not depend on y, the level of the unobservable process Y. From (A.2), it is the solution of the boundary value problem

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma}) P^{(0)} &= 0 \quad \text{in } x > B \text{ and } t < T, \\ P^{(0)}(T,x) &= (x-K)^+, \\ P^{(0)}(t,B) &= 0. \end{aligned} \tag{2.8}$$

We can obtain a formula for $P^{(0)}(t,x)$ by the method of images (see [18], for example):

$$P^{(0)}(t,x) = C^{BS}(t,x;\bar{\sigma}) - \left(\frac{x}{B}\right)^{1-k} C^{BS}(t, B^2/x; \bar{\sigma}), \tag{2.9}$$

where $k = 2r/\bar{\sigma}^2$ and $C^{BS}(t,x)$ is the Black-Scholes pricing formula for *a call option*, with the volatility parameter $\bar{\sigma}$:

$$\begin{aligned} C^{BS}(t,x;\bar{\sigma}) &= xN(d_1) - Ke^{-r(T-t)}N(d_2), \\ d_1 &= \frac{\log(x/K) + \left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}} \\ d_2 &= d_1 - \bar{\sigma}\sqrt{T-t}, \end{aligned}$$

and N denotes the cumulative normal distribution function,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

2.2. First-Order Correction. From (A.5), the stochastic volatility correction $\widetilde{P^{(1)}}(t,x)$, which is of order $\sqrt{\varepsilon}$, satisfies the PDE problem

$$\begin{aligned}\mathcal{L}_{BS}(\bar\sigma)\widetilde{P^{(1)}} &= \mathcal{A}P^{(0)} \quad \text{in } x>B \text{ and } t<T,\\ \widetilde{P^{(1)}}(T,x) &= 0,\\ \widetilde{P^{(1)}}(t,B) &= 0,\end{aligned}$$

where $\mathcal{A}$ is defined as

$$\mathcal{A} = V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right) + V_2^\varepsilon x^2\frac{\partial^2}{\partial x^2}, \tag{2.10}$$

$$V_2^\varepsilon = -\frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}\langle\Lambda\phi'\rangle, \tag{2.11}$$

$$V_3^\varepsilon = \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\langle f\phi'\rangle, \tag{2.12}$$

and $\phi(y)$ is a solution of $\mathcal{L}_0\phi(y) = f(y)^2-\bar\sigma^2$. As shown in [6], the boundedness assumptions on f and γ imply that we can choose ϕ to have bounded first derivative.

The interpretation of the two market constants above are as follows: V_2^ε contains the effect of the market price of volatility risk; V_3^ε contains the effect of the correlation, or skew, ρ. In the case of zero correlation, $V_3^\varepsilon = 0$, and our correction formulas (2.19), (3.11) and (4.19) below collapse and do not require numerical integration. However, in equity markets, ρ is typically estimated to be negative.

In practice, we do not use the homogenization formulas (2.11) and (2.12) to obtain V_3^ε and V_2^ε from a specific stochastic volatility model. Rather, they are calibrated from liquid European options prices, or the implied volatility surface using the LMMR formula (1.1). As computed in [6], V_3^ε and V_2^ε are obtained from a and b in (1.1), and from the long-run mean historical volatility $\bar\sigma$ estimated from stock returns, by

$$\begin{aligned}V_2^\varepsilon &= -\bar\sigma\left(a(r-\frac{1}{2}\bar\sigma^2)+(b-\bar\sigma)\right),\\ V_3^\varepsilon &= -a\bar\sigma^3.\end{aligned}$$

The problem of solving this boundary value problem with a source term can be simplified to a one-dimensional integral by defining

$$\hat{P}(t,x) = \widetilde{P^{(1)}} + \frac{V_3^\varepsilon}{\bar\sigma}xP^{(0)}_{x\bar\sigma} + \frac{V_2^\varepsilon}{\bar\sigma}P^{(0)}_{\bar\sigma},$$

for $x\geq B$. Then $\hat{P}(t,x)$ solves

$$\begin{aligned}\mathcal{L}_{BS}(\bar\sigma)\hat{P}(t,x) &= 0 \quad \text{in } x>B \text{ and } t<T,\\ \hat{P}(T,x) &= 0,\\ \hat{P}(t,B) &= \frac{V_3^\varepsilon}{\bar\sigma}g(t),\end{aligned} \tag{2.13}$$

where we define

$$g(t) = xP^{(0)}_{x\bar\sigma}\big|_{x=B}. \tag{2.14}$$

This is because the barrier option Vega $\mathcal{V} = P^{(0)}_{\bar\sigma}$ solves the PDE problem

$$\begin{aligned} \mathcal{L}_{BS}(\bar\sigma)\mathcal{V} &= -\bar\sigma x^2 P^{(0)}_{xx} \quad \text{in } x > B \text{ and } t < T, \\ \mathcal{V}(T,x) &= 0, \\ \mathcal{V}(t,B) &= 0, \end{aligned}$$

as can be seen by formally differentiating (2.8) with respect to $\bar\sigma$. Differentiating again with respect to x, we can see that the Vega of the hedge $U = xP^{(0)}_{x\bar\sigma}$ satisfies

$$\mathcal{L}_{BS}(\bar\sigma)U = -\bar\sigma x \frac{\partial}{\partial x}\left(x^2 P^{(0)}_{xx}\right), \qquad U(T,x) = 0,$$

but $U(t,B) \neq 0$ in general.

2.3. Interpretation of the Greeks. In the case of a regular option without a barrier boundary condition, the correction to the price is given by

$$\widetilde{P^{(1)}} = -\frac{V_3^\varepsilon}{\bar\sigma} x P^{(0)}_{x\bar\sigma} - \frac{V_2^\varepsilon}{\bar\sigma} P^{(0)}_{\bar\sigma},$$

which corresponds to the alternative formulation

$$\widetilde{P^{(1)}} = -(T-t)\left(V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right) + V_2^\varepsilon x^2 \frac{\partial^2}{\partial x^2}\right) P^{(0)}$$

given in [6] because

$$\mathcal{V} = \bar\sigma (T-t) x^2 P^{(0)}_{xx} \tag{2.15}$$

$$x P^{(0)}_{x\bar\sigma} = \bar\sigma (T-t) x \frac{\partial}{\partial x}\left(x^2 P^{(0)}_{xx}\right). \tag{2.16}$$

While it is convenient for intuition to present the asymptotic correction in terms of the so-called "Greeks" $P^{(0)}_{\bar\sigma}$ and $P^{(0)}_{x\bar\sigma}$, the intuition can be misleading because, here, these terms are evaluated at the long-run mean volatility $\bar\sigma$, and not at (an estimate of) the current volatility level $f(Y_t)$. In other words, these terms represent sensitivity to the global mean volatility rather than local sensitivity, as is how the Greeks are usually employed in practice. The asymptotic calculation has highlighted the Vega and Vega of the Delta $P^{(0)}_{x\bar\sigma}$ as primary measures of the effect of stochastic volatility on pricing in the fast mean-reversion limit, but the current volatility level is unimportant to this order. It is analogous to a central limit theorem correction to a law of large numbers.

In the case of path-dependent options considered here, these Greek terms do not comprise the whole correction, and the term $\hat P$, which can be represented as a boundary integral as we shall see below, plays an important role.

2.3.1. Calculation. The problem (2.13) can be transformed to a constant coefficient backward heat equation by the simple transformations

$$\begin{aligned} \eta &= \log\frac{x}{B} \\ \hat P(t,x) &= \frac{V_3^\varepsilon}{\bar\sigma} v(t,\eta) \exp\left(-\frac{1}{8}\bar\sigma^2(1+k)^2(T-t) + \frac{1}{2}(1-k)\eta\right). \end{aligned}$$

Then $v(t,\eta)$ solves

$$v_t + \frac{1}{2}\bar{\sigma}^2 v_{\eta\eta} = 0 \quad \text{in } \eta > 0 \text{ and } t < T, \qquad (2.17)$$
$$v(T,\eta) = 0,$$
$$v(t,0) = \tilde{g}(t),$$

where

$$\tilde{g}(t) = e^{\frac{1}{8}\bar{\sigma}^2(1+k)^2(T-t)} B^{-\frac{1}{2}(1-k)} g(t).$$

The probabilistic representation of v is simply

$$v(t,\eta) = \mathbb{E}\{\tilde{g}(\tau)\mathbf{1}_{\{\tau \le T\}} \mid B_t = \eta > 0\},$$

where (B_t) is a Brownian motion with $\langle B \rangle_t = \bar{\sigma}^2 t$, and τ is the first time after t that it hits 0. Using the distribution of the hitting time τ (see e.g. [16, Chapter 2, Proposition 8.5]), the solution is given by the one-dimensional integral

$$v(t,\eta) = \frac{1}{\bar{\sigma}\sqrt{2\pi}} \int_t^T \frac{\eta}{(s-t)^{3/2}} e^{-\eta^2/2\bar{\sigma}^2(s-t)} \tilde{g}(s)\,\mathrm{d}s. \qquad (2.18)$$

The boundary condition $v(t,0) = \tilde{g}(t)$ holds in the following sense: $v(t,\eta)$ is the convolution of $\tilde{g}$ with the kernel $t \mapsto \eta t^{-3/2} e^{-\eta^2/2\bar{\sigma}^2 t}$. As $\eta \to 0$, the kernel tends weakly to the Dirac mass at $t=0$ and $v(t,\eta) \to \tilde{g}(t)$ pointwise as $\eta \to 0$.

We obtain the correction to the barrier price as

$$\widetilde{P^{(1)}}(t,x) = -\frac{V_3^\varepsilon}{\bar{\sigma}} x P^{(0)}_{x\bar{\sigma}}(t,x) - \frac{V_2^\varepsilon}{\bar{\sigma}} P^{(0)}_{\bar{\sigma}}(t,x) \qquad (2.19)$$
$$+ \frac{V_3^\varepsilon}{\bar{\sigma}} \frac{x}{B} \log\left(\frac{x}{B}\right) \frac{1}{\bar{\sigma}\sqrt{2\pi}} \int_t^T e^{-\frac{1}{2}d_B(s-t)^2} \frac{g(s)}{(s-t)^{3/2}}\,\mathrm{d}s,$$

where

$$d_B(\tau) = \frac{\log(x/B)}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}(1+k)\bar{\sigma}\sqrt{\tau}.$$

Explicit formulas for $g(t)$ and $\widetilde{P^1}(t,x)$ are given in Appendix B. These are illustrated in Figures 2.1 and 2.2.

2.4. Convergence. In the case of a smooth payoff-at-maturity function, the proof of the convergence result

$$|P(t,x,y) - (P^{(0)}(t,x) + \widetilde{P^{(1)}}(t,x))| = \mathcal{O}(\varepsilon)$$

at a fixed point (t,x,y) is obtained by an adaptation of the proof given in [6, Section 5.4]. The error $Z^\varepsilon(t,x,y)$ defined by

$$P = P^{(0)} + \sqrt{\varepsilon}P^{(1)} + \varepsilon P^{(2)} + \varepsilon^{3/2}P^{(3)} - Z^\varepsilon$$

satisfies

$$\mathcal{L}^\varepsilon Z^\varepsilon = \varepsilon(\mathcal{L}_1 P^{(3)} + \mathcal{L}_2 P^{(2)}) + \varepsilon^{3/2}\mathcal{L}_2 P^{(3)}$$
$$Z^\varepsilon(T,x,y) = \varepsilon P^{(2)}(T,x,y) + \varepsilon^{3/2}P^{(3)}(T,x,y),$$
$$Z^\varepsilon(t,B,y) = \varepsilon P^{(2)}(t,B,y) + \varepsilon^{3/2}P^{(3)}(t,B,y),$$

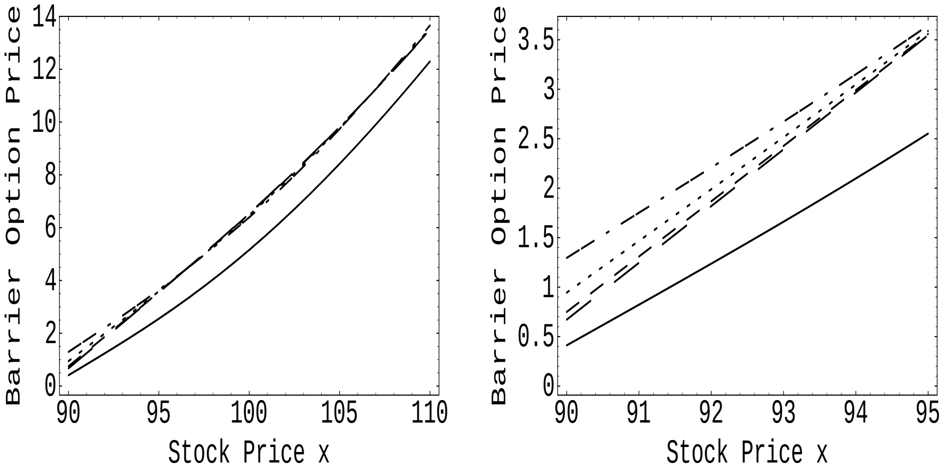

FIG. 2.1. Effect of changing the slope of the skew a on down and out call option price. The parameters used for pricing the contract are K=100, B=89, T=.5, $\bar{\sigma}$=0.17,b= 0.23. As shown more closely in the right figure, near the barrier, making a more negative increases the price. This effect reverses at higher stock prices. In the figures,the solid line shows the corresponding Black-Scholes price. In the right figure, the values of a reading upwards after Black-Scholes pricing curve are $a = -0.02, -0.04, -0.09, -.18$.

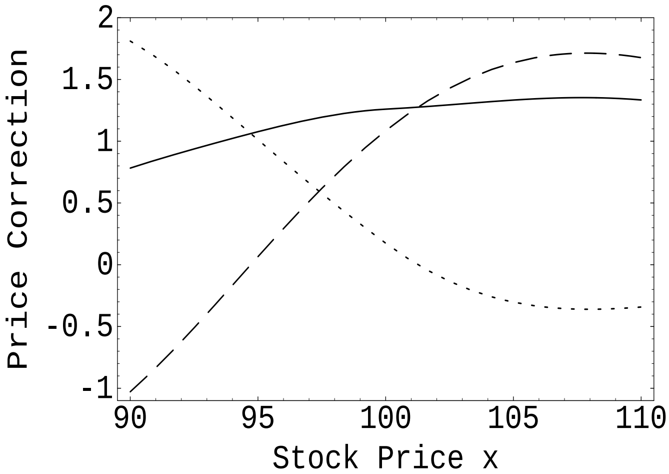

FIG. 2.2. The first order correction for down and out call option at time t=0. The parameters used for pricing the contract are as in Figure 2.1 with $a = -0.154$. The solid line shows the first order correction $\widetilde{P^{(1)}}$, the dotted line is $\hat{P}$, and the dashed line is the contribution of the Greek terms to $\widetilde{P^{(1)}}$ in (2.19).

using the definitions of $P^{(0)}(t,x)$, $P^{(1)}(t,x)$, and choosing $P^{(2)}(t,x,y)$ and $P^{(3)}(t,x,y)$ as solutions of (A.1) and (A.3) respectively. The latter can be chosen to be at most logarithmically growing in y by the properties of the Poisson equations (A.1) and (A.3) and the assumed boundedness of f and Λ. The result follows from the Maximum Principle because smoothness of the payoff implies $P^{(2)}$ and $P^{(3)}$ are smooth with bounded derivatives.

When the payoff is only continuous as in the case of the barrier call option here, the argument of [10] can be adapted to show that

$$|P(t,x,y) - (P^{(0)}(t,x) + \widetilde{P^{(1)}}(t,x))| = \mathcal{O}(\varepsilon^{1-p})$$

for any $p > 0$. This involves a regularization of the payoff, which can be conveniently done by replacing the nonsmooth call payoff $(x-K)^+$ by the Black-Scholes barrier option price $P^{(0)}(T-\delta, x; \bar{\sigma})$ a small time $\delta > 0$ from maturity. This payoff is smooth and zero at the barrier $x = B$, and we can utilize the explicit Black-Scholes barrier option pricing formula (2.9) to easily estimate the blow-up rates of derivatives at $x = K$ as $\delta \downarrow 0$ and $t \to T$.

The important point is that the barrier price $P^{(0)}$ is smooth in $x > B$ and its derivatives have finite limits as $x \to B^+$. Therefore the presence of the knock-out barrier introduces no further complications.

The only further adaptation to the proof in [10] that needs to be made is in showing that the solution of the regularized problem converges to the solution of the unregularized problem as $\delta \downarrow 0$ at a rate independent of ε. This can be achieved by a rotation of co-ordinates so that the two solutions can be written as expectations of functionals of independent processes (ξ, Y), where $\xi = X - F(Y)$ and $F' = \frac{\sqrt{\varepsilon}\rho}{\nu\sqrt{2}} f$, stopped on a curved boundary. (Such a transformation is not computationally convenient but is useful to derive regularity properties). The result follows by conditioning on the subordinating process Y and ε-independent moments of this process.

3. Lookback Options. *Lookback options* are path-dependent options whose payoff depends on the realized maximum or minimum of the underlying asset price during the life of the option. One example of this class of options is the *floating strike lookback put* which pays the difference of the realized maximum of the underlying asset during the option's life and the asset price itself at the expiration time T. Its payoff can be expressed as

$$h(X_T) = J_T - X_T,$$

where we define the running maximum J_t as

$$J_t = \max_{0 \le s \le t} X_s.$$

Pricing equations for lookback options in the Black-Scholes constant volatility model were first given and solved in [11]. A combination of a lookback call (paying the difference between the terminal stock price and the minimum) and a lookback put can be used to model trading strategies employed by many trend-following hedge funds, as discussed in [3] for example.

In a stochastic volatility environment, the price $P(t,x,J,y)$ of this option satisfies

$$\mathcal{L}^\varepsilon P = 0 \quad \text{in } x < J \text{ and } t < T,$$

with a terminal condition $P(T,x,J,y) = J - x$, and a boundary condition at $x = J$, $P_J(t,J,J,y) = 0$. The derivation of the boundary condition is given in [11], and it expresses the fact that the price of the lookback option for $X_t = J_t$ is insensitive to the small changes in J_t because the realized maximum at time T is larger than the realized maximum at time t with probability one.

The problem of finding $P(t,x,J,y)$ can be reduced to a two (space) dimensional boundary value problem with the following similarity reduction:

$$\xi = x/J, \quad \text{and} \quad P(t,x,J,y) = JQ(t,\xi,y).$$

We can express $Q(t,\xi,y)$ as the solution of

$$\begin{aligned} \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) Q &= 0, \quad \text{for } \xi < 1 \text{ and } t < T, \\ Q(T,\xi,y) &= 1-\xi, \\ (Q_\xi - Q)(t,1,y) &= 0, \end{aligned}$$

where, in a slight abuse of notation, we redefine $\mathcal{L}_1$ and $\mathcal{L}_2$ as the same as (2.6) and (2.7), but with ξ replacing x.

3.1. Asymptotic Approximation. Our approximation for the lookback price is

$$Q(t,\xi,y) \approx Q^{(0)}(t,\xi) + \widetilde{Q^{(1)}}(t,\xi),$$

where $P^{(0)}(t,x,J) = JQ^{(0)}(t,x/J)$ is the Black-Scholes price of the option with volatility parameter $\bar{\sigma}$. That is, following the argument in Appendix A, $Q^{(0)}$ solves

$$\begin{aligned} \text{(3.1)}\ \langle\mathcal{L}_2\rangle Q^{(0)} = Q_t^{(0)} + \frac{1}{2}\bar{\sigma}^2\xi^2 Q_{\xi\xi}^{(0)} + r\left(\xi Q_\xi^{(0)} - Q^{(0)}\right) &= 0 \quad \text{in } \xi < 1 \text{ and } t < T, \\ Q^{(0)}(T,\xi) &= 1-\xi, \\ \left(Q_\xi^{(0)} - Q^{(0)}\right)(t,1) &= 0. \end{aligned}$$

The correction term solves the analog of (A.5), namely

$$\text{(3.2)} \qquad \widetilde{Q_t^{(1)}} + \frac{1}{2}\bar{\sigma}^2\xi^2\widetilde{Q_{\xi\xi}^{(1)}} + r\left(\xi\widetilde{Q_\xi^{(1)}} - \widetilde{Q^{(1)}}\right) = \mathcal{A}Q^{(0)} \quad \text{in } \xi < 1 \text{ and } t < T,$$

with $\widetilde{Q^{(1)}}(T,\xi) = 1-\xi$ and

$$\left(\widetilde{Q^{(1)}} - \widetilde{Q_\xi^{(1)}}\right)(t,1) = 0.$$

Here, the operator $\mathcal{A}$ is as in (2.10), but with ξ replacing x.

3.2. Zero-Order Term. Although the pricing formula $P^{(0)}(t,x,J)$ for a lookback put is well-known, we will start by deriving $P^{(0)}(t,x,J)$, as the transformations will also be useful in the derivation of $\widetilde{P^{(1)}}(t,x,J) = J\widetilde{Q^{(1)}}(t,x/J)$.

The PDE problem (3.1) can be transformed to a PDE with constant coefficients by using logarithmic variables. That is defining

$$\eta = \log\xi, \quad u^{(0)}(t,\eta) = Q^{(0)}(t,\xi),$$

we find $u^{(0)}(t,\eta)$ to satisfy

$$\text{(3.3)} \qquad u_t^{(0)} + \frac{1}{2}\bar{\sigma}^2 u_{\eta\eta}^{(0)} + \left(r - \frac{1}{2}\bar{\sigma}^2\right)u_\eta^{(0)} - ru^{(0)} = 0 \quad \text{in } \eta < 0 \text{ and } t < T,$$

with the conditions

$$u^{(0)}(T,\eta) = 1 - e^{\eta},$$
$$\left(u_\eta^{(0)} - u^{(0)}\right)(t,0) = 0.$$

We first find $w^{(0)}(t,\eta) = u_\eta^{(0)}(t,\eta) - u^{(0)}(t,\eta)$ which solves the following (Dirichlet) boundary value problem,

$$w_t^{(0)} + \frac{1}{2}\bar{\sigma}^2 w_{\eta\eta}^{(0)} + \left(r - \frac{1}{2}\bar{\sigma}^2\right) w_\eta^{(0)} - rw^{(0)} = 0 \quad \text{in } \eta < 0 \text{ and } t < T,$$

with the conditions

$$w^{(0)}(T,\eta) = -1,$$
$$w^{(0)}(t,0) = 0.$$

The solution for $w^{(0)}(t,\eta)$ can be found via method of images:

$$w^{(0)}(t,\eta) = e^{-r(T-t)}\left[e^{(1-k)\eta} N\left(c_1(T-t)\right) - N\left(c_2(T-t)\right)\right] \tag{3.4}$$

where

$$c_1(\tau) = \frac{\eta}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}(1-k)\bar{\sigma}\sqrt{\tau} \quad \text{and} \quad c_2(\tau) = \frac{-\eta}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}(1-k)\bar{\sigma}\sqrt{\tau}.$$

To recover $u^{(0)}(t,\eta)$ from (3.4), we use the relationship

$$u^{(0)}(t,\eta) = \int_0^\eta e^{\eta - z} w^{(0)}(t,z)\mathrm{d}z + e^{\eta} u^{(0)}(t,0), \tag{3.5}$$

and to find the initial condition $u^{(0)}(t,0)$, we substitute (3.5) into (3.3), set $\eta = 0$, and conclude that $u^{(0)}(t,0)$ should satisfy

$$u_t^{(0)}(t,0) = -\frac{1}{2}\bar{\sigma}^2 w_\eta^{(0)}(t,0).$$

Therefore $u^{(0)}(t,\eta)$ is given by

$$\begin{aligned} u^{(0)}(t,\eta) = {} & e^{-r(T-t)}\left[-k^{-1}e^{(1-k)\eta} N\left(c_1(T-t)\right) + N\left(c_2(T-t)\right)\right] \\ & + e^{\eta}\left[(1+k^{-1})N\left(c_3(T-t)\right) - 1\right], \end{aligned} \tag{3.6}$$

where

$$c_3(\tau) = \frac{\eta}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}\bar{\sigma}(1+k)\sqrt{\tau}.$$

Restoring all other transformations, we get, in the notation of [18],

$$P^{(0)}(t,x,J) = -x + x(1+k^{-1})N(d_7) + Je^{-r(T-t)}\left(N(d_5) - k^{-1}\left(\frac{x}{J}\right)^{1-k} N(d_6)\right),$$

where

$$d_5 = \frac{\log(J/x) - \left(r - \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$
$$d_6 = \frac{\log(x/J) - \left(r - \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$
$$d_7 = \frac{\log(x/J) + \left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}}.$$

3.3. First-Order Correction. Analogous to the zero-order calculation, we define

$$\eta = \log \xi, \qquad u^{(1)}(t,\eta) = \widetilde{Q^{(1)}}(t,\xi),$$

and, from (3.2), find $u^{(1)}(t,\eta)$ to satisfy

$$u_t^{(1)} + \frac{1}{2}\bar{\sigma}^2 u_{\eta\eta}^{(1)} + \left(r - \frac{1}{2}\bar{\sigma}^2\right) u_\eta^{(1)} - r u^{(1)} = \tilde{\mathcal{A}} u^{(0)} \quad \text{in } \eta < 0 \text{ and } t < T, \tag{3.7}$$
$$u^{(1)}(T,\eta) = 0,$$
$$\left(u_\eta^{(1)} - u^{(1)}\right)(t,0) = 0,$$

where

$$\tilde{\mathcal{A}} = V_3^\varepsilon \left(\frac{\partial^3}{\partial \eta^3} - \frac{\partial^2}{\partial \eta^2}\right) + V_2^\varepsilon \left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial}{\partial \eta}\right).$$

Defining $\mathcal{L}_{LB}$ by

$$\mathcal{L}_{LB} = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2}{\partial \eta^2} + \left(r - \frac{1}{2}\bar{\sigma}^2\right)\frac{\partial}{\partial \eta} - r\cdot,$$

we can verify by differentiating (3.3) with respect to $\bar{\sigma}$ that

$$\mathcal{L}_{LB} u_{\bar{\sigma}}^{(0)} = -\bar{\sigma}(u_{\eta\eta}^{(0)} - u_\eta^{(0)}) \quad \text{and} \quad \mathcal{L}_{LB} u_{\eta\bar{\sigma}}^{(0)} = -\bar{\sigma}(u_{\eta\eta\eta}^{(0)} - u_{\eta\eta}^{(0)})$$

with $u_{\bar{\sigma}}^{(0)}(T,\eta) = (u_{\eta\bar{\sigma}}^{(0)} - u_{\bar{\sigma}}^{(0)})(t,0) = 0$, and with $u_{\eta\bar{\sigma}}^{(0)}(T,\eta) = 0$, but $(u_{\eta\eta\bar{\sigma}}^{(0)} - u_{\eta\bar{\sigma}}^{(0)})(t,0) \neq 0$ in general.

This motivates us to define $\hat{u}(t,\eta)$ by

$$\frac{V_3^\varepsilon}{\bar{\sigma}}\hat{u} = u^{(1)} + \frac{1}{\bar{\sigma}}\left(V_3^\varepsilon u_{\eta\bar{\sigma}}^{(0)} + V_2^\varepsilon u_{\bar{\sigma}}^{(0)}\right).$$

We find that $\hat{u}$ solves

$$\hat{u}_t + \frac{1}{2}\bar{\sigma}^2 \hat{u}_{\eta\eta} + \left(r - \frac{1}{2}\bar{\sigma}^2\right)\hat{u}_\eta - r\hat{u} = 0 \quad \text{in } \eta < 0 \text{ and } t < T,$$
$$\hat{u}(T,\eta) = 0,$$
$$(\hat{u}_\eta - \hat{u})(t,0) = g(t),$$

where we define

$$g(t) = (u_{\eta\eta\bar{\sigma}}^{(0)} - u_{\eta\bar{\sigma}}^{(0)})\big|_{\eta=0} = w_{\eta\bar{\sigma}}^{(0)}\big|_{\eta=0}. \tag{3.8}$$

Defining $\hat{w} = \hat{u}_\eta - \hat{u}$, $\hat{w}(t,\eta)$ solves the Dirichlet boundary value problem

$$\hat{w}_t + \frac{1}{2}\bar{\sigma}^2 \hat{w}_{\eta\eta} + \left(r - \frac{1}{2}\bar{\sigma}^2\right)\hat{w}_\eta - r\hat{w} = 0 \quad \text{in } \eta < 0 \text{ and } t < T,$$
$$\hat{w}(T,\eta) = 0,$$
$$\hat{w}(t,0) = g(t).$$

Following the analysis leading to (2.18), we can write

$$\hat{w}(t,\eta) = -\frac{\eta e^\eta}{\bar{\sigma}\sqrt{2\pi}} \int_t^T e^{-\frac{1}{2}c_3(s-t)^2} \frac{g(s)}{(s-t)^{3/2}}\,\mathrm{d}s. \tag{3.9}$$

This formula, together with a Taylor expansion of g, yields

$$\hat{w}_\eta(t,0) = \frac{1}{2}(1-k)g(t).$$

To recover $\hat{u}(t,\eta)$, we use

$$\hat{u}(t,\eta) = \int_0^\eta e^{\eta - z}\hat{w}(t,z)\,\mathrm{d}z + e^\eta h(t), \tag{3.10}$$

where

$$h'(t) = -\frac{1}{2}\bar{\sigma}^2 \hat{w}_\eta(t,0) - rg(t), \qquad h(T) = 0.$$

Therefore

$$h(t) = \frac{1}{2}\left(r + \frac{1}{2}\bar{\sigma}^2\right)\int_t^T g(s)\,ds.$$

After some computation, we obtain

$$\begin{aligned}\hat{u}(t,\eta) = {} & \bar{\sigma}e^\eta \int_t^T \frac{g(s)}{\sqrt{2\pi(s-t)}}\left(e^{-\frac{1}{2}c_3(s-t)^2} - e^{-\frac{1}{2}\tilde{c}_3(s-t)^2}\right)\mathrm{d}s \\ & - \frac{1}{2}(k+1)\bar{\sigma}^2 \int_t^T \left[N(c_3(s-t)) - N(\tilde{c}_3(s-t))\right] g(s)\mathrm{d}s \\ & + \frac{1}{\bar{\sigma}}(k+1)e^\eta \left[1 - e^{-r(T-t)}N(\tilde{c}_4(T-t)) - N(\tilde{c}_3(T-t))\right],\end{aligned}$$

where

$$\tilde{c}_3(\tau) = \frac{1}{2}(1+k)\bar{\sigma}\sqrt{\tau} \quad \text{and} \quad \tilde{c}_4(\tau) = \frac{1}{2}(1-k)\bar{\sigma}\sqrt{\tau}.$$

In the original variables we can write

$$\widetilde{P^{(1)}}(t,x,J) = -\frac{V_3^\varepsilon}{\bar{\sigma}} x P^{(0)}_{x\bar{\sigma}}(t,x) - \frac{V_2^\varepsilon}{\bar{\sigma}} P^{(0)}_{\bar{\sigma}}(t,x) + \frac{V_3^\varepsilon}{\bar{\sigma}} J\hat{u}(t,\log(x/J)). \tag{3.11}$$

Explicit formulas for the Greeks and g are given in Appendix C.

Figure 3.1 illustrates the effect of the two parts of the correction, and for various skew slopes.

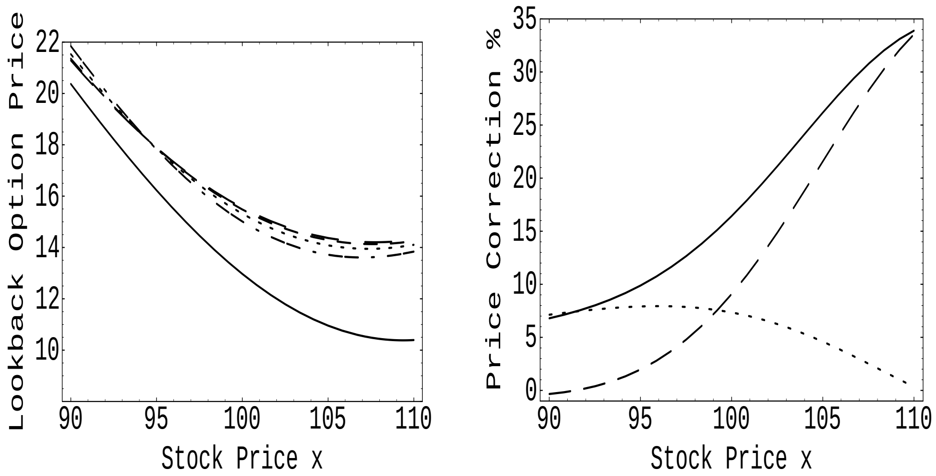

FIG. 3.1. The left graph shows the effect of changing the slope of the skew a on the lookback put option price. The parameters of the contract are T=.5, $\bar{\sigma}$=0.17,b= 0.23. The current running maximum is J=111. The solid line shows the corresponding Black-Scholes price, the values of a reading downwards at the right of the graph are $a = -0.02, -0.04, -0.09, -.18$. When the stock price is near to its running maximum, making a more negative decreases the option price. The right graph shows the percentage of first order correction to the Black-Scholes price for the lookback put option at time t=0. The parameters of the contract are as in the left figure with $a = -0.154$. The solid line shows the whole first order correction, the dashed line shows the contribution of the Greek terms in (3.11) and the dotted line shows the remainder, i.e. the boundary correction.

3.4. Convergence. From (3.6), second and higher derivatives of $u^{(0)}$ with respect to η blow up as $t \to T$ and $\eta \to 0$, similar to the Black-Scholes price of a European call option with log strike price equal to zero. Therefore the proof of a convergence result of the form

$$|P(t,x,J,y) - (P^{(0)}(t,x,J) + \widetilde{P^{(1)}}(t,x,J))| = \mathcal{O}(\varepsilon^{1-p})$$

for any $p > 0$ at a fixed point (t,x,J,y) requires the regularization techniques in [10], discussed in Section 2.4.

4. Passport Options. A passport option allows its holder to trade the stock continuously, starting with initial capital v, and collect his or her profit at the expiration date T, if any, with losses written off. Its price is studied by Hyer et. al. [13] where they assumed a log-normal process for the underlying. They derive and solve the Hamilton-Jacobi-Bellman equation for the price. Shreve and Vecer [17] used probabilistic techniques to price this option as well as other variants. Henderson and Hobson [12] analyzed passport option pricing under stochastic volatility models where they assume independence of the volatility driving process from the stock price process. They give the price analytically using power series expansion methods for different volatility models.

Let $(q_t)_{0\le t\le T}$ be a possible trading strategy, where q_t is the number of stocks held in the trading account at time t. Additionally,

$$-1 \le q_t \le 1$$

at all times, so the trader is restricted to be at most long or short one stock at any time. Let (V_t) be the value of the holder's portfolio so that

$$dV_t = rV_t\,dt + q_t f(Y_t)X_t\,dW_t^\star, \tag{4.1}$$

written in terms of the risk-neutral Brownian motion $W^\star$ because cash flows are priced under $\mathbb{P}^{(\gamma)}$. The payoff of the passport option is simply V_T^+ and and so the *no arbitrage* pricing function $P(t,x,y,v)$ of the contract is given by

$$P(t,x,y,v) = \sup_{|q|\le 1} \mathbb{E}^{\star(\gamma)}\{e^{-r(T-t)}V_T^+ \mid X_t = x, Y_t = y, V_t = v\}.$$

This is conjectured to solve the Hamilton-Jacobi-Bellman PDE

$$\frac{\partial P}{\partial t} + \sup_{|q|\le 1} \mathcal{L}^{(q)}_{x,y,v} P = 0,$$

where $\mathcal{L}^{(q)}_{x,y,v}$ is the infinitesimal generator of (X,Y,V), plus the discounting term:

$$\begin{aligned}\mathcal{L}^{(q)}_{x,y,v} &= \frac{1}{2}f(y)^2x^2\left(\frac{\partial^2}{\partial x^2} + 2q\frac{\partial^2}{\partial x\partial v} + q^2\frac{\partial^2}{\partial v^2}\right)\\ &+\rho\frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}f(y)x\left(\frac{\partial^2}{\partial x\partial y} + q\frac{\partial^2}{\partial y\partial v}\right) + \frac{1}{2}\frac{\nu^2}{\varepsilon}\frac{\partial^2}{\partial y^2}\\ &+r\left(x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}\right) + \left(\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial}{\partial y} - r\,\cdot\,.\end{aligned}$$

The terminal condition is

$$P(T,x,y,v) = v^+,$$

and the domain is $t < T$, $x > 0$, $-\infty < y, v < \infty$.

4.1. Similarity Reduction. We first take advantage of a natural homogeneity in the problem. From (2.2) and (4.1), we see that scaling X and V by a common factor, say θ

$$X \mapsto \theta X \qquad V \mapsto \theta V$$

does not change those equations. In other words,

$$P(t,\theta x,y,\theta v) = \theta P(t,x,y,v),$$

and so we look for a solution of the form

$$P(t,x,y,v) = xQ(t,\xi,y); \qquad \xi = v/x,$$

for some function Q.

Substituting this form gives that $Q(t,\xi,y)$ solves the PDE problem

$$\begin{aligned}Q_t + \sup_{|q|\le 1}&\left\{\frac{1}{2}f(y)^2(q-\xi)^2Q_{\xi\xi} + \rho\frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}f(y)(q-\xi)Q_{\xi y}\right\}\\ &+\frac{1}{2}\frac{\nu^2}{\varepsilon}Q_{yy} + \left[\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(y) + \rho\frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}f(y)\right]Q_y = 0,\\ &Q(T,\xi,y) = \xi^+,\end{aligned} \tag{4.2}$$

in the domain $t < T$, $-\infty < \xi, y < \infty$. Interestingly, r has disappeared from the problem.

Consider the quadratic (in q) term in (4.2)

$$\frac{1}{2} f(y)^2 (q-\xi)^2 Q_{\xi\xi} + \rho \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} f(y)(q-\xi) Q_{\xi y}.$$

Assuming that $Q_{\xi\xi} > 0$ for $t < T$, the maximum of this quadratic over $q \in [-1, 1]$ is at the boundaries:

$$q^* = \pm 1$$

at each point in the domain. Therefore $(q^*)^2 = 1$ and the sup term in (4.2) can be replaced by

$$\frac{1}{2} f(y)^2 (1+\xi^2) Q_{\xi\xi} - \rho \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} f(y) \xi Q_{\xi y} + f(y)^2 \left| \xi Q_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon} f(y)} Q_{\xi y} \right|.$$

Let $R(t, \xi, y)$ be the solution to PDE (4.2) with terminal condition $R(T, \xi, y) = |\xi|$. It is then straightforward to verify that the function $\frac{1}{2}(\xi + R(t, \xi, y))$ verifies both the PDE and the terminal condition in (4.2), so we have

$$Q(t, \xi, y) = \frac{1}{2}(\xi + R(t, \xi, y)). \tag{4.3}$$

The PDE problem for R is therefore:

$$\begin{aligned} R_t + \frac{1}{2} f(y)^2 (1+\xi^2) R_{\xi\xi} - \rho \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} f(y) \xi R_{\xi y} + \frac{1}{2} \frac{\nu^2}{\varepsilon} R_{yy} \\ + \left[\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(y) + \rho \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} f(y) \right] R_y \\ + f(y)^2 \left| \xi R_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon} f(y)} R_{\xi y} \right| = 0. \\ R(T, \xi, y) = |\xi|. \end{aligned} \tag{4.4}$$

Observe that (4.4) is unchanged by the transformation $\xi \mapsto -\xi$. As a consequence, $R(t, \xi, y)$ is an even function of ξ. This property carries over to the first two terms of our expansion, where we will take advantage of it.

4.2. Asymptotic Expansion. We re-derive the asymptotic expansion for this option to highlight the differences with the calculation for the previous two cases given in Appendix A. See [15, 14] for approximations for related stochastic control problems.

The expansion is written here for R, but applies, with obvious modifications to the terminal condition, for Q. Under the usual fast mean-reversion scaling, we rewrite the PDE (4.2) as

$$\left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) R + \mathrm{NL}^\varepsilon = 0,$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are the linear differential operators

$$\begin{aligned}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \\
\mathcal{L}_1 &= -\sqrt{2}\,\rho\nu f(y)\xi \frac{\partial^2}{\partial \xi \partial y} + \sqrt{2}\,\nu(\rho f(y) - \Lambda(y))\frac{\partial}{\partial y} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 (1+\xi^2)\frac{\partial^2}{\partial \xi^2}
\end{aligned}$$

and NL^ε is the nonlinear term

$$\mathrm{NL}^\varepsilon = f(y)^2 \left| \xi R_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}\, f(y)} R_{\xi y} \right| .$$

We look for an expansion

$$R = R^{(0)} + \sqrt{\varepsilon} R^{(1)} + \varepsilon R^{(2)} + \cdots ,$$

valid for small $\varepsilon > 0$. Inserting the series and comparing terms of $\mathcal{O}(\varepsilon^{-1})$ gives

$$\mathcal{L}_0 R^{(0)} = 0,$$

which implies that $R^{(0)}$ does not depend on y, as explained in Appendix A. Hence in the expansion of NL^ε

$$\mathrm{NL}^\varepsilon = f(y)^2 \left| -\frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}\, f(y)} R^{(0)}_{\xi y} + \xi R^{(0)}_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{f(y)} R^{(1)}_{\xi y} + \mathcal{O}(\sqrt{\varepsilon}) \right| ,$$

the $\mathcal{O}(\varepsilon^{-1/2})$ disappears.

Comparing terms of $\mathcal{O}(\varepsilon^{-1/2})$ therefore gives

$$\mathcal{L}_0 R^{(1)} + \mathcal{L}_1 R^{(0)} = 0.$$

Since $\mathcal{L}_1$ takes y-derivatives, this reduces to

$$\mathcal{L}_0 R^{(1)} = 0,$$

which implies that $R^{(1)}$ also does not depend on y. Now

$$\mathrm{NL}^\varepsilon = f(y)^2 \left| \xi R^{(0)}_{\xi\xi} + \sqrt{\varepsilon} \left(\xi R^{(1)}_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{f(y)} R^{(2)}_{\xi y} \right) + \mathcal{O}(\varepsilon) \right| , \tag{4.5}$$

including the next order. The $\mathcal{O}(1)$ terms of the expansion in the PDE give

$$\mathcal{L}_0 R^{(2)} + \mathcal{L}_1 R^{(1)} + \mathcal{L}_2 R^{(0)} + f(y)^2 \left|\xi\right| R^{(0)}_{\xi\xi} = 0, \tag{4.6}$$

where we have assumed that $R^{(0)}_{\xi\xi} \geq 0$, that is the leading term inherits the convexity of R in ξ. The second term $\mathcal{L}_1 R^{(1)} = 0$ because $R^{(1)}$ does not depend on y. We view (4.6) as a Poisson equation for $R^{(2)}$. For there to be a solution, the source term must be centered with respect to the invariant distribution of the OU process (Y_t), namely

$$\left\langle \widetilde{\mathcal{L}_2} \right\rangle R^{(0)} = 0, \tag{4.7}$$

where we define

$$\widetilde{\mathcal{L}_2} = \mathcal{L}_2 + f(y)^2|\xi|\frac{\partial^2}{\partial\xi^2}$$
$$= \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2(1+|\xi|)^2\frac{\partial^2}{\partial\xi^2}.$$

The averaged operator simply replaces $f(y)^2$ by the constant $\bar{\sigma}^2$. Therefore $R^{(0)}$, when transformed back, gives the passport option pricing function with the constant long-run average volatility $\bar{\sigma}$. To move to the next order, we formally linearize the expression (4.5) for NL^ε:

$$\mathrm{NL}^\varepsilon = f(y)^2\left(|\xi|R^{(0)}_{\xi\xi} + \sqrt{\varepsilon}\ \mathrm{sgn}(\xi)\left[\xi R^{(1)}_{\xi\xi} - \frac{\rho\nu\sqrt{2}}{f(y)}R^{(2)}_{\xi y}\right]\right) + \mathcal{O}(\varepsilon).$$

Now comparing terms in the expanded PDE of $\mathcal{O}(\sqrt{\varepsilon})$ gives

$$\mathcal{L}_0R^{(3)} + \mathcal{L}_1R^{(2)} + \mathcal{L}_2R^{(1)} + \ \mathrm{sgn}(\xi)\left[\xi f(y)^2R^{(1)}_{\xi\xi} - \rho\nu\sqrt{2}f(y)R^{(2)}_{\xi y}\right] = 0.$$

This is a Poisson equation for $R^{(3)}$ whose solvability condition gives

$$\left\langle\widetilde{\mathcal{L}_2}\right\rangle R^{(1)} = -\left\langle(\mathcal{L}_1 - \rho\nu\sqrt{2}f\ \mathrm{sgn}(\xi)\frac{\partial^2}{\partial\xi\partial y})R^{(2)}\right\rangle$$
$$= \nu\sqrt{2}\left\langle\left(\rho f\ \mathrm{sgn}(\xi)(1+|\xi|)\frac{\partial^2}{\partial\xi\partial y} - (\rho f - \Lambda)\frac{\partial}{\partial y}\right)R^{(2)}\right\rangle. \tag{4.8}$$

As in Section 2.2 let $\phi(y)$ be a solution to $\mathcal{L}_0\phi = f(y)^2 - \bar{\sigma}^2$. Then (4.6) gives

$$R^{(2)} = -\frac{1}{2}\phi(y)(1+|\xi|)^2R^{(0)}_{\xi\xi} + D(t,\xi),$$

for some function D that does not depend on y. Substituting and computing the right-side (4.8) gives a combination of second- and third-derivatives of $R^{(0)}$ in the ξ variable.

As usual, we absorb the $\sqrt{\varepsilon}$ term into the correction and call

$$\widetilde{R^{(1)}} = \sqrt{\varepsilon}R^{(1)}.$$

Then $\widetilde{R^{(1)}}(t,\xi)$ solves

$$\widetilde{R^{(1)}_t} + \frac{1}{2}\bar{\sigma}^2(1+|\xi|)^2\widetilde{R^{(1)}_{\xi\xi}} = -(V_3^\varepsilon - V_2^\varepsilon)(1+|\xi|)^2R^{(0)}_{\xi\xi} - V_3^\varepsilon\ \mathrm{sgn}(\xi)(1+|\xi|)^3R^{(0)}_{\xi\xi\xi}, \tag{4.9}$$

where V_2^ε and V_3^ε are the market group parameters defined in (2.11) and (2.12). The terminal condition is

$$\widetilde{R^{(1)}}(T,\xi) = 0.$$

4.3. Zero-Order Term. Again we start by finding the zero-order approximation. We work with $R(t,\xi,y)$ and recover $Q(t,\xi,y)$ using (4.3). Thus $R^{(0)}$ satisfies

$$R_t^{(0)} + \frac{1}{2}\bar{\sigma}^2(1+|\xi|)^2 R_{\xi\xi}^{(0)} = 0 \quad \text{in } -\infty < \xi < \infty \text{ and } t < T, \tag{4.10}$$

$$R^{(0)}(T,\xi) = |\xi|. \tag{4.11}$$

Thus $R^{(0)}(t,\cdot)$ is even at all times, so by the smoothing properties of (4.10) we have $R_\xi^{(0)}(t,0) = 0$ for $t < T$. Hence we can solve

$$\begin{aligned} R_t^{(0)} + \frac{1}{2}\bar{\sigma}^2(1+\xi)^2 R_{\xi\xi}^{(0)} &= 0 \quad \text{in } \xi > 0 \text{ and } t < T, \\ R^{(0)}(T,\xi) &= \xi \\ R_\xi^{(0)}(t,0) &= 0 \end{aligned} \tag{4.12}$$

and obtain the solution in $\xi < 0$ by the even extension.

We transform to constant coefficients via

$$\eta = \log(1+\xi), \quad R^{(0)}(t,\xi) = u^{(0)}(t,\eta).$$

Then $u^{(0)}(t,\eta)$ solves the Neumann boundary value problem

$$\begin{aligned} u_t^{(0)} + \frac{1}{2}\bar{\sigma}^2(u_{\eta\eta}^{(0)} - u_\eta^{(0)}) &= 0 \quad \text{in } \eta > 0 \text{ and } t < T, \\ u^{(0)}(T,\eta) &= e^\eta - 1, \\ u_\eta^{(0)}(t,0) &= 0. \end{aligned} \tag{4.13}$$

We first find the partial derivative $w^{(0)} = u_\eta^{(0)}$, which solves a Dirichlet boundary value problem:

$$\begin{aligned} w_t^{(0)} + \frac{1}{2}\bar{\sigma}^2\left(w_{\eta\eta}^{(0)} - w_\eta^{(0)}\right) &= 0 \quad \text{in } \eta > 0 \text{ and } t < T, \\ w^{(0)}(T,\eta) &= e^\eta, \\ w^{(0)}(t,0) &= 0. \end{aligned} \tag{4.14}$$

By the method of images,

$$w^{(0)}(t,\eta) = e^\eta N(c_5(T-t)) - N(c_6(T-t)),$$

where

$$c_5(\tau) = \frac{\eta}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}\bar{\sigma}\sqrt{\tau} \quad \text{and} \quad c_6(\tau) = \frac{-\eta}{\bar{\sigma}\sqrt{\tau}} + \frac{1}{2}\bar{\sigma}\sqrt{\tau}. \tag{4.15}$$

As in the lookback option case, $u^{(0)}(t,\eta)$ can be recovered from $w^{(0)}(t,\eta)$ to give

$$\begin{aligned} u^{(0)}(t,\eta) = {}& e^\eta N(c_5(T-t)) - N(-c_6(T-t)) \\ & + \bar{\sigma}\sqrt{T-t}\left(N'(c_6(T-t)) + c_6(T-t)N(c_6(T-t))\right). \end{aligned}$$

Restoring all the transformations gives $P^{(0)}(t,x,v)$ as

$$P^{(0)} = \frac{1}{2}\left[v + xu^{(0)}\left(t, \log\left(1 + \frac{|v|}{x}\right)\right)\right]$$

which can be written in the notation of [17] as

$$P^{(0)} = \frac{1}{2}\left[v + (x+|v|)N(d_+) - xN(d_-) + x\bar{\sigma}\sqrt{T-t}\,N'(d_-) - x\bar{\sigma}\sqrt{T-t}\,d_-N(-d_-)\right],$$

where

$$d_\pm = \frac{\log\left(1+\frac{|v|}{x}\right)}{\bar{\sigma}\sqrt{T-t}} \pm \frac{1}{2}\bar{\sigma}\sqrt{T-t}.$$

4.4. First-Order Correction. The first order correction $\widetilde{R^{(1)}}$ satisfies the PDE

$$\widetilde{R^{(1)}_t} + \frac{1}{2}\bar{\sigma}^2(1+|\xi|)^2\widetilde{R^{(1)}_{\xi\xi}} = -(V_3^\varepsilon - V_2^\varepsilon)(1+|\xi|)^2 R^{(0)}_{\xi\xi} - V_3^\varepsilon(1+|\xi|)^3 R^{(0)}_{\xi\xi\xi},$$

with terminal condition $\widetilde{R^{(1)}}(T,\xi) = 0$. Thus $\widetilde{R^{(1)}}$ is again an even function of ξ,and we can solve

$$\widetilde{R^{(1)}_t} + \frac{1}{2}\bar{\sigma}^2(1+\xi)^2\widetilde{R^{(1)}_{\xi\xi}} = -(V_3^\varepsilon - V_2^\varepsilon)(1+\xi)^2 R^{(0)}_{\xi\xi} - V_3^\varepsilon(1+\xi)^3 R^{(0)}_{\xi\xi\xi},$$

with the terminal condition $\widetilde{R^{(1)}}(T,\xi) = 0$ and the boundary condition $\widetilde{R^{(1)}_\xi}(t,0) = 0$ in $\xi > 0, t < T$. We will subtract the "particular solution" at a later stage when it becomes easier to identify.

Applying the same set of transformations, namely

$$\eta = \log(1+\xi), \qquad \widetilde{R^{(1)}}(t,\xi) = u^{(1)}(t,\eta),$$

we get from (4.6) that

$$\begin{aligned} u^{(1)}_t + \frac{1}{2}\bar{\sigma}^2\left(u^{(1)}_{\eta\eta} - u^{(1)}_\eta\right) &= \tilde{\mathcal{A}}u^{(0)} \quad \text{in } \eta > 0 \text{ and } t < T, \\ u^{(1)}(T,\eta) &= 0, \\ u^{(1)}_\eta(t,0) &= 0, \end{aligned} \tag{4.16}$$

where

$$\tilde{\mathcal{A}} = -V_3^\varepsilon\left(\frac{\partial^3}{\partial\eta^3} - \frac{\partial^2}{\partial\eta^2}\right) + (V_2^\varepsilon + V_3^\varepsilon)\left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial}{\partial\eta}\right).$$

Defining $\mathcal{L}_p$ by

$$\mathcal{L}_p = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2\left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial}{\partial\eta}\right),$$

we can verify by differentiating (4.13) that

$$\mathcal{L}_p u^{(0)}_{\bar{\sigma}} = -\bar{\sigma}(u^{(0)}_{\eta\eta} - u^{(0)}_\eta) \quad \text{and} \quad \mathcal{L}_p u^{(0)}_{\eta\bar{\sigma}} = -\bar{\sigma}(u^{(0)}_{\eta\eta\eta} - u^{(0)}_{\eta\eta}).$$

Moreover, $u^{(0)}_{\bar{\sigma}} = u^{(0)}_{\eta\bar{\sigma}} = u^{(0)}_{\eta\eta\bar{\sigma}} = 0$ for $t = T$ and $u^{(0)}_{\eta\bar{\sigma}}(t,0) = 0$ but $u^{(0)}_{\eta\eta\bar{\sigma}}(t,0) \neq 0$ in general. This motivates us to define $\hat{u}$ by

$$\frac{V_3^\varepsilon}{\bar{\sigma}}\hat{u} = u^{(1)} - \frac{V_3^\varepsilon}{\bar{\sigma}}u^{(0)}_{\eta\bar{\sigma}} - \left(\frac{V_2^\varepsilon}{\bar{\sigma}} + \frac{V_3^\varepsilon}{\bar{\sigma}}\right)u^{(0)}_{\bar{\sigma}}.$$

Further, defining $\hat{w} = \hat{u}_\eta$ to reduce to a Dirichlet boundary value problem, we find that $\hat{w}(t,\eta)$ solves

$$\begin{aligned} \hat{w}_t + \frac{1}{2}\bar{\sigma}^2 \left(\hat{w}_{\eta\eta} - \hat{w}_\eta\right) = 0 \quad \text{in } \eta > 0 \text{ and } t < T, \\ \hat{w}(T,z) = 0, \\ \hat{w}(t,0) = g(t), \end{aligned}$$

where

$$g(t) = -u^{(0)}_{\eta\eta\bar{\sigma}}\big|_{\eta=0} = -w^{(0)}_{\eta\bar{\sigma}}\big|_{\eta=0}. \tag{4.17}$$

We now follow the analysis in Section 3.3. First, the analogue of (3.9) is

$$\hat{w}(t,\eta) = \frac{\eta}{\bar{\sigma}\sqrt{2\pi}} \int_t^T e^{-\frac{1}{2}c_6(s-t)^2} \frac{g(s)}{(s-t)^{3/2}} \mathrm{d}s,$$

which in particular gives

$$\hat{w}_\eta(t,0) = \frac{1}{2} g(t).$$

The solution for $\hat{u}(t,\eta)$ is recovered from

$$\hat{u}(t,\eta) = \int_0^\eta \hat{w}(t,z)\,\mathrm{d}z + h(t), \tag{4.18}$$

where

$$h'(t) = \frac{1}{2}\bar{\sigma}^2 \left(g(t) - \hat{w}_\eta(t,0)\right), \qquad h(T) = 0.$$

Thus

$$h(t) = -\frac{1}{4}\bar{\sigma}^2 \int_t^T g(s)\mathrm{d}s.$$

The solution for $\hat{u}(t,\eta)$ can be obtained as

$$\begin{aligned} \hat{u}(t,\eta) = \bar{\sigma} \int_t^T & \frac{g(s)}{\sqrt{2\pi(s-t)}} \left(e^{-\frac{1}{2}\tilde{c}_6(s-t)^2} - e^{-\frac{1}{2}c_6(s-t)^2}\right) \mathrm{d}s \\ &+ \frac{1}{2}\bar{\sigma}^2 \int_t^T g(s)\left(N(-c_6(s-t)) - N(\tilde{c}_6(s-t))\right) \mathrm{d}s \\ &+ \frac{1}{\bar{\sigma}} \left(\frac{1}{2} - N(-\tilde{c}_6(T-t))\right), \end{aligned}$$

where c_6 is defined in (4.15) and $\tilde{c}_6(\tau) = -\frac{1}{2}\bar{\sigma}\sqrt{\tau}$.

By restoring all transformations we obtain the first-order correction for the passport option as

$$\begin{aligned} \widetilde{P^{(1)}}(t,x,v) = &- \frac{V_3^\varepsilon}{\bar{\sigma}} \left(1 + \frac{x}{|v|}\right) x P^{(0)}_{x\bar{\sigma}} - \left(\frac{V_2^\varepsilon}{\bar{\sigma}} - \frac{V_3^\varepsilon}{\bar{\sigma}}\frac{x}{|v|}\right) P^{(0)}_{\bar{\sigma}} \\ &+ \frac{1}{2}\frac{V_3^\varepsilon}{\bar{\sigma}} x \hat{u}\left(t, \log\left(1 + \frac{|v|}{x}\right)\right). \end{aligned} \tag{4.19}$$

Explicit formulas for the Greeks and g are given in Appendix D.

Figures 4.1 and 4.2 illustrate the effect of the correction and the slope of the implied volatility skew.

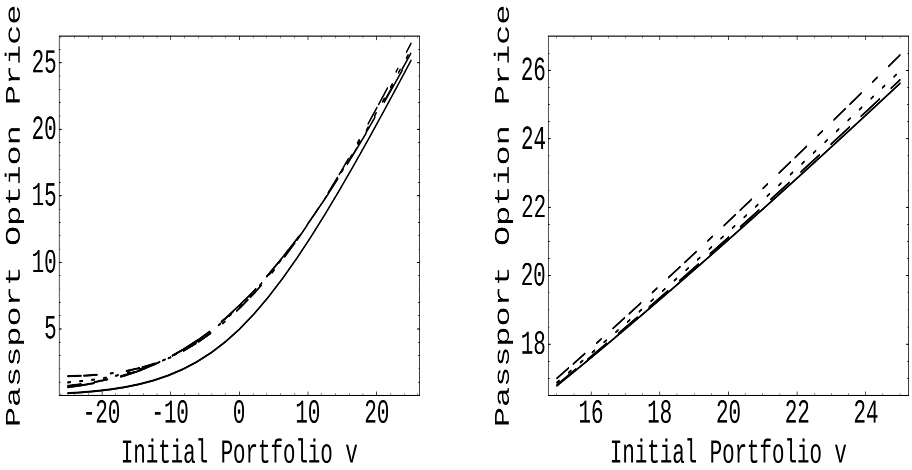

FIG. 4.1. The left graph illustrates the effect of changing the slope of the skew a on the passport option price. The parameters of the contract are $x = 100$, $T = .5$, $\bar{\sigma} = 0.17$, and $b = 0.23$. As $|v|$ gets larger, making a more negative increases the option value, while this effect reverses as $|v|$ gets closer to 0. The right figure shows more closely the upper right corner of the left figure. The solid line shows the corresponding Black-Scholes price, the values of a reading upwards after the Black-Scholes pricing curve are $a = -0.02, -0.04, -0.09, -.18$.

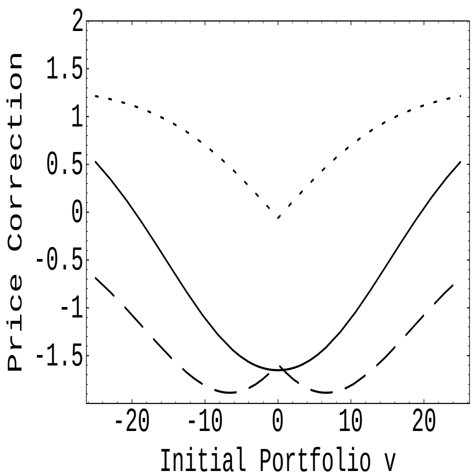

FIG. 4.2. The first order correction for passport option at time t=0. The parameters of the contract are as in Figure 4.1 with $a = -0.154$. The solid line shows the full first order correction, the dashed line shows the contribution of the Greek terms in (4.19) and the dotted line shows the remainder, i.e. the boundary correction.

4.5. Convergence. The existence of a smooth solution to (4.4) is an open question. If such a solution exists so that $R_\xi = 0$ at $\xi = 0$ for $t < T$, then one can write a *nonlinear* error equation for Z^ε defined by

$$R = R^{(0)} + \sqrt{\varepsilon}R^{(1)} + \varepsilon R^{(2)} + \varepsilon^{3/2}R^{(3)} - Z^\varepsilon$$

with suitable terminal and boundary conditions. Under strong regularity hypotheses on the solution, a *weak* convergence result may be obtained by introducing test functions in (ξ, y). The convergence is necessarily weak in this case because of the sense in which the expansion of the $|\cdot|$ function can be used.

Appendix A. Asymptotic Calculation. We present the formal asymptotic expansion that yields the PDE problems to be solved for the Black-Scholes price and stochastic volatility correction for the first two exotic options considered here. Further details are presented in [6]. In the case of the passport option, the argument is more involved because of the embedded nonlinear optimization problem.

Let $P(t,x,y)$ solve $\mathcal{L}^\varepsilon P = 0$ in the domain $t < T$, $y \in I\!R$ and $x > x_0$, where $\mathcal{L}^\varepsilon$ is of the form (2.4). The key features are

- $\mathcal{L}_0$, given in (2.5), is the generator of an OU process;
- $\mathcal{L}_1$ takes derivatives in y and so kills any function that does not depend on y;
- $\mathcal{L}_2$ takes derivatives in t and x, but not y. It has coefficients that depend on y and an associated boundary condition at $x = x_0$ which does not depend on ε or y.
- The terminal condition at $t = T$ does not depend on ε or y.

Inserting an expansion $P = P_0 + \sqrt{\varepsilon}P^{(1)} + \varepsilon P^{(2)} + \cdots$, and comparing powers of ε^{-1} gives $\mathcal{L}_0 P^{(0)} = 0$. This is an ODE in y, and from the properties of $\mathcal{L}_0$, the only solutions with reasonable growth at infinity are constants in y. Therefore we take $P^{(0)} = P^{(0)}(t,x)$. Similarly, comparing terms of order $\varepsilon^{-1/2}$, we conclude $P^{(1)}$ also does not depend on y.

The order 1 terms give

$$\mathcal{L}_0 P^{(2)} + \mathcal{L}_2 P^{(0)} = 0, \tag{A.1}$$

which is a Poisson equation in y for $P^{(2)}$. By the Fredholm alternative, $\mathcal{L}_2 P^{(0)}$ must be orthogonal to the null space of the adjoint of $\mathcal{L}_0$, which here is spanned by the invariant density $\mathcal{N}(m, \nu^2)$ of Y. Denoting by $\langle \cdot \rangle$ averaging with respect to this density, $P^{(0)}(t,x)$ must solve

$$\langle \mathcal{L}_2 \rangle P^{(0)} = 0, \tag{A.2}$$

with the associated boundary condition at $x = x_0$. In the case of the barrier, the averaged operator $\langle \mathcal{L}_2 \rangle$ is given by

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right),$$

where $\bar{\sigma}^2 = \langle f^2 \rangle$. We also write $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma})$. In the case of the lookback, we have the same operator with ξ replacing x.

Comparing terms of $\mathcal{O}(\sqrt{\varepsilon})$ in the PDE, we find

$$\mathcal{L}_0 P^{(3)} = -\left(\mathcal{L}_1 P^{(2)} + \mathcal{L}_2 P^{(1)}\right), \tag{A.3}$$

which we look at as a Poisson equation for $P^{(3)}(t,x,y)$. Just as the Fredholm solvability condition for $P^{(2)}$ determined the equation for $P^{(0)}$, the solvability for (A.3) will give us the equation for $P^{(1)}(t,x)$. Substituting for $P^{(2)}(t,x,y)$ with

$$P^{(2)} = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle\right) P^{(0)} + c(t,x), \tag{A.4}$$

for some function c not depending on y, this condition is

$$\left\langle \mathcal{L}_2 P^{(1)} - \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle\right) P^{(0)} \right\rangle = 0,$$

where

$$\langle \mathcal{L}_2 P^{(1)} \rangle = \langle \mathcal{L}_2 \rangle P^{(1)}$$

since $P^{(1)}$ does not depend on y. Absorbing the $\sqrt{\varepsilon}$ factor and calling

$$\widetilde{P^{(1)}} = \sqrt{\varepsilon} P^{(1)},$$

gives that

$$\langle \mathcal{L}_2 \rangle \widetilde{P^{(1)}} = \mathcal{A} P^{(0)}, \tag{A.5}$$

where we define

$$\mathcal{A} = \sqrt{\varepsilon} \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle\right) \right\rangle,$$

plus the homogeneous boundary condition at $x = x_0$ and homogeneous terminal condition at $t = T$.

In the case of the barrier option, direct computation leads to the formula (2.10), and similarly for the lookback with ξ replacing x.

Appendix B. Formulas for Barrier Option Correction. Here, we include the explicit formulas for the terms in (2.19): $g(t)$, defined in (2.14) is given by

$$g(t) = -\left[\frac{2 \log \frac{B}{K}}{\bar{\sigma}^2 (T-t)} \mathcal{V}^{BS}(t,B) + \frac{4r}{\bar{\sigma}^3} C^{BS}(t,B) \right],$$

and Greek terms for the barrier option are given by

$$P^{(0)}_{\bar{\sigma}}(t,x) = \mathcal{V}^{BS}(t,x) - \left(\frac{x}{B}\right)^{1-k} \mathcal{V}^{BS}\left(t, \frac{B^2}{x}\right) - \frac{4r}{\bar{\sigma}^3} \log \frac{x}{B} \left(\frac{x}{B}\right)^{1-k} C^{BS}\left(t, \frac{B^2}{x}\right),$$

$$\begin{aligned} P^{(0)}_{x\bar{\sigma}}(t,x) = {} & C^{BS}_{x\bar{\sigma}}(t,x) + \left(\frac{x}{B}\right)^{-k} \left(\frac{k-1}{B} \mathcal{V}^{BS}\left(t, \frac{B^2}{x}\right) + \frac{B}{x} C^{BS}_{x\bar{\sigma}}\left(t, \frac{B^2}{x}\right) \right) \\ & - \frac{4r}{\bar{\sigma}^3} \left(\frac{x}{B}\right)^{-k} \frac{1}{B} \left(1 - k \log \frac{x}{B}\right) C^{BS}\left(t, \frac{B^2}{x}\right) \\ & + \frac{4r}{\bar{\sigma}^3} \left(\frac{x}{B}\right)^{-(k+1)} \log \frac{x}{B} \Delta^{BS}\left(t, \frac{B^2}{x}\right), \end{aligned}$$

where $\Delta^{BS}(t,x)$, $\mathcal{V}^{BS}(t,x)$, $C^{BS}_{x\bar{\sigma}}(t,x)$ are the Greeks of a call option that has the same parameters:

$$\begin{aligned} \Delta^{BS}(t,x) &= C^{BS}_x(t,x) = N(d_1), \\ \mathcal{V}^{BS}(t,x) &= C^{BS}_{\bar{\sigma}}(t,x;\bar{\sigma}) = x e^{-\frac{1}{2} d_1^2} \frac{\sqrt{T-t}}{\sqrt{2\pi}}, \\ C^{BS}_{x\bar{\sigma}}(t,x) &= \frac{\mathcal{V}^{BS}(t,x)}{x} \left(1 - \frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right). \end{aligned}$$

Appendix C. Formulas for Lookback Option Correction. The formula for $g(t)$ in the case of the lookback option, defined in (3.8) is

$$g(t) = \frac{2k}{\bar\sigma} e^{-r(T-t)} N\left(\frac{1}{2}(1-k)\bar\sigma\sqrt{T-t}\right) - 2\frac{e^{-\frac{1}{8}(1+k)^2\bar\sigma^2(T-t)}}{\bar\sigma^2\sqrt{2\pi(T-t)}}.$$

And the Greek terms for this option referred in (3.11) are

$$P_{\bar\sigma}^{(0)}(t,x) = -Je^{-r(T-t)}\left(\frac{x}{J}\right)^{1-k}\left(\frac{\bar\sigma}{r} + \frac{2}{\bar\sigma}\log\frac{x}{J}\right)N(d_6(T-t)) + \frac{\bar\sigma}{r}xN(d_7(T-t)),$$

$$P_{x\bar\sigma}^{(0)}(t,x) = \left(\frac{4r}{\bar\sigma^3}\log\frac{x}{J} - \frac{\bar\sigma}{r} - \frac{2}{\bar\sigma}\log\frac{x}{J}\right)\frac{J}{x}N(d_6(T-t))$$
$$-\frac{2\log\frac{x}{J}}{\bar\sigma^2\sqrt{T-t}}\frac{J}{x}N'(d_6(T-t)) + \frac{\bar\sigma}{r}\frac{J}{x}N(d_7(T-t)).$$

Appendix D. Formulas for Passport Option Correction. In the case of the passport option, $g(t)$ defined in (4.17) is simply given by

$$g(t) = 2\frac{e^{-\frac{1}{8}\bar\sigma^2(T-t)}}{\bar\sigma^2\sqrt{2\pi(T-t)}}.$$

The Greek terms used in (4.19) are as follows

$$P_{\bar\sigma}^{(0)}(t,x) = x\bar\sigma(T-t)N(d_-) + 2x\sqrt{T-t}N'(d_-),$$
$$P_{x\bar\sigma}^{(0)}(t,x) = \frac{P_{\bar\sigma}^{(0)}(t,x)}{x} - \frac{2|v|\log(1+\frac{|v|}{x})}{(x+|v|)\bar\sigma^2\sqrt{T-t}}N'(d_-).$$

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