Implied Volatility of Leveraged ETF Options

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Abstract

This paper studies the problem of understanding implied volatilities from options written on leveraged exchanged-traded funds (LETFs), with an emphasis on the relations between LETF options with different leverage ratios. We first examine from empirical data the implied volatility skews for LETF options based on the S&P 500. In order to enhance their comparison with non-leveraged ETFs, we introduce the concept of *moneyness scaling* and provide a new formula that links option implied volatilities between leveraged and unleveraged ETFs. Under a multiscale stochastic volatility framework, we apply asymptotic techniques to derive an approximation for both the LETF option price and implied volatility. The approximation formula reflects the role of the leverage ratio, and thus allows us to link implied volatilities of options on an ETF and its leveraged counterparts. We apply our result to quantify matches and mismatches in the level and slope of the implied volatility skews for various LETF options using data from the underlying ETF option prices. This reveals some apparent biases in the leverage implied by the market prices of different products, long and short with leverage ratios two times and three times.

Keywords: exchange-traded funds; leverage; implied volatility; stochastic volatility; moneyness scaling
Mathematics Subject Classification (2010): 35C20, 91G80, 93E10
JEL Classification: G12, G13, G17

1 Introduction

Exchange-traded funds (ETFs) are products that track the returns on various financial quantities. For example, the SPDR S&P 500 ETF (SPY) tracks the S&P 500 index, so a share in the SPY should yield the identical daily return as that index. Other ETFs track commodities and even volatility indices like the VIX. Except for an expense fee, they are a low-cost and simple tool for speculation that have become increasingly popular. Their speculative use has been attributed for instance to a recent rise in the value of commodities in the past few years. In the US, the ETF market has grown from a dozen funds with one billion dollars in assets in 1995 to over a thousand funds with $1.5 trillion in assets as of January 2012. For many investors, ETFs provide various desirable features such as liquidity, diversification, low expense ratios and tax efficiency.

Some ETFs, called leveraged ETFs (LETFs), promise a fixed leverage ratio with respect to a given underlying asset or index. Among others, the ProShares Ultra S&P500 (SSO) is an LETF

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on the S&P 500 with *leverage ratio* $\beta = +2$, which is supposed to generate twice the daily return of the S&P 500 index, minus an expense fee. The leverage ratio can also be negative, meaning the LETF is taking a short position in the underlying's returns. In other words, by longing an inverse LETF, an investor can short the underlying without conducting a short-selling transaction or facing the associated margin requirement. As an example, the ProShares UltraShort S&P500 (SDS) is an LETF on the S&P 500 with $\beta = -2$. There are also triple leveraged LETFs, such as ProShares UltraPro S&P 500 (UPRO) with $\beta = +3$, and ProShares UltraPro Short S&P 500 (SPXU) with $\beta = -3$. Surprisingly, even during the political climate for deleveraging in the aftermath of the 2008 Financial Crisis, markets for options on LETFs, which contribute an additional layer of leveraging, have grown, and it is important to analyze how these markets reflect volatility risk.

We analyze options (puts and calls) written on various LETFs. Since different LETFs are supposed to track the same underlying index or ETF, these funds and their associated options have very similar sources of randomness. This gives rise to the question of consistent pricing of options on ETFs and LETFs. Specifically, we discuss the dynamics of leveraged ETFs and analyze the implied volatility surfaces derived from options prices on leveraged and unleveraged ETFs through stochastic volatility models. Our empirical evidence suggests potential price discrepancies among ETF options with different leverage ratios. We also introduce the concept of moneyness scaling, and provide a new formula that links option implied volatilities between leveraged and unleveraged ETFs. Moreover, we discuss another interpretation of moneyness scaling through the notion of matching the dual Deltas between LETF and ETF options.

In existing literature, Cheng and Madhavan (2009) analyze the return dynamics of LETFs in a discrete-time model. They illustrate that, due to daily re-balancing, the LETF value depends on the realized volatility of the underlying index. This path-dependence can lead to value erosion over time. In a related study, Avellaneda and Zhang (2010) discuss the performance and potential tracking errors of LETFs and demonstrate the path-dependent characteristics under both discrete-time and continuous-time frameworks. In a risk analysis on LETFs, Leung and Santoli (2012) discuss the admissible leverage ratios induced by a risk measure, for example, value-at-risk (VaR) and conditional VaR. In the context of constant proportion trading, Haugh (2011) derives similar price dynamics of LETFs under a jump-diffusion model and examines the tracking errors of LETFs. On the other hand, there is relatively little literature on pricing LETF options. In their theses, Russell (2009) analyzes LETF implied volatilities with the CEV and other models, and Zhang (2010) provides numerical results for pricing LETF options assuming the underlying follows the Heston model. However, they do not compare LETF option prices with different leverage ratios. Most recently, Ahn et al. (2012) propose an approximation to compute LETF option prices with Heston stochastic volatility and jumps for the underlying.

Our study utilizes asymptotic methods for multiscale stochastic volatility models; see, for example, Fouque et al. (2011), and references therein. This asymptotic approach has been applied to analyze, for example, interest rate derivatives (Cotton et al., 2004), commodities derivatives (Jaimungal and Hikspoors, 2008), and defaultable equity options (Bayraktar, 2008). In Fouque and Kollman (2011) and Fouque and Tashman (2012), the authors propose a number of continuous-time CAPM models with stochastic volatility. A key feature of their models is the *CAPM Beta* coefficient that represents the covariance of the asset and market returns divided by the market variance. Using this terminology, one can also view LETFs as aiming to achieve a pre-specified CAPM Beta by maintaining a fixed leverage ratio.

The main objective of the current paper is to apply the asymptotic methodology to reveal the role of leverage ratio in the implied volatilities of LETF options. In particular, we obtain a mathematical connection between the approximated unleveraged ETF implied volatility skew and its leveraged counterparts. This allows us to first calibrate the coefficients that appear in the asymptotic expansion using unleveraged ETF option data, such as those on SPY, and infer the approximated LETF options implied volatility. This procedure is practically important since the index or unleveraged ETF option data tends to be significantly richer than the associated LETF option data. Empirically, we observe that the unleveraged market typically has better liquidity and more contracts with different strikes and maturities traded. Alternatively, we also calibrate the approximated LETF IVs using the LETF option data directly, and compare the calibration results from the two approaches.

The rest of the paper is organized as follows. In Section 2, we give an overview of pricing LETF options under the Black-Scholes, local volatility, and stochastic volatility models. In Section 3.1, we examine the empirical implied volatility term structures for various LETF options based on the S&P 500. Motivated by the market observations, we introduce in Section 3.2 the concept of moneyness scaling for linking LETF implied volatilities with different leverage ratios. In Section 4, we consider the pricing of LETF options in a multiscale stochastic volatility framework, and derive an approximation formula for implied volatilities. Then in Section 5, we analyze market option data using these results. Finally, concluding remarks are provided in Section 6.

2 LETF Option Prices and Implied Volatilities

We begin our analysis by studying LETF option prices and their implied volatilities. Therefore, our objective is to relate the implied volatilities of LETF options to the implied volatilities of (unleveraged) ETF options, and understand the role of leverage ratio β.

2.1 LETF valuation in the Black-Scholes Model

In order to define implied volatility for LETF options, we first consider their pricing under the constant volatility Black–Scholes model. Let $(X_t)_{t\geq 0}$ be the reference index being tracked by an ETF or its leveraged counterparts. Examples of the underlying index include the indices Nasdaq 100, S&P 500, Russell 2000, etc. Under the unique risk-neutral pricing measure $\mathbb{P}^\star$, the underlying asset price X follows the lognormal process:

$$\frac{dX_t}{X_t} = r\,dt + \sigma\,dW_t^\star, \tag{1}$$

where $W^\star$ is a standard Brownian motion under $\mathbb{P}^\star$, along with constant interest rate r and constant volatility σ.

A long-leveraged ETF $(L_t)_{t\geq 0}$ based on X with a constant leverage ratio $\beta \geq 1$ is constructed as follows. At any time t, the cash amount of βL_t (β times the fund value) is invested in X, and the amount $(\beta-1)L_t$ is borrowed at the risk-free rate $r \geq 0$. As is typical for all ETFs, a small expense rate $c \geq 0$ is incurred. For simplicity, we assume perfect tracking. Under the risk-neutral pricing measure $\mathbb{P}^\star$, the fund price process $(L_t)_{t\geq 0}$ satisfies the SDE

$$\frac{dL_t}{L_t} = \beta\left(\frac{dX_t}{X_t}\right) - ((\beta-1)r + c)\,dt \tag{2}$$

$$= (r-c)\,dt + \beta\sigma\,dW_t^\star\,. \tag{3}$$

For a short-leveraged fund $\beta \leq -1$, the cash amount βL_t is shorted on X, and $(1-\beta)L_t$ is kept in the money market account. To model the fund price $(L_t)_{t\geq 0}$, SDEs (2)-(3) also apply but with a negative β. In practice, most typical leverage ratios are $\beta = 1, 2, 3$ (long) and $-1, -2, -3$ (short).

From SDEs (1) and (3), the β-LETF can be written in terms of its reference index:

$$\frac{L_t}{L_0} = \left(\frac{X_t}{X_0}\right)^{\beta} e^{-(r(\beta-1)+c)t - \frac{\beta(\beta-1)}{2}\sigma^2 t}. \tag{4}$$

This means that the growth in the LETF L_t/L_0 over time period length t is the underlying's growth X_t/X_0 to the power β, *but* with time decays due to the expense fee and, most crucially, the build up of volatility (because for $\beta > 1$ or $\beta < 0$, $\beta(\beta-1) > 0$). Over longer times, the volatility will lead to significant attrition in fund value, even if the underlying is performing well.

2.2 LETF Option Prices & Greeks

The no-arbitrage price of a European call option on L with terminal payoff $(L_T - K)^+$ on date T is given by

$$C_{BS}^{(\beta)}(t, L; K, T) := e^{-r(T-t)} \mathbb{E}^{\star}\{(L_T - K)^+ \,|\, L_t = L\} = C_{BS}(t, L; K, T, r, c, |\beta|\sigma), \quad 0 \le t \le T \tag{5}$$

where $C_{BS}(t, L; K, T, r, c, \sigma)$ is the standard Black-Scholes formula for a call option with strike K and expiration date T when the interest rate is r, dividend rate is c, and volatility parameter is σ. We observe from equation (5) that theoretical call prices on the LETF L can be expressed in terms of the traditional Black-Scholes formula with volatility scaled by the absolute value of the leverage ratio β.

Since the LETF price L is again a lognormal process, the computation of all Greeks for LETF options follows in the same vein as in the Black-Scholes model. Here, we comment on the role of leverage ratio β on Delta only. The Delta represents the holding in the LETF L in the dynamic hedging portfolio, and is defined by the partial derivative with respect to the LETF value L:

$$\frac{\partial C_{BS}^{(\beta)}}{\partial L}(t, L; K, T, r, c, \sigma) = e^{-c(T-t)} N(d_1^{(\beta)}),$$

where N is the standard normal c.d.f. and

$$d_1^{(\beta)} = \frac{\log(L/K) + (r - c + \frac{\beta^2\sigma^2}{2})(T-t)}{|\beta|\sigma\sqrt{T-t}}. \tag{6}$$

One interesting question is how the Greeks of an ETF option is related to that of a β-LETF option since these options share the same source of randomness. We shall address this in Proposition 2 below.

2.3 LETF Implied Volatility

One useful way to compare option prices, especially across strikes and maturities, is to express each of them in terms of its implied volatility (IV). Given the observed market price $C^{obs}$of a call on L with strike K and expiration date T, the corresponding implied volatility $I^{(\beta)}$ is given in terms of the inverse of the Black-Scholes formula for the LETF option:

$$I^{(\beta)}(K, T) = (C_{BS}^{(\beta)})^{-1}(C^{obs}) = \frac{1}{|\beta|} C_{BS}^{-1}(C^{obs}). \tag{7}$$

Note we normalize by the $|\beta|^{-1}$ factor in our definition of implied volatility so that they remain on the same scale (*i.e.* the implied volatilities from the 2x LETF options are not typically double those from options on unleveraged ETFs). As a hypothetical special case, if the observed option price coincides with the Black-Scholes model price, i.e. $C^{obs} = C_{BS}^{(\beta)}(t, L; K, T)$, then it follows from (5) that the implied volatility reduces to

$$I^{(\beta)}(K,T) = \frac{1}{|\beta|} C_{BS}^{-1}\left(C_{BS}(t, L; K, T, r, c, |\beta|\sigma)\right) = \frac{1}{|\beta|}|\beta|\sigma = \sigma.$$

As a result, if market prices follow the Black-Scholes model, all calls written on ETFs and LETFs, regardless of the different leverage ratios, strikes and expiration dates, must have identical implied volatility. The same conclusion follows for put options. Nevertheless, we will observe from empirical data the well-known implied volatility skews for LETF options. The key question is how the level of skews (slope) depends on the leverage ratio.

To this end, let us first draw insight from a model-free bound on the slope of the LETF option implied volatility. Assuming the current LETF price is L and holding r, T, and β fixed, we consider the implied volatility curve $I^{(\beta)}(K)$ as a function of strike K.

Proposition 1 *The slope of the implied volatility curve admits the following bound:*

$$-\frac{e^{-(r-c)(T-t)}}{|\beta| L\sqrt{T-t}} \frac{(1 - N(d_2^{(\beta)}))}{N'(d_1^{(\beta)})} \leq \frac{\partial I^{(\beta)}(K)}{\partial K} \leq \frac{e^{-(r-c)(T-t)}}{|\beta| L\sqrt{T-t}} \frac{N(d_2^{(\beta)})}{N'(d_1^{(\beta)})}, \tag{8}$$

where d_1^β is given in (6) with $\sigma = I^\beta(K)$ and $d_2^{(\beta)} = d_1^{(\beta)} - |\beta| I^{(\beta)}(K)\sqrt{T-t}$.

Proof. By definition, the implied volatility $I(K)$, as a function of K, satisfies the equality $C^{obs} = C_{BS}(t, L; K, T, r, c, |\beta| I^{(\beta)}(K))$. In the absence of arbitrage, the observed and Black-Scholes call option prices must be decreasing in strike K. By chain rule, we have

$$\frac{\partial C^{obs}}{\partial K} = \frac{\partial C_{BS}}{\partial K}(|\beta| I^{(\beta)}(K)) + |\beta| \frac{\partial C_{BS}}{\partial \sigma}(|\beta| I^{(\beta)}(K)) \frac{\partial I^{(\beta)}(K)}{\partial K} \leq 0. \tag{9}$$

Since the Black-Scholes Vega $\frac{\partial C_{BS}}{\partial \sigma} > 0$ for calls and puts, we rearrange to obtain the upper bound on the slope of the implied volatility curve:

$$\frac{\partial I^{(\beta)}(K)}{\partial K} \leq -\frac{1}{|\beta|} \frac{\partial C_{BS}/\partial K}{\partial C_{BS}/\partial \sigma}(|\beta| I^{(\beta)}(K)), \tag{10}$$

where the partial derivatives are evaluated with volatility parameter $(|\beta| I^{(\beta)}(K))$. Similarly, the fact that put prices are increasing in strike K implies the lower bound:

$$\frac{\partial I^{(\beta)}(K)}{\partial K} \geq -\frac{1}{|\beta|} \frac{\partial P_{BS}/\partial K}{\partial P_{BS}/\partial \sigma}(|\beta| I^{(\beta)}(K)), \tag{11}$$

where P_{BS} is the Black-Scholes put option price. Recall that puts and calls with the same K and T also have the same implied volatility $I^{(\beta)}(K)$. Hence, combining (10) and (11) gives (8). ■

The bound (8) means that the slope of the implied volatility curve cannot be too negative or too positive. It also reflects that a higher leverage ratio in absolute value will scale both the upper and lower bounds, making the bounds more stringent. Intuitively, this suggests that the implied volatility curve for LETF options would be flatter than that for ETF options. Our empirical analysis (for instance, Figure 2) also confirm this pattern over a typical range of strikes. We observe however that the leverage ratio β also appears in $d_1^{(\beta)}$ and $d_2^{(\beta)}$, so the overall scaling effect is nonlinear.

3 LETF Implied Volatilities and Moneyness Scaling

We now turn to data and compare the implied volatilities from options on various ETFs and LETFs that track the S&P 500 index (SPX). For each expiration date, the implied volatility curve is plotted as a function of log-moneyness

$$\mathrm{LM} = \log\left(\frac{\text{strike}}{\text{(L)ETF price}}\right).$$

Based on the observed market IVs, we will then propose the method of *moneyness scaling* to view and compare these IV curves in Section 3.2.

3.1 Empirical Implied Volatilities

First, we look at the IVs for options on SPX and those on the ETF SPY that tracks SPX. The implied volatilities are remarkably close: for August 23, 2010, we show the first eight maturities in Figure 1.

Next, we look at how implied volatilities on SPY and the 2x LETF SSO compare. On August 23, 2010, SSO has only five maturities, but generally strike prices over a broader range. Figure 2 shows the implied volatilities at each of the common maturities and the ratio of the implied volatilities (interpolating the SPY IVs in LM to match them up). In Figure 2(f), we illustrate the IV ratios between SSO and SPY options, for long and short maturities. As the log-moneyness increases, the ratio increases from about 0.8 to around 1.2. The ratio is closest to 1 when the log-moneyness is slightly greater than zero (near-the-money).

We now turn to the implied volatilities on the -2x LETF SDS. Figure 3 shows the implied volatilities at each of the available maturities on August 23, 2010, as well as the ratios of the implied volatilities. In contrast to the SSO IVs, the IV ratio between SDS and SPY increases from about 0.4 to around 1.8 as the log-moneyness increases, and the range is wider than that between SPY and SSO. Again, the ratio is closest to 1 when the log-moneyness is slightly greater than zero (near-the-money).

Our first observation is that the implied volatilities from leveraged and unleveraged ETF options of the same log-moneyness and maturity are clearly not the same. The SSO IVs decrease with LM, have a shallower negative skew as compared with SPY, and cross near-the-money. On the other hand, SDS IVs *increase* with LM, as they are short in the S&P 500, and they also cross the SPY IV skew near the money. In addition, for both SSO:SPY and SDS:SPY ratios, they have an apparent structure of *increasing* as LM increases, being closest near-the-money. In the next section, we will analyze how well theses structures can be captured in the framework of multiscale stochastic volatility models.

3.2 Implied Volatility Comparison via Moneyness Scaling

The most salient features of the empirical implied volatilities in Figures 1-3 are: (i) the IV curve for SSO ($\beta = 2$) appears to be flatter than that for SPY ($\beta = 1$), and (ii) the IV curve for SDS ($\beta = -2$) is upward sloping as opposed to downward sloping like those for SPY ($\beta = 1$) and SSO ($\beta = 2$). The second observation is intuitive because higher implied volatility is commonly associated with more bearish options positions, such as OTM puts, written on the S&P 500 index X. For $\beta > 0$, this corresponds to the LETF puts with strikes less than the spot LETF price. For $\beta < 0$, the bearish positions correspond to the LETF *calls* with strike greater than the spot LETF price, giving rise to an upward sloping implied volatility. In fact, these features are persistent across other dates, including a year-long dataset which we will analyze in Section 5.

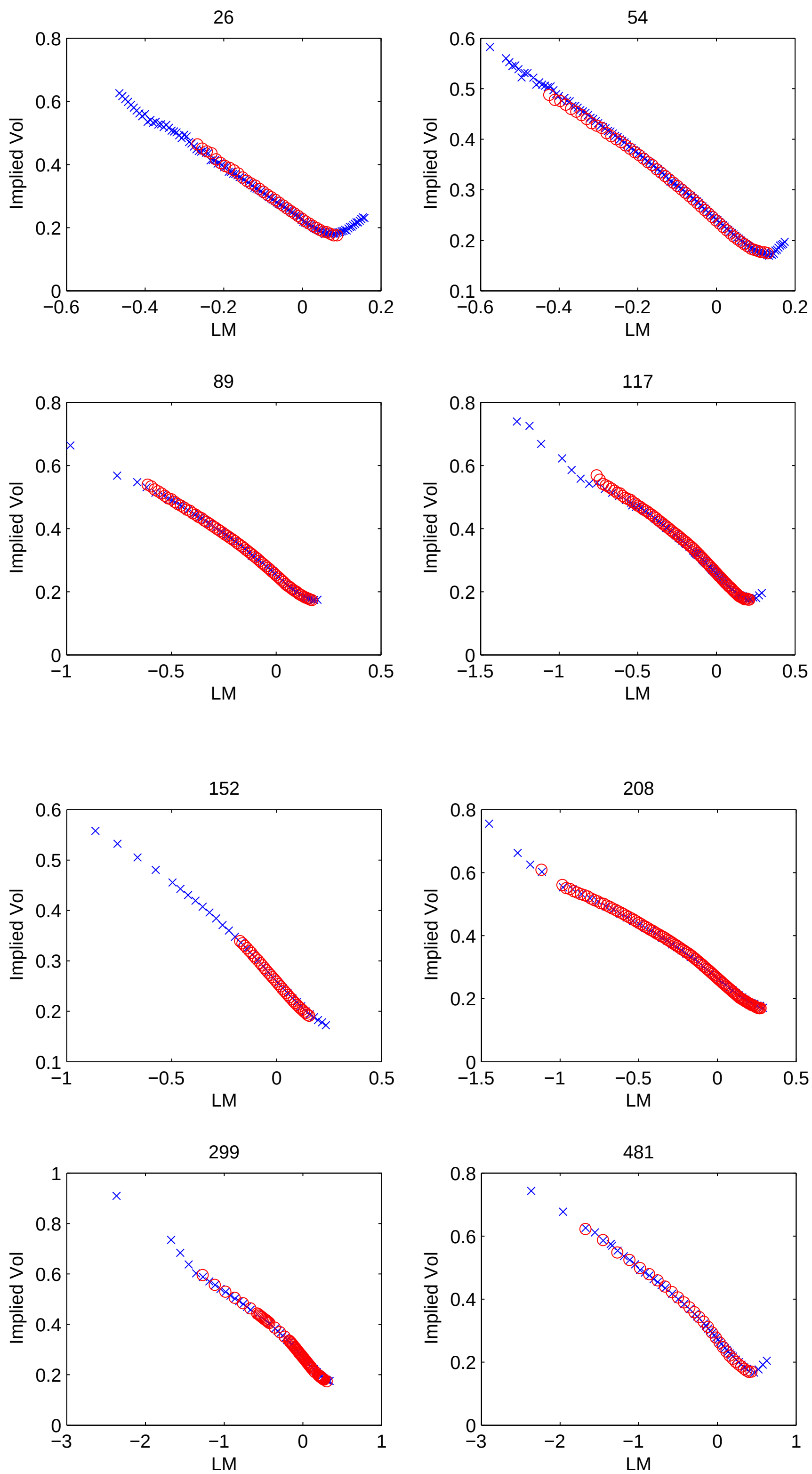
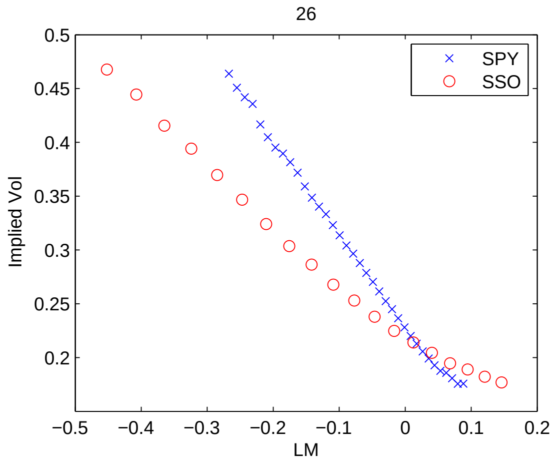
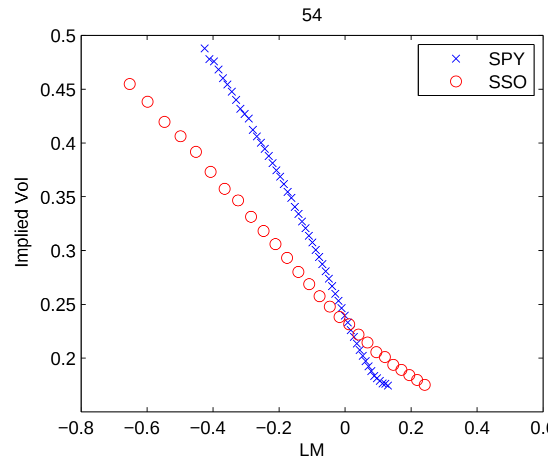


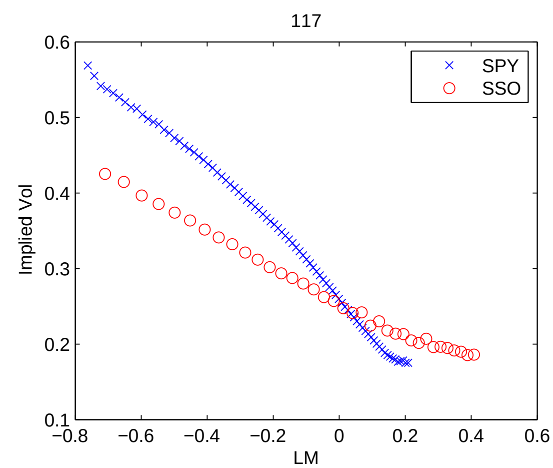
Figure 1: *SPX (blue cross) and SPY (red circles) implied volatilities on August 23, 2010 for different maturities (from 26 to 481 days) plotted against log-moneyness.*



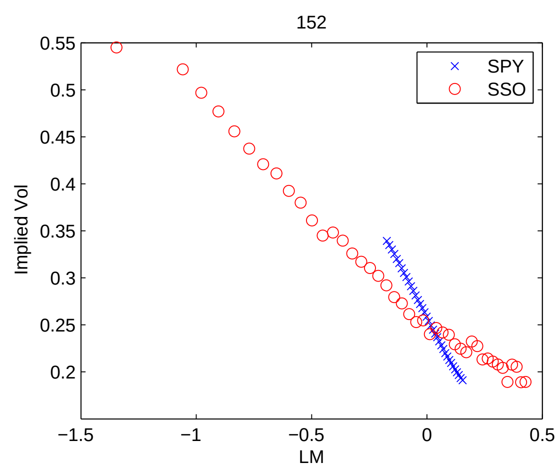
(a) 26 days to maturity



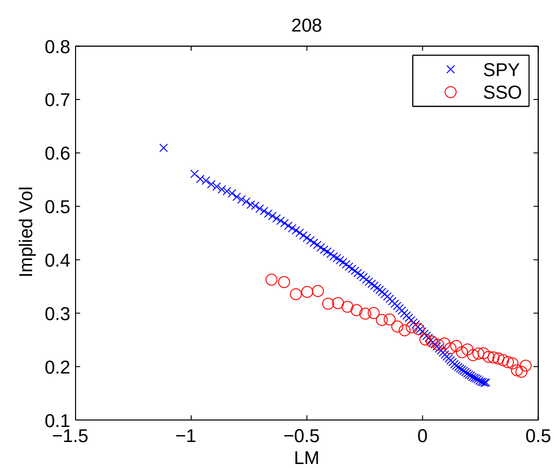
(b) 54 days to maturity



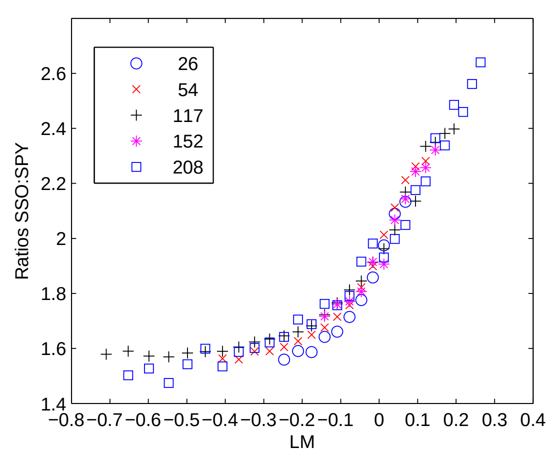
(c) 117 days to maturity



(d) 152 days to maturity

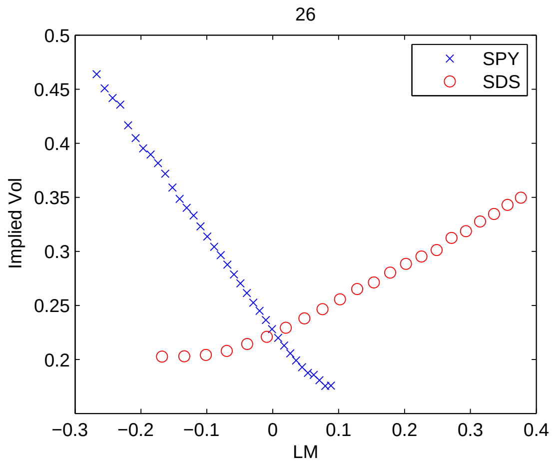


(e) 208 days to maturity

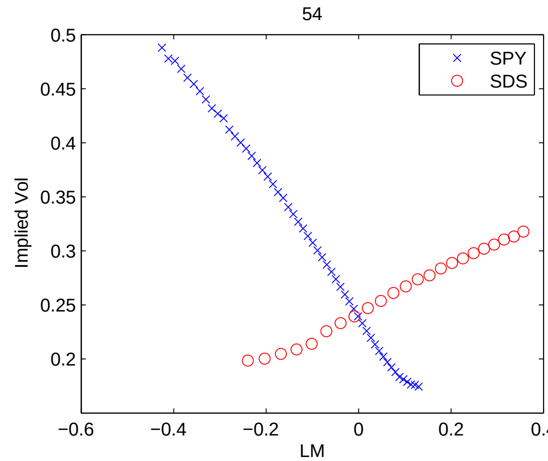


(f) Ratios SSO:SPY

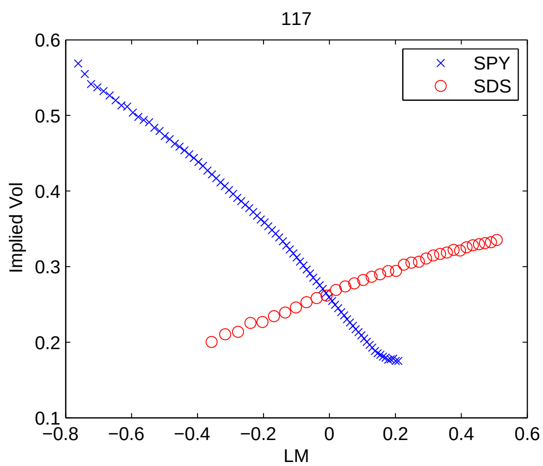
Figure 2: *SPY (blue cross) and SSO (red circles) implied volatilities against log-moneyness (LM) for increasing maturities. Panel (f) shows SSO:SPY implied volatility ratios for different maturities.*



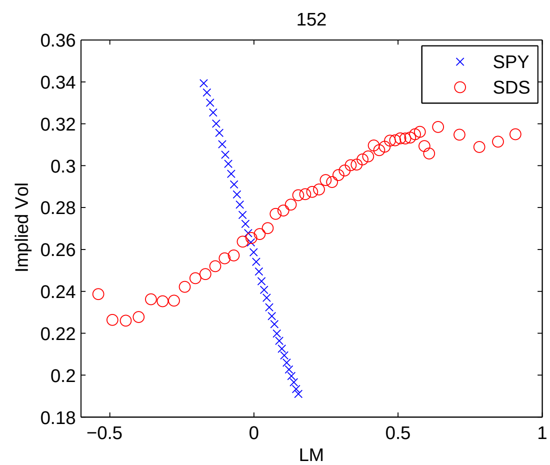
(a) 26 days to maturity



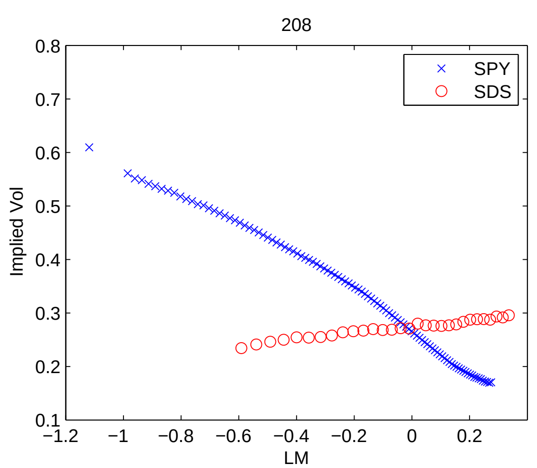
(b) 54 days to maturity



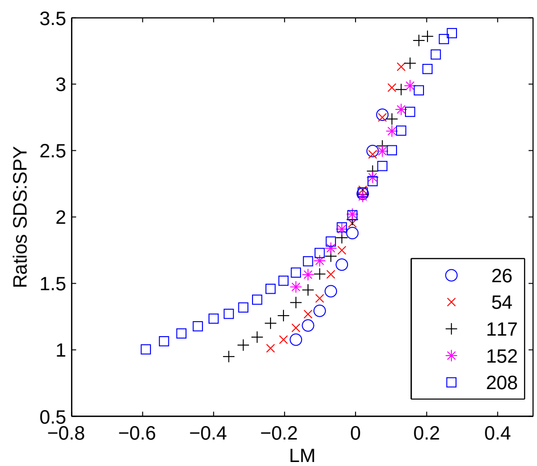
(c) 117 days to maturity



(d) 152 days to maturity



(e) 208 days to maturity



(f) Ratios SDS:SPY

Figure 3: *SPY (blue cross) and SDS (red circles) implied volatilities against log-moneyness (LM) for increasing maturities. Figure 3(f) shows SDS:SPY implied volatility ratios for different maturities.*

In order to explain these patterns, we introduce the idea of linking implied volatilities between ETF and LETF options via the method of *moneyness scaling*. This will be useful for traders to compare the option prices across not only different strikes but also different leverage ratios, and potentially identify option price discrepancies. One plausible explanation for the different IV patterns is that different leverage ratios correspond to different underlying dynamics. Therefore, the distribution of the terminal price of any β-LETF naturally depends on the leverage ratio β.

In a general stochastic volatility model, we can write the log LETF price as

$$\log\left(\frac{L_T}{L_0}\right) = \beta\log\left(\frac{X_T}{X_0}\right) - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\int_0^T \sigma_t^2 dt,$$

where $(\sigma_t)_{t\geq 0}$ is the stochastic volatility process. The terms $\log\left(\frac{L_T}{L_0}\right)$ and $\log\left(\frac{X_T}{L_0}\right)$ are the log-moneyness of the terminal LETF value and terminal ETF value. Next, we condition on that the terminal (random) log-moneyness $\log\left(\frac{X_T}{X_0}\right)$ equal to some constant $LM^{(1)}$, and compute the conditional expectation of the β-LETF log-moneyness. This leads us to define

$$LM^{(\beta)} := \beta LM^{(1)} - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\mathbb{E}^\star\left\{\int_0^T \sigma_t^2 dt \,\Big|\, \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right\}. \tag{12}$$

In other words, $LM^{(\beta)}$ is the best estimate for the terminal log-moneyness of L_T, given the terminal value $\log\left(\frac{X_T}{X_0}\right) = LM^{(1)}$. This equation strongly suggests a relationship between the log-moneyness between ETF and LETF options, though the conditional expectation in (12) is non-trivial.

In order to obtain a more explicit and useful relationship, let us consider the simple case with a constant σ as in the Black-Scholes model. Then, equation (12) reduces to

$$LM^{(\beta)} = \beta\, LM^{(1)} - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\sigma^2 T. \tag{13}$$

By this formula, the β-LETF log-moneyness $LM^{(\beta)}$ is expressed as an affine function of the unleveraged ETF log-moneyness $LM^{(1)}$. The leverage ratio β not only scales the log-moneyness $LM^{(1)}$, but also plays a role in two constant terms.

Interestingly, the moneyness scaling formula (13) can also be derived using the idea of *dual Delta matching* under the Black-Scholes model. To this end, the dual Deltas of the LETF call and put options are defined, respectively, by the partial derivatives of their prices with respect to strike (see Reiss and Wystup (2001)):

$$\frac{\partial C_{BS}^{(\beta)}}{\partial K}(t,L;K,T,r,c,\sigma) = -e^{-r(T-t)}N(d_2^{(\beta)}), \quad \frac{\partial P_{BS}^{(\beta)}}{\partial K}(t,L;K,T,r,c,\sigma) = e^{-r(T-t)}N(-d_2^{(\beta)}), \tag{14}$$

where $d_2^{(\beta)} = d_1^{(\beta)} - |\beta|\sigma\sqrt{T-t}$ (see (6)).

Proposition 2 *Under the Black-Scholes model, the dual Delta of a β-LETF call (resp. put), with log-moneyness $LM^{(\beta)}$ defined in* (13)*, coincides with*
(i) the dual Delta of an ETF call (resp. put) with log-moneyness $LM^{(1)}$ if $\beta > 0$, or
(ii) the negative of the dual Delta of an ETF put (resp. call) with log-moneyness $LM^{(1)}$ if $\beta < 0$.

Proof. Applying (13) to the dual Delta of the β-LETF call, we get

$$\begin{aligned}
-e^{-r(T-t)}N(d_2^{(\beta)}) &= -e^{-r(T-t)}N\left(\frac{-LM^{(\beta)} + (r - c - \frac{\beta^2\sigma^2}{2})(T-t)}{|\beta|\sigma\sqrt{T-t}}\right) \\
&= -e^{-r(T-t)}N\left(\frac{\beta}{|\beta|}\cdot\frac{-LM^{(1)} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= -e^{-r(T-t)}N\left(\mathrm{sign}(\beta)\, d_2^{(1)}\right). \qquad (15)
\end{aligned}$$

For $\beta > 0$, equality (15) indeed yields the dual delta for the ETF *call* with log-moneyness $LM^{(1)}$. For $\beta < 0$, then $-e^{-r(T-t)}N(-d_2^{(1)})$ in (15) is the negative of the dual delta of an ETF *put* with log-moneyness $LM^{(1)}$ (see (14)). The LETF put case follows the same line of proof. ■

As a related result, the risk-neutral probability of expiring in-the-money for an ETF call with log-moneyness $LM^{(\beta)}$ is the same as that for an LETF call for $\beta > 0$, or an LETF put for $\beta < 0$, with log-moneyness $LM^{(\beta)}$. Precisely, from (4) and (13) we see that

$$\mathbb{Q}\left\{\log(X_T/X_0) > LM^{(1)}\right\} = \mathbb{Q}\left\{\log(L_T/L_0) \gtrless LM^{(\beta)}\right\}, \qquad \text{for} \quad \beta \gtrless 0.$$

This also means that, under the Black-Scholes model, the price of an ETF *binary call* with log-moneyness $LM^{(1)}$, if traded, should be equal to the price of an LETF *binary call* with log-moneyness $LM^{(\beta)}$ for $\beta > 0$, and an LETF *binary put* with log-moneyness $LM^{(\beta)}$ for $\beta < 0$.

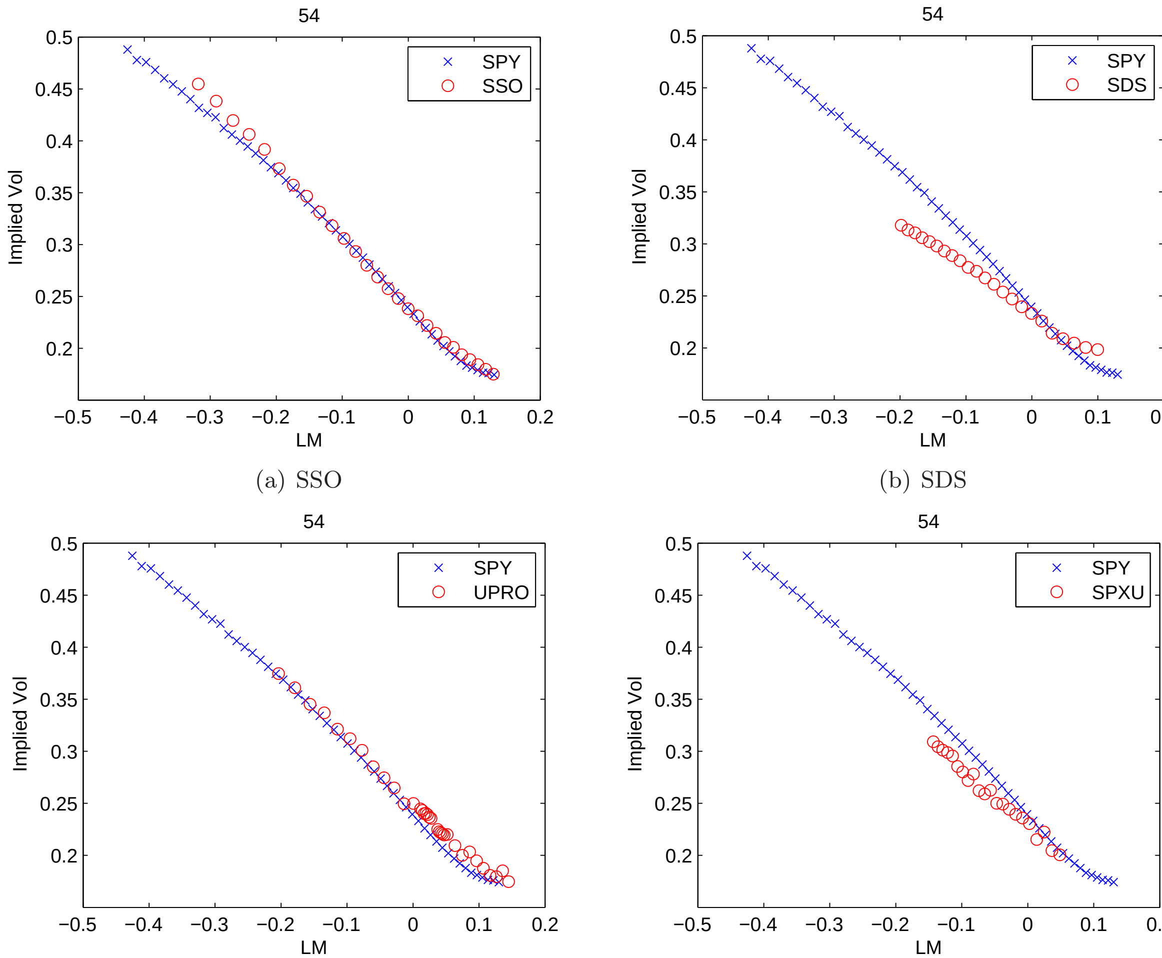
In the same spirit, we can apply (13) to link the log-moneyness pair $(LM^{(\beta)}, LM^{(\hat{\beta})})$ of LETFs with different leverage ratios $(\beta, \hat{\beta})$, namely,

$$LM^{(\beta)} = \frac{\beta}{\hat{\beta}}\left(LM^{(\hat{\beta})} + (r(\hat{\beta}-1)+\hat{c})T + \frac{\hat{\beta}(\hat{\beta}-1)}{2}\sigma^2 T\right) - (r(\beta-1)+c)T - \frac{\beta(\beta-1)}{2}\sigma^2 T, \quad (16)$$

where $\hat{c}$ is the fee associated with the $\hat{\beta}$-LETF.

In summary, the proposed moneyness scaling method is a simple and intuitive way to identify the appropriate pairing of ETF and β-LETF IVs. In other words, for each β-LETF IV, the associated log-moneyness $LM^{(\beta)}$ can be viewed as a function of the log-moneyness $LM^{(1)}$ associated with the unleveraged ETF. Consequently, we can plot the IVs for both the β-LETF and ETF over the same axis of log-moneyness $LM^{(1)}$ (see Figure 4). As a result, we can compare the LETF IVs with the ETF IVs by plotting them over the same log-moneyness axis. In practical calibration, we do not observe or assume a constant volatility σ. In order to apply moneyness scaling formula (13), we replace σ with the average IV across all available strikes for the ETF options. A more cumbersome alternative way is to estimate the conditional expectation of the integrated variance in (12). The expected variance σ_t^2 at time $t \leq T$ conditioned on the *terminal* value of the underlying is not easily computable even for specific models. It becomes explicit in the Black-Scholes model with constant volatility, as stated in (13).

Next, we apply the log-moneyness scaling to the empirical IVs. In Figure 4, we plot both the ETF and LETF IVs after moneyness scaling on August 23, 2010 with 54 days to maturity. As we can see, the mapping works remarkably well as the LETF and ETF IVs overlap significantly, except in the SPXU case. In contrast to Figures 1-3, the IVS for the short LETFs, SDS and SPXU, are now *downward* sloping and are almost parallel with the SPY IVs. As a result, the log-moneyness scaling allows us to visibly discern the potential mismatch between ETF and LETF IVs.



(a) SSO

(b) SDS

(c) UPRO

(d) SPXU

Figure 4: *SPY (blue cross) and LETF (red circles) implied volatilities after moneyness scaling on August 23, 2010 with 54 days to maturity, plotted against log-moneyness of SPY options.*

4 Implied Volatility Asymptotics for LETF Options

In this section, we study LETF option implied volatilities within a multiscale stochastic volatility framework. An asymptotic approximation will allow us to examine the role played by the leverage ratio β in a parsimonious fashion. Starting in the class of two-factor diffusion stochastic volatility models described by their characteristic time scales of fluctuation (one fast mean-reverting, one slowly varying), the implied volatility approximation allows us to trace how β impacts the level and skew structure, as well as the term-structure from LETF options, as compared to the unleveraged ETF options. Just as implied volatility can be seen as a synoptic quantity that is useful in visualizing the departure of market options data from a constant volatility theory, the *group parameters*[1] identified by the asymptotics provides a lens through which to analyze relationships between leveraged and unleveraged implied volatility surfaces.

[1]These are parameters that group together some of the stochastic volatility model parameters. We refer to Fouque et al. (2011).

Under a risk neutral measure $I\!P^\star$, the reference index X, the LETF L, fast volatility factor Y and slow volatility factor Z are described by the system of SDEs:

$$dX_t = rX_t\,dt + f(Y_t,Z_t)X_t\,dW_t^{(0)\star} \tag{17}$$

$$dL_t = (r-c)L_t\,dt + \beta f(Y_t,Z_t)L_t\,dW_t^{(0)\star} \tag{18}$$

$$dY_t = \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\eta(Y_t)\Lambda_1(Y_t,Z_t)\right)dt + \frac{1}{\sqrt{\varepsilon}}\eta(Y_t)\,dW_t^{(1)\star} \tag{19}$$

$$dZ_t = \left(\delta\,\ell(Z_t) - \sqrt{\delta}\,g(Z_t)\Lambda_2(Y_t,Z_t)\right)dt + \sqrt{\delta}\,g(Z_t)\,dW_t^{(2)\star}. \tag{20}$$

Here, the standard $I\!P^\star$-Brownian motions $\left(W^{(0)\star}, W^{(1)\star}, W^{(2)\star}\right)$ are correlated as follows:

$$d\langle W^{(0)\star}, W^{(1)\star}\rangle_t = \rho_1\,dt, \quad d\langle W^{(0)\star}, W^{(2)\star}\rangle_t = \rho_2\,dt, \quad d\langle W^{(1)\star}, W^{(2)\star}\rangle_t = \rho_{12}\,dt\,,$$

where $|\rho_1|, \rho_2|, |\rho_{12}| < 1$, and $1+2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, in order to ensure positive definiteness of the associated covariance matrix. The correlation coefficients capture skew effects that are an important feature of typical implied volatility surfaces. Under the pricing measure $I\!P^\star$, $\Lambda_1(y,z)$ and $\Lambda_2(y,z)$ are the associated market prices of volatility risk. The dynamics of Y and Z under the historical measure $I\!P$ are given by setting Λ_1 and Λ_2 to zero in (19) and (20) (and replacing the Brownian motions by $I\!P$-Brownian motions).

The small positive parameter ε corresponds to the short mean-reversion time scale of the *fast* volatility factor Y. The coefficients $\alpha(y)$ and $\eta(y)$ need not be specified for our asymptotic analysis, as long as the corresponding process Y is mean-reverting (ergodic) under $I\!P$, and has a unique invariant distribution denoted by Φ that is independent of ε. We assume that the volatility function f is positive, smooth in z, and such that $f^2(\cdot,z)$ is integrable with respect to Φ. While specifying ε is not needed here (it will be contained in terms calibrated from data), one typically thinks of this fast factor as capturing mean-reversion in volatility on the scale of a few days.

On the other hand, the small positive parameter δ characterizes the *slow* volatility factor Z. The coefficients $\ell(z)$ and $g(z)$ need not be specified for our analysis. Again δ does not need to be specified at this stage (it will appear in parameters calibrated from data), but one may think of the slow factor as describing volatility fluctuations on a time scale of a few months.

This class of models has been applied to price equity options (that is, on X) in Fouque et al. (2011), where further technical details and empirical motivation are provided. Next, we investigate the impact of leverage β on the pricing of LETF options and their implied volatilities.

4.1 Options on Index or Unleveraged ETF

We first consider the pricing of a European option on X and its implied volatility. The no arbitrage price of a European option with terminal payoff $h(X_T)$, representing either a call or put option, is given by

$$P^{\varepsilon,\delta}(t,X_t,Y_t,Z_t) = I\!E^\star\left\{e^{-r(T-t)}h(X_T) \mid X_t,Y_t,Z_t\right\}.$$

The function $P^{\varepsilon,\delta}(t,x,y,z)$ solves a high-dimensional partial differential equation. This is very computationally intensive to solve and does not lead to efficient calibration procedures. Alternatively, we apply the perturbation theory, as discussed in Fouque et al. (2011), in order to simplify the calibration problem and shed light on the implied volatilities.

We summarize the main steps. In the following, we denote by $\langle\cdot\rangle$ averaging with respect to the invariant distribution Φ. We define:

1. The *averaged effective volatility* $\bar{\sigma}(z)$ is defined by

$$\bar{\sigma}^2(z) = \langle f^2(\cdot, z)\rangle = \int f^2(y,z)\Phi(dy). \tag{21}$$

2. The corrected volatility parameter $\sigma^\star(z)$ is composed of $\bar{\sigma}$ adjusted by a small term from the market price of volatility risk from the fast factor Y:

$$\sigma^\star(z) = \sqrt{\bar{\sigma}^2(z) + 2V_2^\varepsilon(z)}, \tag{22}$$

where $V_2^\varepsilon(z)$ is of order $\sqrt{\varepsilon}$ and is given by

$$V_2^\varepsilon(z) = \frac{\sqrt{\varepsilon}}{2}\left\langle \eta\Lambda_1 \frac{\partial\phi}{\partial y}\right\rangle. \tag{23}$$

Here $\phi(y,z)$ is a solution to the Poisson equation:

$$\frac{1}{2}\eta^2(y)\frac{\partial^2\phi}{\partial y^2} + \alpha(y)\frac{\partial\phi}{\partial y} = f^2(y,z) - \bar{\sigma}^2(z). \tag{24}$$

Note that ϕ is defined up to an arbitrary additive function of z, but this will play no role in the approximation below.

3. The three *group parameters*: V_3^ε containing the correlation ρ_1 between shocks to the fast factor Y and shocks to the index X is given by

$$V_3^\varepsilon(z) = -\frac{\rho_1\sqrt{\varepsilon}}{2}\left\langle \eta f \frac{\partial\phi}{\partial y}\right\rangle. \tag{25}$$

It is of order $\sqrt{\varepsilon}$. Next, V_0^δ contains the market price of volatility risk from the slow factor Z:

$$V_0^\delta(z) = -\frac{g(z)\sqrt{\delta}}{2}\langle\Lambda_2\rangle\bar{\sigma}'(z), \tag{26}$$

and V_1^δ contains the correlation ρ_2 between shocks to the slow factor Z and shocks to the index X

$$V_1^\delta(z) = \frac{\rho_2 g(z)\sqrt{\delta}}{2}\langle f\rangle\bar{\sigma}'(z), \tag{27}$$

and both V_0^δ and V_1^δ are of order $\sqrt{\delta}$.

Proposition 3 *For fixed (t,x,y,z), the European option price $P^{\varepsilon,\delta}(t,x,y,z)$ is approximated by $P^\star(t,x,z)$, where*

$$P^\star = P^\star_{BS} + \left\{\tau V_0^\delta + \tau V_1^\delta\left(x\frac{\partial}{\partial x}\right) + \frac{V_3^\varepsilon}{\sigma^\star}\left(x\frac{\partial}{\partial x}\right)\right\}\frac{\partial P^\star_{BS}}{\partial\sigma}, \tag{28}$$

and the order of accuracy is given by

$$P^{\varepsilon,\delta} = P^\star + \mathcal{O}(\varepsilon\log|\varepsilon| + \delta). \tag{29}$$

Here, $P^\star_{BS}$ is the Black-Scholes call price with time-to-maturity $\tau = T - t$ and the corrected volatility parameter $\sigma^\star$.

We refer to (Fouque et al., 2011, Chapters 4-5) for the derivation and accuracy proof of these formulas. Note that the stochastic volatility correction terms in (28) are comprised of Greeks of the Black-Scholes price $P^\star_{BS}$, in particular the Vega and the Delta-Vega.

In order to compute this approximated price $P^\star$, only the *group market parameters* $(\sigma^\star, V_0^\delta, V_1^\delta, V_3^\varepsilon)$ need to be estimated from the term structure of implied volatilities obtained from European call and put options. The price approximation (28) can be used to derive the implied volatility I associated with $P^{\varepsilon,\delta}$ defined by $P_{BS}(I) = P^{\varepsilon,\delta}$. To do so, we define the variable *Log-Moneyness to Maturity Ratio* by

$$\text{LMMR} \quad = \quad \frac{\log(K/x)}{\tau}. \tag{30}$$

Proposition 4 *The first-order approximation for the implied volatility is given by*

$$I = b^\star + \tau b^\delta + \left(a^\varepsilon + \tau a^\delta\right) LMMR + \mathcal{O}(\varepsilon \log|\varepsilon| + \delta), \tag{31}$$

where the parameters $(b^\star, b^\delta, a^\varepsilon, a^\delta)$ are defined in terms of $(\sigma^\star, V_0^\delta, V_1^\delta, V_3^\varepsilon)$ by

$$b^\star = \sigma^\star + \frac{V_3^\varepsilon}{2\sigma^\star}\left(1 - \frac{2r}{\sigma^{\star 2}}\right), \qquad a^\varepsilon = \frac{V_3^\varepsilon}{\sigma^{\star 3}}, \tag{32}$$

$$b^\delta = V_0^\delta + \frac{V_1^\delta}{2}\left(1 - \frac{2r}{\sigma^{\star 2}}\right), \qquad a^\delta = \frac{V_1^\delta}{\sigma^{\star 2}}. \tag{33}$$

We refer to (Fouque et al., 2011, Section 5.1) for the derivation of these formulas. Note that V_3^ε is of order $\sqrt{\varepsilon}$ whereas V_0^δ and V_1^δ are both of order $\sqrt{\delta}$. The implied volatility approximation shows that the intercept $b^\star$ is of order one, the slope coefficients a^ε and a^δ are of order $\sqrt{\varepsilon}$ and $\sqrt{\delta}$ respectively, and that b^δ is of order $\sqrt{\delta}$.

Remark 5 *Proposition 4 describes an IV approximation that holds for a large class of multiscale stochastic volatility models. Calibrating $(\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon)$ from implied volatilities reveals some averages with respect to the functions f, Λ_1, and Λ_2. But since f, Λ_1, and Λ_2 are unspecified, this allows for an enormous degree of freedom associated with these functions, and so the relationships (22)-(27) do not impose constraints among $(\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon)$ in the calibration.*

The approximations in Propositions 3 and 4 are valid with reasonably small ϵ and δ. The values of ϵ and δ in (29) *cannot be chosen to be arbitrarily small a priori. In fact, they are embedded in the group parameters obtained from calibration. Since the parameters $(\sigma^*)^2, V_0^\delta, V_1^\delta$, and V_3^ϵ are proportional to square root of δ and ϵ, they are expected to be small (Fouque et al., 2011, Chap. 4-5). One can back out these four parameters from* (32)-(33)*, and they can be used to price exotic options to the same degree of approximation within the multiscale stochastic volatility models (Fouque et al., 2011, Chap. 6).*

4.2 Options on Leveraged ETF

We now proceed to investigate the role of β in the coefficients and parameters in the first-order approximation of LETF option prices. First, the risk-neutral price of a European LETF option with terminal payoff $h(L_T)$ is given by

$$P_\beta^{\varepsilon,\delta}(t, L_t, Y_t, Z_t) \quad = \quad I\!E^\star\left\{e^{-r(T-t)}h(L_T) \mid L_t, Y_t, Z_t\right\}.$$

The main goal is to trace how the β introduced in the scaling of the volatility term in (18) enters the various parts of the asymptotics. For simplicity, we assume the expense fee $c = 0$.

Proposition 6 *For fixed (t,x,y,z), the LETF option price $P_\beta^{\varepsilon,\delta}(t,x,y,z)$ is approximated by*

$$P_\beta^{\varepsilon,\delta} = P_\beta^\star + \mathcal{O}(\varepsilon \log|\varepsilon| + \delta),$$

where

$$P_\beta^\star = P_{BS}^{\star,\beta} + \left\{ \tau V_{0,\beta}^\delta + \tau V_{1,\beta}^\delta \left(x\frac{\partial}{\partial x} \right) + \frac{V_{3,\beta}^\varepsilon}{\sigma_\beta^\star} \left(x\frac{\partial}{\partial x} \right) \right\} \frac{\partial P_{BS}^{\star,\beta}}{\partial \sigma}.$$

Here the β-dependent group market parameters are:

$$V_{0,\beta}^\delta = |\beta| V_0^\delta, \quad V_{1,\beta}^\delta = \beta|\beta| V_1^\delta, \quad V_{3,\beta}^\varepsilon = \beta^3 V_3^\varepsilon, \quad \sigma_\beta^\star = |\beta|\sigma^\star, \tag{34}$$

and $P_{BS}^{\star,\beta}$ is the Black-Scholes call option price with time-to-maturity $\tau = T - t$ and the corrected volatility parameter $\sigma_\beta^\star$.

Proof. These follow from repeating Proposition 3 with f replaced by βf. Indeed we have the following replacements: from (21), $\bar\sigma^2 \mapsto \beta^2\bar\sigma^2$ and so $\bar\sigma \mapsto |\beta|\bar\sigma$; from (24), $\phi \mapsto \beta^2\phi$; from (23), $V_2^\varepsilon \mapsto \beta^2 V_2^\varepsilon$, and so from (22), $\sigma^\star \mapsto |\beta|\sigma^\star$; from (25), $V_3^\varepsilon \mapsto \beta^3 V_3^\varepsilon$; from (26), $V_0^\delta \mapsto |\beta| V_0^\delta$; and finally from (27), $V_1^\delta \mapsto \beta|\beta| V_1^\delta$. ∎

Note that β enters the group parameters in different ways rather than just simply scaling them. The next goal is to see how β impacts the calibration parameters that are fit to the implied volatility surface.

Proposition 7 *The implied volatility of an LETF option $I^{(\beta)}$ is defined as in (7) by*

$$I^{(\beta)} = \frac{1}{|\beta|} P_{BS}^{-1}(P_\beta),$$

where the inversion of the Black-Scholes formula is with respect to volatility. Its first-order approximation, to order of accuracy $\mathcal{O}(\varepsilon \log|\varepsilon| + \delta)$, is

$$I^{(\beta)} \approx b_\beta^\star + \tau b_\beta^\delta + \left(a_\beta^\varepsilon + \tau a_\beta^\delta \right) LMMR, \tag{35}$$

where the skew slopes are given in terms of the unleveraged ETF skew slopes by

$$a_\beta^\varepsilon = \frac{1}{\beta} a^\varepsilon, \qquad a_\beta^\delta = \frac{1}{\beta} a^\delta, \tag{36}$$

and where the level parameters $(b_\beta^\star, b_\beta^\delta)$ are given in terms of the unleveraged ETF group parameters $(\sigma^\star, V_0^\delta, V_1^\delta, V_3^\varepsilon)$ by

$$b_\beta^\star = \sigma^\star + \frac{\beta V_3^\varepsilon}{2\sigma^\star}\left(1 - \frac{2r}{\beta^2\sigma^{\star 2}}\right), \qquad b_\beta^\delta = V_0^\delta + \frac{\beta V_1^\delta}{2}\left(1 - \frac{2r}{\beta^2\sigma^{\star 2}}\right). \tag{37}$$

Proof. From Proposition 4, the approximation for $|\beta| I^{(\beta)}$ is given by

$$|\beta| I^{(\beta)} \approx \widehat{b_\beta^\star} + \tau \widehat{b_\beta^\delta} + \left(\widehat{a_\beta^\varepsilon} + \tau \widehat{a_\beta^\delta} \right) \text{LMMR},$$

where $LMMR = \log(K/L_t)/\tau$ is defined w.r.t. the LETF spot price L_t, and

$$\widehat{b^\star_\beta} = \sigma^\star_\beta + \frac{V^\varepsilon_{3,\beta}}{2\sigma^\star_\beta}\left(1 - \frac{2r}{\sigma^{\star 2}_\beta}\right) \quad , \quad \widehat{a^\varepsilon_\beta} = \frac{V^\varepsilon_{3,\beta}}{\sigma^{\star 3}_\beta},$$

$$\widehat{b^\delta_\beta} = V^\delta_{0,\beta} + \frac{V^\delta_{1,\beta}}{2}\left(1 - \frac{2r}{\sigma^{\star 2}_\beta}\right) \quad , \quad \widehat{a^\delta_\beta} = \frac{V^\delta_{1,\beta}}{\sigma^{\star 2}_\beta}.$$

Substituting from (34) and dividing by $|\beta|$ leads to (35) and the relations (36)-(37). ■

We note first that the first-order approximation reveals that the skew slope parameters have the direct relationship (36), and so in particular the short leveraged ETFs implied volatilities have the opposite signed slopes as we saw in Figure 3 for $\beta = -2$. The relationship between the level parameters (the b's) is not direct, but is given in (37) after passing through the V's. These relations can be used to assess how less liquid LETF implied volatility surfaces relate to the more liquid ETF IV surface within the framework of multiscale stochastic volatility, and we look at data in the next section.

5 Analysis of ETF & LETF Implied Volatility Surfaces

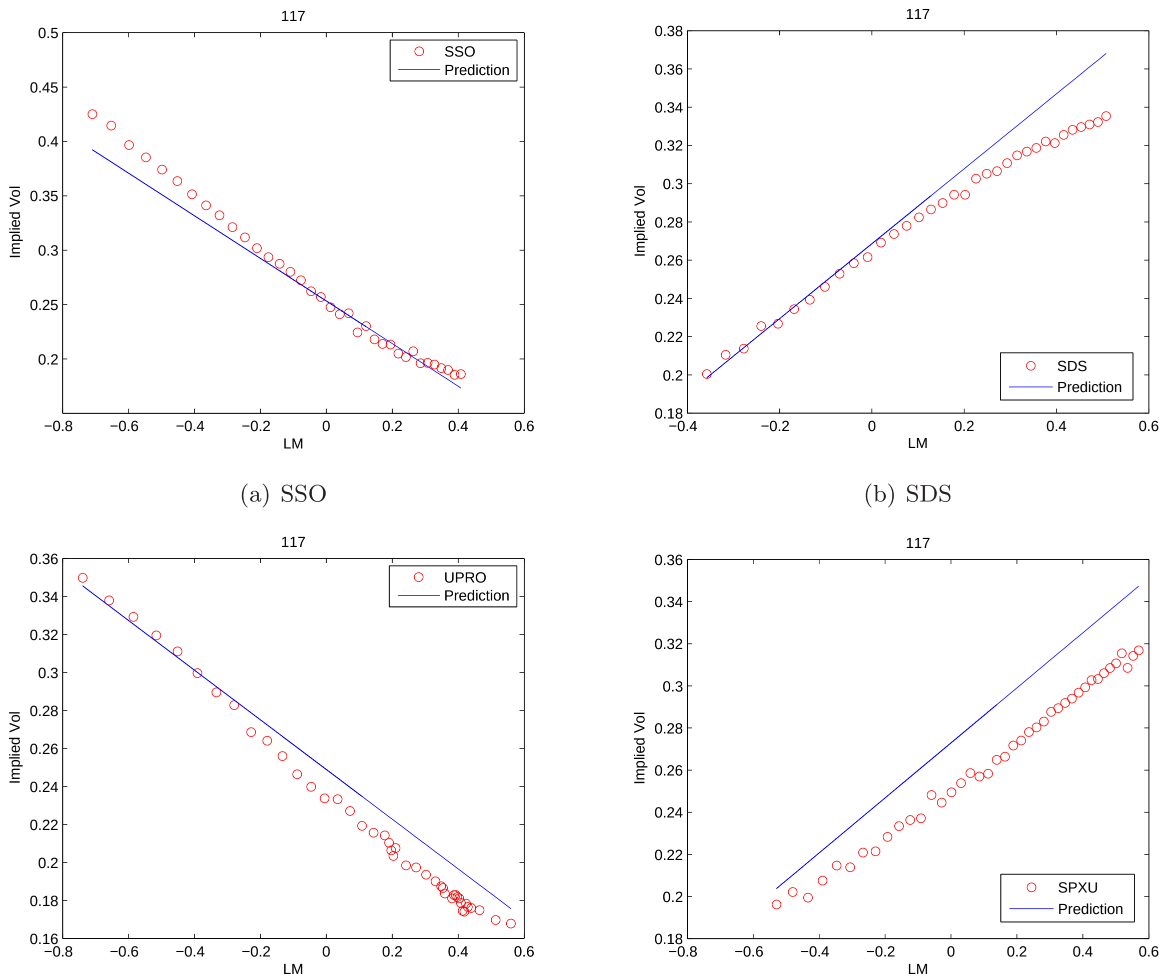
Proposition 7 suggests a link between the approximated IVs of an ETF and its leveraged counterparts within the multiscale stochastic volatility framework. The objective of this section is to examine from market data the relationships between the calibrated implied volatility surfaces of ETF and LETF options of different leverage ratios.

We consider two calibration approaches. On one hand, one can first calibrate the stochastic volatility group parameters $(\sigma^\star, V^\delta_0, V^\delta_1, V^\varepsilon_2, V^\varepsilon_3)$ using unleveraged ETF option data, such as those on SPY. This will give the estimated $(a^\varepsilon_\beta, b^\star_\beta, a^\delta_\beta, b^\delta_\beta)$ for the approximated LETF options implied volatility (35) for all available leverage ratios. This procedure is useful in practice especially when the index/ETF option data is significantly richer than the LETF option data. Indeed, the unleveraged market typically has better liquidity and more contracts with different strikes and maturities traded. On the other hand, one can directly use the LETF option data to calibrate the approximated IVs. We shall compare the two approaches to better under the IVs of LETF options.

5.1 From SPY IVs to LETF IVs

We first apply our IV approximation using option data from SPY to four associated leveraged ETFs, namely, SSO, SDS, UPRO, and SPXU. Approximation (31) is fitted to the SPY IV surface to obtain least-squares estimates of $(a^\varepsilon, b^\star, a^\delta, b^\delta)$. Through (32)-(33), their values give $(\sigma^*, V^\delta_0, V^\delta_1, V^\epsilon_3)$, which are applied to (36) and (37) to yield our predictions for $(a^\varepsilon_\beta, b^\star_\beta, a^\delta_\beta, b^\delta_\beta)$. This calibration procedure provides a way to compare IV surfaces across leverage ratios, all with reference to the unleveraged ETF IV data.

In Figure 5, we overlay the IVs of LETF options from August 23, 2010 with the IV approximation based on SPY option data. The LETF options shown have 117 days to maturity. The calibration that leads to the predicted approximation involves using SPY options data of all available maturities. Similarly, Figure 6 shows the data and predictions for 208-day maturity LETF options. In these examples, we observe that the SPY-predicted slope is close to the LETF IV, except for one case in panel 6(b) where the SDS IV looks relatively flat. The matching of IV slopes suggests that the β-effect on the correlation or skew is quite consistent between the SPY and LETF data. On the



(a) SSO (b) SDS

(c) UPRO (d) SPXU

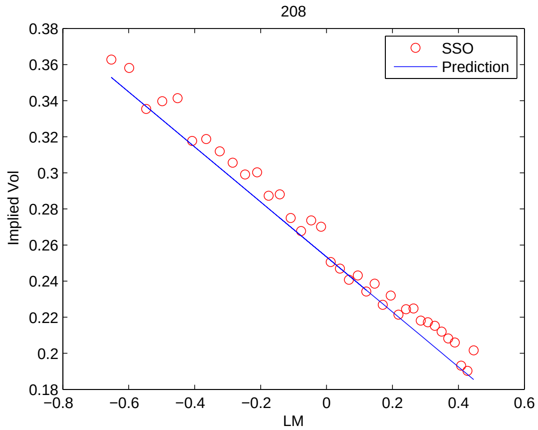
Figure 5: *IVs of LETF options with 117 days to maturity and prediction from SPY calibration.*

other hand, there seems to be greater mismatch in the IV level between data and the prediction. Since, in the theory, this is related through the b's (see (37)) to the market prices of volatility risk, it suggests differing risk premia across the different products.

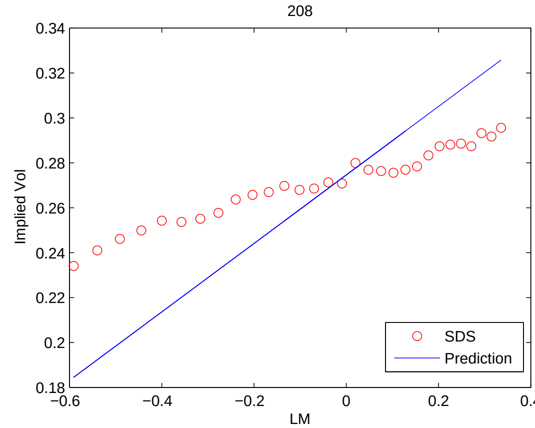
Next, we examine the difference between prediction from SPY calibration and LETF option data by repeating this procedure every day from September 1st 2009 to September 1st 2010. In Figure 7, we show the relative errors in the predicted slopes and intercepts compared with a direct least-squares fit to each LETF IV skew for each maturity.

Note that the prediction uses data from all maturities from one product, namely SPY, and its performance is measured against a direct affine fit to a single maturity skew on another product. The mean and standard deviation of the relative errors are reported in Table 1.

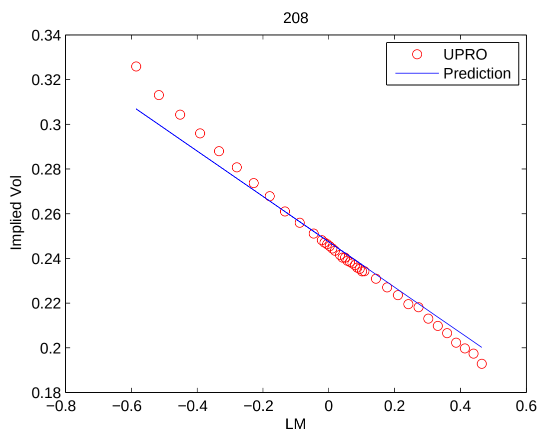
From these, we see that the slope errors are negatively biased, except for the SDS (where the standard deviation of the slope fit error is greater by an order of magnitude than those for the other LETFs). For the intercept, the ± 2 LETFs have better consistency with SPY than the more leveraged ± 3 LETFs where the risk premium mismatch seems greater.



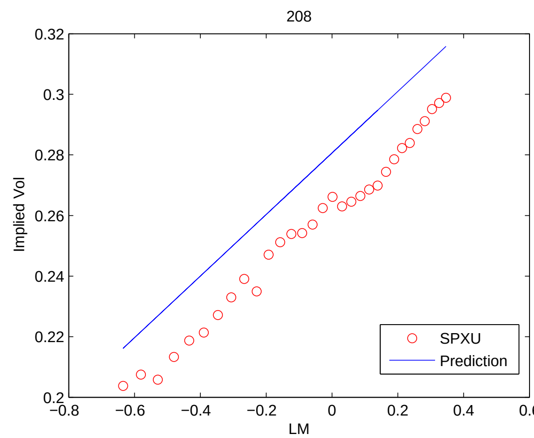
(a) SSO



(b) SDS



(c) UPRO



(d) SPXU

Figure 6: *IVs of LETF options with 208 days to maturity and prediction from SPY calibration.*

		SSO	SDS	UPRO	SPXU
Intercept Relative Error	mean	0.13%	-0.07%	1.41%	7.49%
	std dev.	2.36%	3.36%	4.83%	20.60%
Slope Relative Error	mean	-15.51%	15.04%	-20.00%	-3.76%
	std dev.	8.94%	42.55%	9.51%	6.74%

Table 1: *The relative errors of the SPY-predicted intercept and slope for the IVs for SSO, SDS, UPRO, and SPXU, over the period September 2009 to September 2010.*

5.2 Direct Fitting to LETF IV Surfaces

Now we fit the implied volatility approximation according to formula (35) of Proposition 7 directly to LETF IV data across all maturities. In contrast to the SPY options, there are fewer traded contracts on LETFs. As seen in Figure 8, the first-order approximations based on the multiscale stochastic volatility, while not perfect, capture the implied volatility data across strikes and maturities reasonably well, and they provide an visualization of the slope of the IV.

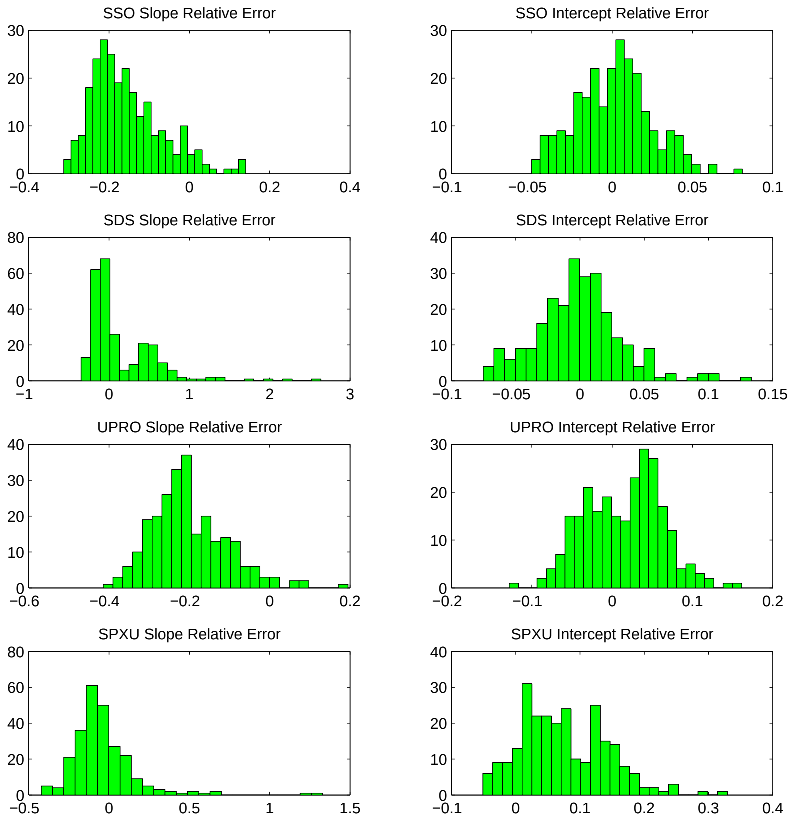
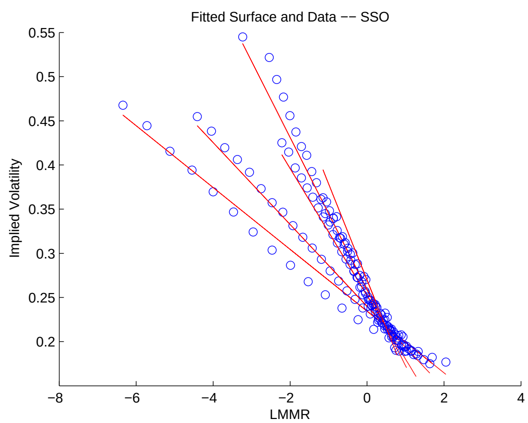
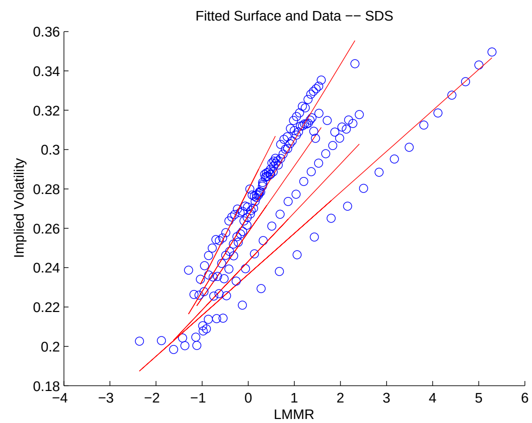


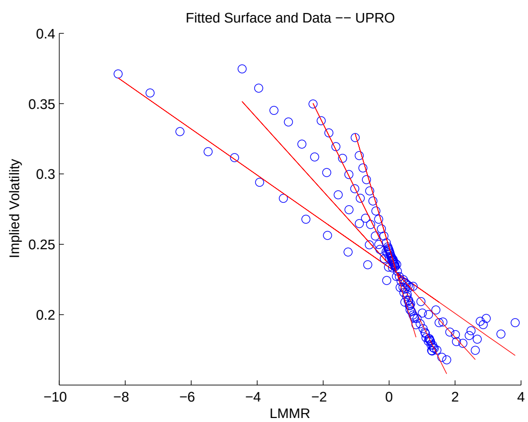
Figure 7: *Histogram of relative errors of SPY-calibrated predictions of the IV slopes and intercepts for SSO, SDS, UPRO, SPXU, collected over the period September 2009 to September 2010.*



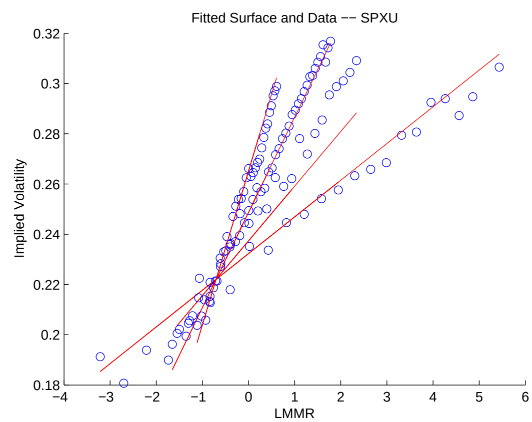
(a) SSO



(b) SDS



(c) UPRO



(d) SPXU

Figure 8: *LETF options IVs and their calibrated first-order approximations for all available maturities on August 23, 2010.*

Next, we consider how the empirical skew slopes respect the relations (36). To do so, we calibrate daily the skew parameters $(a^{\varepsilon}_{\beta}, a^{\delta}_{\beta})$ for $\beta \in \{1, \pm 2, \pm 3\}$. Then we fix an arbitrary mid-range maturity, $\tau = 100$ days, and look at the ratios

$$R_{ij} := \frac{a^{\varepsilon}_{\beta_i} + \tau a^{\delta}_{\beta_i}}{a^{\varepsilon}_{\beta_j} + \tau a^{\delta}_{\beta_j}}, \qquad i, j \in \{1, \pm 2, \pm 3\},\ i \neq j,$$

for the 10 pairwise combinations. In theory, the ratios should be

$$R_{ij} = \beta_j / \beta_i.$$

Of course this does not hold so cleanly in the data, and we are interested in any systematic deviation from these relationships. Since it is not clear which data might be "preferentially correct", we give the five datasets equal weighting and calculate estimates of each β_i as if β_j were correct, and then average over the four estimates for each β_i.

That is, for each daily observation R_{ij}, we can compute estimated

$$\widehat{\beta}_{j|i} := \beta_i R_{ij}, \qquad \widehat{\beta}_{i|j} := \beta_j / R_{ij}. \tag{38}$$

The histograms of the daily averaged estimates

$$\bar{\beta}_i = \frac{1}{4} \sum_{j \neq i} \widehat{\beta}_{i|j}$$

are presented in Figure 9, and the means and standard deviations are in Table 2.

	SPY	SSO	SDS	UPRO	SPXU
$\bar{\beta}$ mean	1.0512	1.9840	-2.6314	2.4705	-2.9264
$\bar{\beta}$ standard dev.	0.1092	0.1596	0.5101	0.2605	0.4780

Table 2: *The mean and standard deviation of estimated $\bar{\beta}$ over September 2009-September 2010 for SPY, SSO, SDS, UPRO, and SPXU.*

These results reflect the relative leverage ratios implied by the market. In particular, the SPY, SSO and, surprisingly, the SPXU implied volatilities are consistent with their leverage ratios of $1, 2$ and -3 respectively. However, implied volatility skews from SDS ($\beta = -2$) appear to systematically *overestimate* the magnitude of leverage ratio, as if the LETF was more short than it is actually supposed to be. In contrast, the skews from UPRO ($\beta = 3$) systematically *underestimate* the leverage ratio, as if the LETF was not so ultra-leveraged. While these are findings are dependent on our framework for analyzing the data, the systematic discrepancies are quite striking.

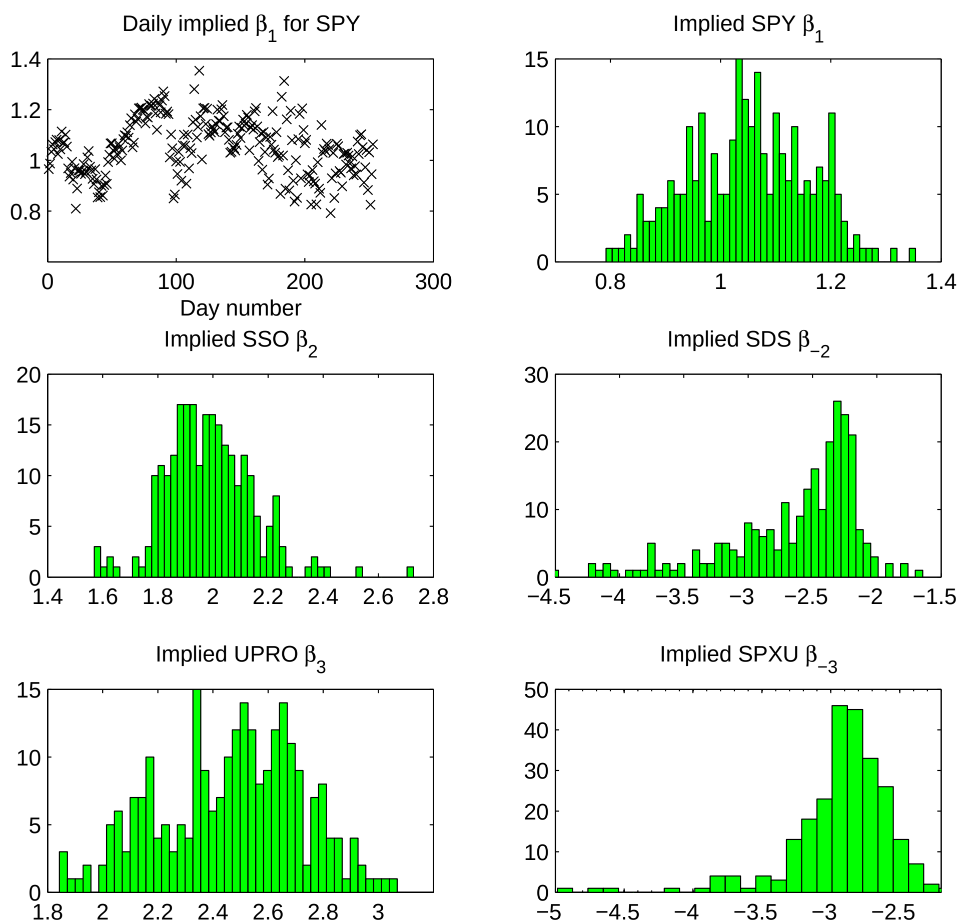


Figure 9: *Histograms of $\bar{\beta}_i$, the implied β from September 2009-September 2010 LETF implied volatilities.*

6 Conclusion

The recent growth of leveraged ETFs and their options markets calls for research to understand the inter-connectedness of these markets. In our study, we have investigated from both empirical and theoretical perspectives implied volatilities of options on LETFs with different leverage ratios. Our analytical results have led us to examine implies volatilities in several ways. First, we propose the method of moneyness scaling for enhance the comparison of IVs with different leverage ratios. Then, under the multiscale stochastic volatility framework, we use unleveraged ETF option data to calibrate for the necessary parameters to arrive at a predicted LETF options implied volatility approximation (see Proposition 7). This approach allows us to use the richer unleveraged index/ETF option data to shed light on the less liquid LETF options market. Alternatively, we also calibrate directly from LETF option data to obtain the approximated implied volatilities.

Our calibration examples confirm the *approximate* connection between ETF and LETF IVs revealed by the approximation formulas (31) and (35) within the multiscale stochastic volatility framework, while volatility risk premium may differ across LETF options markets, especially for the double-short and triple-short LETFs. Of course, it will be of future interest to perform the analogous analysis in other regimes that have been studied in the unleveraged case: large strike (Lee (2004); Gulisashvili and Stein (2009)); long maturity (Tehranchi (2009)); or short time-to-expiration (Berestycki et al. (2004)).

Leveraged exchange-traded products are also available for other reference indexes such as Nasdaq 100 and Russell 2000, as well as other asset classes such as commodities and real estate. For volatility trading, there are the S&P 500 Short-Term VIX Futures Index ETN (VXX), and Mid-Term VIX Futures Index ETN (VXZ), as well as options on these two ETNs traded on CBOE. All these should motivate research to investigate the connection between derivatives prices and the consistency thereof, as we have done under the multiscale stochastic volatility framework. In this regard, models that are tractable and capable of capturing the characteristics of inter-connected markets are highly desirable. A related issue is to quantify the leverage risks and the feedback effect on market volatility created by ETFs and LETFs[2], not to mention what might occur if a sizeable LETF option market is not well studied and understood.

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[2]See, for instance, "Volatility, Thy Name Is E.T.F." by Andrew Ross Sorkin, New York Times, October 10, 2011.

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