Portfolio Optimization with Derivatives and Indifference Pricing

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July 26, 2004

Abstract

We study the problem of portfolio optimization in an incomplete market using derivatives as well as basic assets such as stocks. In such markets, an investor may want to use derivatives, as a proxy for trading volatility, for instance, but they should be traded statically, or relatively infrequently, compared with assumed continuous trading of stocks, because of the much larger transaction costs. We discuss the computational tractability obtained by assuming exponential utility, and connection to the method of utility-indifference pricing. In particular, we show that the optimal number of derivatives to invest in is given by the optimizer in the Legendre transform of the indifference price as a function of quantity, evaluated at the market price. This is illustrated in a standard diffusion stochastic volatility model, when the indifference price is the solution of a quasilinear PDE problem. We suggest some asymptotic approximations for the optimal derivative holding, first when it might be small, and second in the case of slowly varying volatility.

Contents

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1 Introduction

In this article, we study the problem of portfolio optimization using stocks and derivatives, where the performance is measured by expected exponential utility.

1.1 Background

Portfolio optimization problems within the context of continuous-time stochastic models of financial variables have been, and continue to be, the subject of much research activity in financial mathematics and engineering. A wide-ranging theory has been developed since the papers of Merton [30, 31] for understanding the issue of optimal asset allocation for maximizing expected utility under various sources of market incompleteness (such as transaction costs, trading constraints or stochastic volatility), different utility prescriptions, the presence of random endowments and so on. Key references, in particular for the duality theory used heavily to obtain existence and uniqueness results in incomplete markets, are [27, 26], and recent extensions can be found in [6, 36].

Typically, these results study the problem of optimizing over *continuous* trading strategies in *primitive* (or underlying) securities such as stocks. However, traders have long been using *derivative securities* as a proxy for some of the untradeable components in an incomplete market. A standard example is a strangle which involves long positions in a European call option with strike K_u and a European put option with the same maturity date T and a lower strike K_l. Both options are out-of the money at the time of their purchase, so we assume the current stock price $S_0 \in (K_l, K_u)$. The terminal payoff as a function of the stock price on date T is shown in Figure 1. Such a position is often described as being "long volatility",

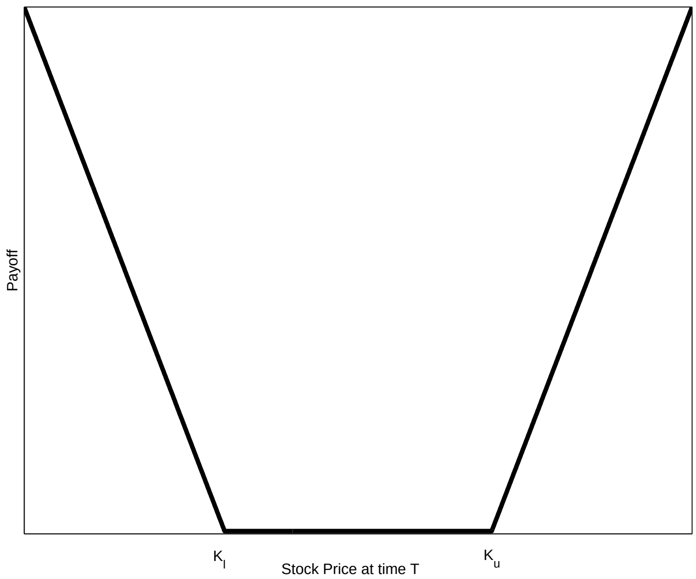

Figure 1: Strangle payoff function.

since the holder is rewarded by a significant move in the stock price, either up or down.

In this paper, we study the problem of incorporating derivatives along with stocks in the investment problem. A crucial difference between the two asset classes is that transaction costs on derivatives trades are significantly higher than on basic stocks. In addition, there may be greater liquidity issues in the less frequently traded derivatives markets. We shall assume, therefore, that stocks can be traded continuously, ignoring transaction costs, but that options can only be bought or sold *statically.*

Other authors have studied a similar problem, but under different assumptions. Carr and Madan [5] assumed the availability for trading of European options of all strikes, thereby completing the market, and in a one-period equilibrium model. Liu and Pan [29] assumed continuous trading of the derivatives, again completing the market in a different way. We shall assume a given finite set of contingent claims available for purchase (or short sale) at given unit market prices. For tractability, we will also assume the investor's preferences to be described by an exponential utility function.

In a complete market, derivatives are redundant because they can be replicated by dynamic trading in the underlying. In that case, the problem studied here is not well-posed. The utility-indifference pricing mechanism, introduced by Hodges and Neuberger [18], asks at what price an investor is indifferent, with respect to maximum expected utility, about a given derivatives position in an incomplete market. It turns out that this question, rephrased in terms of quantity of derivatives for given market prices, yields the answer as to the optimal static position to take in the derivatives.

1.2 The Investment Problem

We describe here the problem in the simplest setting of one stock and one derivative. The more general problem where there are many options available is studied in [20].

We suppose there is an investor with initial investment capital $v > 0$. He can trade dynamically in a stock and a bank account. The stock price process is denoted S, and is defined on a probability space $(\Omega, \mathcal{F}, P)$, where P is the investor's subjective measure. For simplicity, we take the interest rate to be zero throughout. The analysis can be modified for a nonzero interest rate by switching to discounted variables.

He can also trade statically in a derivative security that pays the random amount G on date T. The market price of the derivative is p.

Assumption 1.1 *The payoff G is bounded.*

Remark 1.2 *Although the strangle given as an example in the previous section, and other common strategies involving call option payouts are not bounded, we shall assume they are replaced by their cutoff versions, where the cutoffs are conditioned on sufficiently extreme events so as not to affect practical accuracy. For some relaxations of this assumption, see [9], [3] or [22].*

Assumption 1.3 *The investor has an exponential utility function:*

$$U(x) = -e^{-\gamma x},$$

where $\gamma > 0$ is his risk-aversion parameter.

The investor buys α derivatives for price αp at time zero, and holds them till expiration at time T. With his remaining capital $v - \alpha p$, he trades continuously in the Merton portfolio, that is, the stock and bank account. Let θ_t be the amount held in the stock at time t, and $(X_t)_{0 \leq t \leq T}$ the value of this latter portfolio

$$X_t = v - \alpha p + \int_0^t \theta_t \mathrm{d}t.$$

Then the investor's problem is to maximize over both the dynamic control θ and the static derivative quantity α his expected utility of terminal wealth.

Let

$$u(x, \alpha G, \gamma) = \sup_\theta \mathbb{E}\left\{-e^{-\gamma(X_T + \alpha G)}\right\},$$

the optimal expected utility from trading the stock with initial capital x, and an option payout (or random endowment) αG at the terminal time. Then the investor's problem is to find

$$\max_\alpha u(v - \alpha p, \alpha G, \gamma). \tag{1}$$

This is an optimization of a function which is itself the value function of a stochastic control problem. In the next section, we show that it is closely related to another control problem, namely that of finding the utility-indifference price of the derivatives.

2 Indifference Pricing and the Dual Formulation

We consider a market with two tradeable instruments: the stock, or the risky asset, S and the riskless bond. We assume that S is a locally bounded $(P, \mathbb{F})$-semi-martingale where $\mathbb{F}$ is a filtration on the given probability space satisfying the usual conditions.

2.1 Utility Indifference Prices

We start by defining the set of absolutely continuous (equivalent) local martingale measures $\mathbb{P}_a$ ($\mathbb{P}_e$) as

$$\begin{aligned}\mathbb{P}_a &= \{Q \ll P \,|\, S \text{ is a local } (Q, \mathbb{F})\text{-martingale}\},\\ \mathbb{P}_e &= \{Q \sim P \,|\, S \text{ is a local } (Q, \mathbb{F})\text{-martingale}\}.\end{aligned}$$

The indifference price of the claim G is specified through the solutions of two stochastic control problems. The first is the classical Merton optimal investment problem

$$M(x, \gamma) = \sup_{\Theta} \mathbb{E}\left\{-e^{-\gamma X_T}\right\}, \tag{2}$$

where Θ is a suitable set of trading strategies made precise below. The second is the optimal investment problem for the buyer of the claim,

$$u(x, G, \gamma) = \sup_{\Theta} \mathbb{E}\left\{-e^{-\gamma(X_T+G)}\right\}. \tag{3}$$

Clearly, $u(x, 0, \gamma) = M(x, \gamma)$.

Then the (buyer's) indifference price h of G is defined by

$$u(x - h(G, \gamma), G, \gamma) = M(x, \gamma). \tag{4}$$

To be specific about the permissible trading strategies, we first introduce $\mathbb{P}_f(P)$, the set of measures in $\mathbb{P}_a$ with finite relative entropy with respect to P, where the relative entropy of a measure, $H(Q|P)$ is defined

$$H(Q|P) = \begin{cases} \mathbb{E}\left\{\frac{\mathrm{d}Q}{\mathrm{d}P}\log\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right\}, & Q \ll P,\\ \infty, & \text{otherwise}.\end{cases} \tag{5}$$

Assumption 2.1 *There exists an equivalent local martingale measure with finite relative entropy*

$$\mathbb{P}_f(P) \cap \mathbb{P}_e \neq \emptyset. \tag{6}$$

We denote by Θ the set of S-integrable trading strategies for which the corresponding wealth process is a martingale under all measures in $\mathbb{P}_f(P)$ with respect to the filtration $\mathbb{F}$.

2.2 Dual Problem: Relative Entropy Minimization

It is convenient for interpretation and computation to study the dual of the buyer's stochastic control problem (3). The dual of the maximization of expected exponential utility over trading strategies with a static option position is the problem of minimizing this option's payoff over a space of measures penalized by the relative entropy of the measure. Before stating these results and related references, we sketch the relation between these problems.

2.2.1 Sketch of Duality Theory

Let us start by weakening the martingale equality constraint to an inequality

$$u(x,G,\gamma) = \sup_{\theta} \mathbb{E}\left\{U(X_T+G)\right\}, \tag{7}$$

$$\text{s.t. } \mathbb{E}^Q\left\{X_T\right\} \leq x, \quad \forall\, Q \in \mathbb{P}_f(P). \tag{8}$$

As G is bounded, we define $\xi = X_T + G$. For a measure $Q \in \mathbb{P}_f(P)$, a positive constant Λ, and a trading strategy such that (8) holds,

$$\mathbb{E}\left\{U(\xi)\right\} \leq \mathbb{E}\left\{U(\xi) - \Lambda\frac{\mathrm{d}Q}{\mathrm{d}P}(\xi - G - x)\right\},$$

since we are adding a nonnegative quantity to the right hand side. Taking the supremum over all allowable strategies ω by ω, which is equivalent to taking supremum over ξ's, we get

$$\sup_{\xi} \mathbb{E}\left\{U(\xi)\right\} \leq \mathbb{E}\left\{V\left(\Lambda\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right\} + \Lambda\left(x + \mathbb{E}^Q\{G\}\right), \quad \forall Q \in \mathbb{P}_f(P), \text{ and } \forall \Lambda \geq 0, \tag{9}$$

where $V(\Lambda)$ is the Fenchel-Legendre transform of $U(x)$

$$V(\Lambda) = \sup_{x\in\mathbb{R}} \left(U(x) - \Lambda x\right) = \frac{\Lambda}{\gamma}\log\frac{\Lambda}{\gamma} - \frac{\Lambda}{\gamma}, \quad \Lambda \geq 0. \tag{10}$$

Note that $\mathbb{E}\left\{V\left(\Lambda\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right\}$ is finite for $\Lambda < \infty$ as Q is in $\mathbb{P}_f(P)$. Furthermore, taking the infimum in (9) over all possible Q's and Λ's, we get

$$\sup_{\xi} \mathbb{E}\left\{U(\xi)\right\} \leq \inf_{\Lambda\geq 0}\inf_{Q\in\mathbb{P}_f(P)} \mathbb{E}\left\{V\left(\Lambda\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right\} + \Lambda\left(x + \mathbb{E}^Q\{G\}\right). \tag{11}$$

Writing $V(\Lambda)$ explicitly as defined in (10), the above reduces to

$$\sup_{\pi} \mathbb{E}\left\{U(\xi)\right\} \leq \inf_{\Lambda\geq 0}\inf_{Q\in\mathbb{P}_f(P)} \left(\frac{\Lambda}{\gamma}\log\frac{\Lambda}{\gamma} - \frac{\Lambda}{\gamma} + \Lambda x + \Lambda\left(\frac{1}{\gamma}H(Q|P) + \mathbb{E}^Q\{G\}\right)\right). \tag{12}$$

The optimizing Q in (12) does not depend on Λ, which will not be true for other common utility functions, and in particular log and power.

One part of the problem is showing the existence and uniqueness of $\widehat{Q} \in \mathbb{P}_f(P)$ that minimizes

$$\frac{1}{\gamma}H(Q|P) + \mathbb{E}^Q\{G\}$$

under Assumption 2.1 and the boundedness assumption on G. In [9], Delbaen *et al.* reduce this problem to the case, where $G = 0$ with a measure transformation and they conclude as needed by using the results of Fritelli [16]. Given $\widehat{Q}$, the parameter $\widehat{\Lambda}$ that minimizes the right hand side of (12) is given by

$$\widehat{\Lambda} = \gamma \exp\left(-\gamma\left(x + \mathbb{E}^{\widehat{Q}}\{G\} + \frac{1}{\gamma}H(\widehat{Q}|P)\right)\right)$$

which is strictly positive. From (10), the corresponding maximizer $\widehat{\xi}$ of the right hand side of (9) is given by

$$\widehat{\xi} = -\frac{1}{\gamma}\log\left(\frac{\widehat{\Lambda}}{\gamma}\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}P}\right) = \widehat{X}_T + G.$$

With the above representation, we conclude that $\widehat{X}_T$ is a martingale under $\widehat{Q}$. Concluding that the wealth is a stochastic integral of a trading strategy in Θ with respect to S is more involved, and we refer the reader to Lemma 3.3 of [9], or Proposition 1.2.3 of [3]. Then $\widehat{X}_T$ is optimal for the primal problem, because for any trading strategy in Θ, we have

$$\mathbb{E}\{U(\xi)\} \leq \mathbb{E}\left\{U(\xi) - \widehat{\Lambda}\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}P}(\xi - G + x)\right\} \leq \mathbb{E}\left\{U(\widehat{\xi}) - \widehat{\Lambda}\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}P}(\widehat{\xi} - G + x)\right\} = \mathbb{E}\left\{U(\widehat{\xi})\right\}.$$

2.2.2 Main Duality Formula

For exponential utility, a duality result including a contingent claim in a general semimartingale setting was shown by Delbaen *et al.* [9]. They show the equality of the solutions

$$u(x, G, \gamma) = -\exp\left(-\gamma \inf_{Q\in\mathbb{P}_f(P)}\left(\mathbb{E}^Q\{G\} + \frac{1}{\gamma}H(Q|P)\right) - \gamma x\right), \tag{13}$$

and they conclude that the optimizers in both problems are achieved in their feasible sets. Moreover, the minimizing measure is equivalent to P. They give different theorems corresponding to different feasible sets of strategies.

Rouge and El Karoui [35] studied indifference pricing with a Brownian filtration using backward stochastic differential equations. Kabanov and Stricker [25] showed that some of the assumptions in [9] were superfluous. Becherer [3] extended the results of [9] by using the extensions of [25]. It is worth noting that our set of feasible trading strategies corresponds to Θ_2 in [3]. The duality relation for general utility functions defined on $\mathbb{R}_+$ was considered in [27], and for utility functions defined on $\mathbb{R}$ in [36]. However, these papers do not involve a claim. The results were extended to include a claim in [6] and [33].

2.3 Expressions for Indifference Prices

It is easy to see from the duality formula (13) that the dependence of the value function u in (3) on the initial wealth x is simply through the multiplicative factor $-e^{-\gamma x}$. This is the typical *ansatz* one would make in a dynamic programming approach to solving the problem in Markovian models, and we see that the separation of variables is quite general. Since the indifference price h, defined in (4), is merely an adjustment in initial wealth level for setting G to zero, it immediately follows that

$$h(G,\gamma) = \frac{1}{\gamma}\log\left(\frac{M(0,\gamma)}{u(0,G,\gamma)}\right), \tag{14}$$

and substituting the specific expressions for the duals of the buyer's and Merton problems, we can write

$$h(G,\gamma) = \inf_{Q\in\mathbb{P}_f(P)}\left(\mathbb{E}^Q\left\{G\right\} + \frac{1}{\gamma}H(Q|P)\right) - \inf_{Q\in\mathbb{P}_f(P)}\frac{1}{\gamma}H(Q|P). \tag{15}$$

Notice that h is independent of the initial wealth.

Indifference Price with an Alternative Expression

The entropy terms in (15) can be combined into one entropy term with a different prior measure, the minimal entropy martingale measure, which is the measure minimizing the relative entropy in $\mathbb{P}_f(P)$:

$$Q^0 = \arg\min_{Q\in\mathbb{P}_f(P)} H(Q|P). \tag{16}$$

Results on the existence and uniqueness of this measure can be found in Fritelli [16] and Grandits and Rheinländer [17].

Theorem 2.1 *(Theorem 2.2-5 of Fritelli [16] and Theorem 2.2 of Delbaen et al. [9]) Under assumption (6), Q^0 exists, is unique, is in $\mathbb{P}_f(P)\cap\mathbb{P}_e$ and its density has the form*

$$\frac{\mathrm{d}Q^0}{\mathrm{d}P} = c_0 e^{-\gamma X_T^0}, \tag{17}$$

where X_T^0 is the optimal terminal wealth associated with the solution of the Merton problem (2) and

$$\log c_0 = H(Q^0|P) < \infty.$$

Moreover, X_T^0 is attained by a trading strategy in Θ.

Proposition 2.2 *Assume*

$$\frac{\mathrm{d}Q^0}{\mathrm{d}P} \in L^2(P). \tag{18}$$

The indifference price $h(G,\gamma)$ of the bounded claim G is equal to

$$h(G,\gamma) = \inf_{Q\in\mathbb{P}_f(Q^0)}\left(\mathbb{E}^Q\left\{G\right\} + \frac{1}{\gamma}H(Q|Q^0)\right). \tag{19}$$

Proof: From (17), the relative entropy of a measure $Q \ll P$ with respect to P can be written in terms of its relative entropy with respect to Q^0 as

$$H(Q|P) = H(Q|Q^0) + H(Q^0|P) - \gamma \mathbb{E}^Q \left\{ X_T^0 \right\}. \tag{20}$$

If we choose Q in $\mathbb{P}_f(P)$, the last term on the right hand side of (20) is zero as X_T^0 is a martingale under Q. Moreover, Q is also in $\mathbb{P}_f(Q^0)$ as all terms in (20) except from $H(Q|Q^0)$ are finite and $Q^0 \sim P$.

To deduce the reverse conclusion, we note that if the assumption given in (18) holds, $e^{\gamma |X_T^0|}$ is in $L^1(Q^0)$ because

$$\mathbb{E}^{Q^0} \left\{ e^{\gamma X_T^0} \right\} = \mathbb{E} \left\{ \frac{\mathrm{d}Q^0}{\mathrm{d}P} e^{\gamma X_T^0} \right\} = c_0 < \infty.$$

Using Lemma 3.5 of Delbaen *et al.* [9] for the random variable $|X_T^0|$, we deduce that

$$\mathbb{E}^Q \left\{ |X_T^0| \right\} \leq H(Q|Q^0) + e^{-1} \mathbb{E}^{Q^0} \left\{ e^{\gamma |X_T^0|} \right\}.$$

In other words, X_T^0 is in $L^1(Q)$ for all $Q \in \mathbb{P}_f(Q^0)$. As the last term on the right hand side of (20) is now guaranteed to be finite for all $Q \in \mathbb{P}_f(Q^0)$, we conclude that $\mathbb{P}_f(Q^0) \subset \mathbb{P}_f(P)$. But then the last term on the right hand side of (20) is zero for all $Q \in \mathbb{P}_f(Q^0)$. □

The expression (19) points out buyer's tendency to price the claim with its worst-case expectation penalized by the entropic distance from the prior risk-neutral measure, Q^0.

As $Q^0 \sim P$, we can also apply the duality result to (19), and obtain

$$h(G, \gamma) = -\frac{1}{\gamma} \log \left(-\sup_{\Theta} \mathbb{E}^{Q^0} \left\{ -e^{-\gamma(X_T^0 + G)} \right\} \right). \tag{21}$$

3 Utility Indifference Pricing

The investor who is contemplating buying α options at market price p will maximize her expected terminal utility by choosing the optimal number of options (assuming it exists)

$$\alpha^* = \arg\max_{\alpha} u(x - \alpha p, \alpha G, \gamma). \tag{22}$$

Throughout, we assume a linear pricing rule in the market. From the definition (4) of h, this is equivalent to

$$\begin{aligned} \alpha^* &= \arg\max_{\alpha} M(x - \alpha p + h(\alpha G, \gamma), \gamma), \\ &= \arg\max_{\alpha} -e^{-\gamma(x - \alpha p + h(\alpha G, \gamma)) - H(Q^0|P)}, \end{aligned}$$

using (13) with $G \equiv 0$ and the definition (16) of Q^0. Extracting the terms which depend on α, this reduces to

$$\alpha^* = \arg\max_{\alpha} \left(h(\alpha G, \gamma) - \alpha p \right). \tag{23}$$

In other words, *the optimal derivatives position is found from the Fenchel-Legendre transform of the indifference price as a function of quantity, evaluated at the market price.* From this, it is clear that existence and uniqueness of the solution to our optimization problem (22) will depend on the strict concavity of the indifference price as a function of quantity, and the value of the market price p.

To this end, in the next few sections, we study some properties of the indifference price $h(\alpha G, \gamma)$ as a function of α and γ.

3.1 Dependence on the Risk-Aversion Parameter

Large Risk-Aversion Limit

As investors become more risk-averse, the price they are willing to pay for a contingent claim tends to the subhedging price of the claim

$$\lim_{\gamma \uparrow \infty} h(G, \gamma) = \inf_{Q \in \mathbb{P}_e} \mathbb{E}^Q\{G\}. \tag{24}$$

In fact it is easy to show from (15) that the limit is the infimum of the expected payoff over all measures in $\mathbb{P}_f(P) \cap \mathbb{P}_e$; however, to show that it is also the infimum over all measures in $\mathbb{P}_e$, requires more work. For a proof of the result, we refer the reader to the Corollary 5.1 of [9].

Zero Risk-Aversion Limit

It was proved by Becherer [3] that in the limit as the risk aversion parameter tends to zero, the indifference price goes to the expected payoff under the minimal entropy martingale measure:

$$\lim_{\gamma \downarrow 0} h(G, \gamma) = \mathbb{E}^{Q^0}\{G\}. \tag{25}$$

Monotonicity

Monotonicity of the indifference price as a function of the risk aversion parameter can be seen directly from (15). For γ_1, let us define the measure that attains the minimum in (15) as Q_1, which is guaranteed to exist by the duality result. Then,

$$h(G, \gamma_1) = \mathbb{E}^{Q_1}\{G\} + \frac{1}{\gamma_1}\left(H(Q_1|P) - H(Q^0|P)\right). \tag{26}$$

The last term on the right hand side is nonnegative as Q^0 is the minimal entropy martingale measure. Dividing this positive term by $\gamma_2 > \gamma_1$ instead of γ_1, we only make the right hand side of (26) smaller:

$$h(G, \gamma_1) \geq \mathbb{E}^{Q_1}\{G\} + \frac{1}{\gamma_2}\left(H(Q_1|P) - H(Q^0|P)\right). \tag{27}$$

As $h(G, \gamma_2)$ is the infimum of

$$\mathbb{E}^{Q}\{G\} + \frac{1}{\gamma_2}\left(H(Q|P) - H(Q^0|P)\right)$$

over Q in $\mathbb{P}_f(P)$ and as Q_1 is in this feasible set, we conclude that

$$\mathbb{E}^{Q_1}\{G\} + \frac{1}{\gamma_2}\left(H(Q_1|P) - H(Q^0|P)\right) \geq h(G, \gamma_2). \tag{28}$$

Combining equations (27) and (28), we conclude the monotonicity of the indifference price as a function of the risk aversion parameter: $h(G, \gamma_1) \geq h(G, \gamma_2)$ for $\gamma_2 > \gamma_1$.

The results on the limits of the indifference price and its monotonicity are also given by Rouge and El Karoui [35] in the context of Itô process models. They also show that the indifference price takes all values between these bounds for different levels of the risk-aversion parameter γ.

3.2 Extreme Quantity Asymptotics

Becherer [3] also notes that the indifference price is a decreasing function of the risk aversion parameter and satisfies the property

$$h(\alpha G, \gamma) = \alpha h(G, \alpha\gamma) \text{ for } \alpha > 0, \tag{29}$$

which follows easily from (15). Therefore, the fair price of the claim G, introduced by Davis in [7], and defined as

$$\lim_{\alpha \downarrow 0} \frac{h(\alpha G, \gamma)}{\alpha},$$

the marginal value of introducing G, is given by:

$$\lim_{\alpha \downarrow 0} \frac{h(\alpha G, \gamma)}{\alpha} = \lim_{\gamma \downarrow 0} h(G, \gamma) = \mathbb{E}^{Q^0}\{G\}.$$

Equations (24) and (29) imply that

$$\lim_{\alpha \uparrow \infty} \frac{1}{\alpha} h(\alpha G, \gamma) = \inf_{Q \in \mathbb{P}_e} \mathbb{E}^{Q}\{G\}. \tag{30}$$

Moreover,

$$\lim_{\alpha \downarrow -\infty} \frac{1}{\alpha} h(\alpha G, \gamma) = -\lim_{\alpha \uparrow \infty} \frac{1}{\alpha} h\left(\alpha(-G), \gamma\right) = -\inf_{Q \in \mathbb{P}_e} \mathbb{E}^{Q}\{-G\} = \sup_{Q \in \mathbb{P}_e} \mathbb{E}^{Q}\{G\}. \tag{31}$$

The limit is known as the superhedging price of the claim.

3.3 Differentiability

In this section, we prove the following:

Proposition 3.1 *The derivative of the indifference price of αG with respect to α exists for $\alpha \in \mathbb{R}$ and*

$$\frac{\partial}{\partial \alpha} h(\alpha G, \gamma) = \mathbb{E}^{Q^\alpha}\left\{G\right\}, \tag{32}$$

where Q^α is the measure minimizing entropy with respect to P^α over $\mathbb{P}_f(P^\alpha)$, with

$$\frac{\mathrm{d}P^\alpha}{\mathrm{d}P} = c_\alpha e^{-\gamma\alpha G}, \text{ and } c_\alpha = \left(\mathbb{E}\left\{e^{-\gamma\alpha G}\right\}\right)^{-1}. \tag{33}$$

Proof: We directly calculate the limits in the definition of the derivative:

$$\lim_{\epsilon\downarrow 0} \frac{h((\alpha+\epsilon)G,\gamma) - h(\alpha G,\gamma)}{\epsilon} = \lim_{\epsilon\uparrow 0} \frac{h((\alpha+\epsilon)G,\gamma) - h(\alpha G,\gamma)}{\epsilon} = \mathbb{E}^{Q^\alpha}\{G\}. \tag{34}$$

By (14), the first limit is equal to

$$\lim_{\epsilon\downarrow 0} \frac{1}{\gamma\epsilon} \log\left(\frac{\sup_\Theta \mathbb{E}\{-e^{-\gamma(X_T+\alpha G)}\}}{\sup_\Theta \mathbb{E}\{-e^{-\gamma(X_T+(\alpha+\epsilon)G)}\}}\right). \tag{35}$$

In terms of P^α, (35) can be expressed as

$$\lim_{\epsilon\downarrow 0} \frac{1}{\gamma\epsilon} \log\left(\frac{\sup_\Theta \mathbb{E}^{P^\alpha}\left\{-e^{-\gamma X_T}\right\}}{\sup_\Theta \mathbb{E}^{P^\alpha}\left\{-e^{-\gamma(X_T+\epsilon G)}\right\}}\right). \tag{36}$$

This expression is the limit as ϵ goes to zero of the indifference price per unit of ϵG options, from the viewpoint of an investor with subjective measure P^α (compare with (14)), if we can show that for this investor the set of allowable trading strategies is Θ. In other words, we need to show that $\mathbb{P}_f(P) = \mathbb{P}_f(P^\alpha)$. Using the definition of P^α given in (33) and that G is bounded, the relative entropy of a measure Q with respect to P^α can be written in terms of its entropy with respect to P as

$$H(Q|P^\alpha) = H(Q|P) - \log c_\alpha + \mathbb{E}^Q\{\gamma\alpha G\}. \tag{37}$$

Equality of the sets follows trivially.

As $P^\alpha \sim P$, (6) is satisfied with the new prior P^α, and we can use the duality result to re-write (36) as

$$\lim_{\epsilon\downarrow 0} \inf_{Q\in\mathbb{P}_f(P^\alpha)} \left(\mathbb{E}^Q\{G\} + \frac{1}{\gamma\epsilon} H(Q|P^\alpha)\right) - \inf_{Q\in\mathbb{P}_f(P^\alpha)} \left(\frac{1}{\gamma\epsilon} H(Q|P^\alpha)\right). \tag{38}$$

Taking the limit as ϵ goes to zero is equivalent to taking the limit as the risk aversion parameter goes to zero with the prior P^α fixed, and we conclude by (25) that the limit exists and is equal to $\mathbb{E}^{Q^\alpha}\{G\}$. The result for the second limit in (34) follows similarly. □

3.4 Strict Concavity

The indifference price is concave in α as it is the infimum over $\mathbb{P}_f(P)$ of the affine function of α

$$\alpha\mathbb{E}^Q\{G\} + \frac{1}{\gamma}(H(Q|P) - H(Q^0|P)).$$

Since it has also a well-defined gradient, the indifference price is in fact differentiable. Moreover, the derivative of the indifference price is bounded between the limits given in (30) and (31). Therefore, the existence of α^* as defined in (23) is guaranteed for market prices that are between these limits, in other words for market prices that are between the superhedging and subhedging prices of the option G.

Another interpretation of this result follows from Theorem 5.3 in [37], which states that the interval given by the superhedging and subhedging prices of an option is exactly the interval of no-arbitrage prices of that option. Therefore, the existence of α^* is guaranteed for arbitrage-free market prices, p.

In this theorem, Schachermayer also points out that there are two cases. Either the subhedging price of an option is equal to its superhedging price, in which case the option is replicable and the set of no-arbitrage prices of the option is a single point, or the subhedging price of an option is strictly less than its superhedging price, in which case the set of no-arbitrage prices is the open interval

$$\left(\inf_{Q\in\mathbb{P}_e} \mathbb{E}^Q\{G\}, \sup_{Q\in\mathbb{P}_e} \mathbb{E}^Q\{G\}\right). \tag{39}$$

In the following proposition, we show strict concavity of the indifference price of an option which is not replicable in the sense specified by Schachermayer in [37].

Proposition 3.2 *The indifference price of αG is a strictly concave function of $\alpha \in \mathbb{R}$ if the subhedging price of G is strictly less than its superhedging price.*

Proof: Let us start by fixing $\alpha_2 > \alpha_1$. We will assume that $h(\alpha G, \gamma)$ is a linear function of α on the line segment between α_1 and α_2 and derive a contradiction. We define the measures $P^1 = P^{\alpha_1}$ and $P^2 = P^{\alpha_2}$ as in (33), and Q^1 and Q^2 as the measures that minimize the entropy with respect to P^1 and P^2, respectively. From (37), we get

$$(H(Q^2|P^1) - H(Q^1|P^1)) + ((H(Q^1|P^2) - H(Q^2|P^2)) = \gamma(\alpha_1 - \alpha_2)(\mathbb{E}^{Q^2}\{G\} - \mathbb{E}^{Q^1}\{G\}).$$

As $\mathbb{E}^{Q^i}\{G\}$ is the slope of the the indifference price at α_i, the right hand side is equal to zero, by our linearity assumption. Now, Q^1 is the minimizer of $H(Q|P^{\alpha_1})$ over $\mathbb{P}_f(P)$ which includes Q^2, so the first term in the left hand side is nonnegative. The same conclusion applies to the second term, therefore both terms are zero. Then the uniqueness of the minimal entropy martingale measure (see Theorem 2.1) implies that $Q^1 = Q^2$.

Using (17) and (33), the density of Q^i can be specified as follows

$$\frac{\mathrm{d}Q^i}{\mathrm{d}P} = c^i e^{-\gamma(X_T^i + \alpha_i G)}, \text{ for } i = 1, 2,$$

where X_T^1 and X_T^2 are the optimal terminal wealths in the optimization problems defining $u(0, \alpha_1 G, \gamma)$ and $u(0, \alpha_2 G, \gamma)$. Therefore, these terminal wealths are attained by two trading strategies in Θ. Combining the above density representation with the equality of Q^1 and Q^2, we get

$$(\alpha_2 - \alpha_1)G = \text{const} + X_T^1 - X_T^2.$$

Then for all $Q \in \mathbb{P}_f(P)$, $\mathbb{E}^Q\{G\}$ is a constant (and is equal to the Davis' fair price). However, Corollary 5.1 in Delbaen *et al.* [9] states that the supremum of $\mathbb{E}^Q\{G\}$ over $Q \in \mathbb{P}_e(P)$ is equal to the supremum over $Q \in \mathbb{P}_f(P) \cap \mathbb{P}_e$ and therefore the former is also equal to the Davis fair price. This implies that the set of no-arbitrage prices is a single point and the option is replicable, which is a contradiction. We conclude that the indifference price is a strictly concave function of α. □

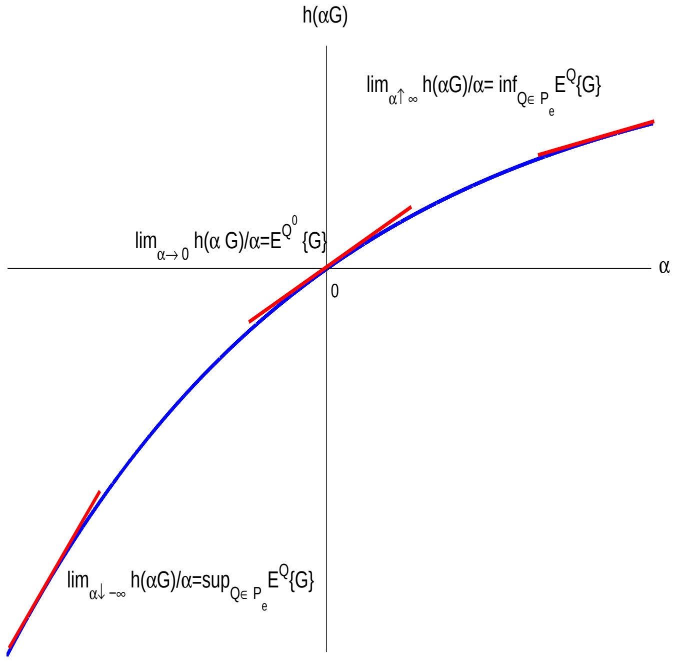

Figure 2: The indifference price is a strictly concave function of the number of derivatives. The limit of the slope as α goes to infinity is the subhedging price of the derivative, and the superhedging price as α goes to minus infinity. The slope at α equal to zero is the Davis fair price.

3.5 Several Contingent Claims

In realistic situations, there are many contingent claims available for an investor to incorporate in her portfolio. Suppose there are N options in the market with bounded payoffs G_i and market prices p_i for $i = 1, .., N$. The optimal number of each option to hold can be formulated in a similar way to the single option case. Let $\alpha = (\alpha_1, .., \alpha_N)$ denote a static position in the options. Then it is clear from our previous analysis that the optimal static position α^* in the

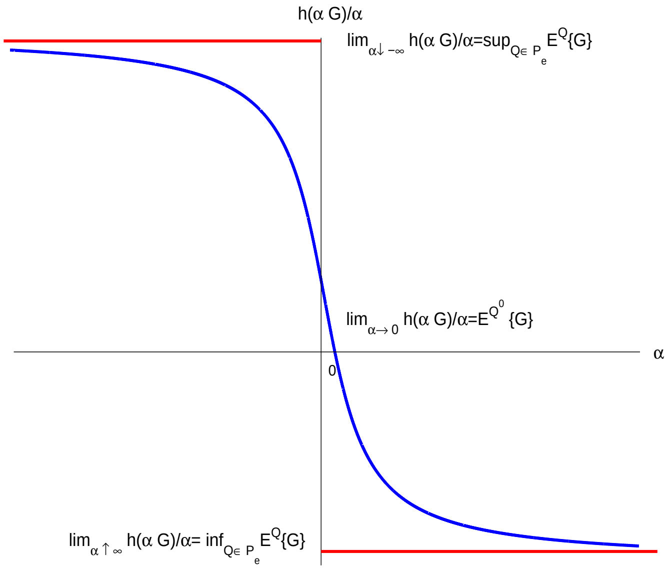

Figure 3: The indifference price over α. The limit of the slope as α goes to infinity is the subhedging price of the derivative, and the superhedging price as α goes to minus infinity. The slope at α equal to zero is the Davis' fair price.

derivatives is given by

$$\alpha^* = \arg\max_{\alpha} \left(h(\alpha \cdot G, \gamma) - \alpha \cdot p \right), \tag{40}$$

where $\alpha \cdot G = \sum_{i=1}^{N} \alpha_i G_i$. The problem is now to find conditions on the indifference price as a function of the vector α and the market price vector p of the derivatives G for existence and uniqueness of an optimal investment strategy. In [20], we show:

- Assuming that none of the claims G_i is redundant (in a sense made precise there), the set of no-arbitrage price vectors is an open convex subset V of $\mathbb{R}^N$.
- The indifference price is a strictly concave function of α with a well-defined gradient.
- For each market price vector p in V, there exists a unique optimal derivatives position α^*.

4 Stochastic Volatility Models

Stochastic volatility models are popular because they capture the deviation of stock price data from the Black-Scholes geometric Brownian motion model in a parsimonious way. They were originally introduced in the late 1980's by Hull and White [19] and others for option pricing. Much of their success derives from their predicted option prices exhibiting the implied volatility skew that is observed in many options markets. See [14], for example, for details.

The risky asset S is modelled by the following SDEs

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma(Y_t) S_t \,\mathrm{d}W_t^1, & (41)\\ \mathrm{d}Y_t &= b(Y_t)\,\mathrm{d}t + a(Y_t)\left(\rho\,\mathrm{d}W_t^1 + \rho'\,\mathrm{d}W_t^2\right), & (42)\end{aligned}$$

where Y is the volatility driving factor correlated with the stock price and $\rho' = \sqrt{1-\rho^2}$. W^1 and W^2 are two independent Brownian motions on the given space and we take the filtration to be the augmented natural filtration of these Brownian motions. We will throughout assume the following.

Assumption 4.1 *i. σ and a are smooth and bounded with bounded derivatives,*

ii. $0 < L \leq \sigma$, for some constant $L < \infty$,

iii. b is smooth with bounded first derivative.

In this class of models, the wealth process satisfies

$$\mathrm{d}X_t = \mu\pi_t \,\mathrm{d}t + \sigma(Y_t)\pi_t \,\mathrm{d}W_t^1, \qquad X_0 = x, \tag{43}$$

where π_t represents the dollar amount invested in the stock at time t. The set of admissible policies Θ in this model is the set of trading strategies that satisfy the integrability constraint $\mathbb{E}\left\{\int_0^T \pi_t^2 \mathrm{d}t\right\} < \infty$. We consider European claims $G = g(S_T, Y_T)$, where g is smooth and bounded with bounded derivatives. However, many path-dependent claims can be treated in a similar manner with additional variables or boundary conditions. For example, indifference pricing of barrier options was studied in [22]. We do not discuss American-style options here.

The indifference price in this model can be characterized as the solution of a quasilinear PDE by using the HJB equations related to the value functions u and M as in [38]. An alternative is solving the corresponding dual control problems given in (15), which is the approach we will follow here. We start by finding the minimal entropy martingale measure Q^0.

4.1 Q^0 within the Stochastic Volatility Model

The so-called minimal martingale measure, P^0, which was introduced in [13], is defined by the following Girsanov transformation

$$\frac{\mathrm{d}P^0}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_s)}\,\mathrm{d}W_s^1 - \frac{1}{2}\int_0^T \frac{\mu^2}{\sigma^2(Y_s)}\,\mathrm{d}s\right).$$

By our assumptions on $\sigma(y)$, P^0 has finite relative entropy and is equivalent to P. Therefore, Q^0 is in $\mathbb{P}_f(P) \cap \mathbb{P}_e$, and Assumption 2.1 is satisfied.

For a given equivalent local martingale measure, P^λ there exists λ with $\int_0^T \lambda_t^2 \mathrm{d}t < \infty$ a.s. such that

$$\frac{\mathrm{d}P^\lambda}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_s)}\,\mathrm{d}W_s^1 + \int_0^T \lambda_s\,\mathrm{d}W_s^2 - \frac{1}{2}\int_0^T \left(\frac{\mu^2}{\sigma^2(Y_s)} + \lambda_s^2\right)\mathrm{d}s\right). \tag{44}$$

For the moment we shall consider λ in $\mathcal{H}^2(P^\lambda)$, where $\mathcal{H}^2(Q)$ consists of all adapted processes u that satisfy the integrability condition $\mathbb{E}^Q\left\{\int_0^T u_t^2\,\mathrm{d}t\right\} < \infty$. The entropy of such a measure P^λ with respect to P is

$$\mathbb{E}^{P^\lambda}\left\{\frac{1}{2}\int_0^T\left(\frac{\mu^2}{\sigma(Y_t)^2}+\lambda_t^2\right)\mathrm{d}t\right\}. \tag{45}$$

We introduce the stochastic control problem related to maximizing the negative of relative entropy

$$\psi(t,y) = \sup_{\lambda\in\mathcal{H}^2(P^\lambda)} \mathbb{E}^{P^\lambda}\left\{-\frac{1}{2}\int_t^T\left(\frac{\mu^2}{\sigma^2(Y_s)}+\lambda_s^2\right)\mathrm{d}s\Big|Y_t=y\right\}. \tag{46}$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with this stochastic control problem is

$$\begin{aligned}\psi_t + \mathcal{L}_y^0\psi + \sup_\lambda\left(\rho' a(y)\psi_y\lambda - \frac{1}{2}\lambda^2\right) &= \frac{\mu^2}{2\sigma^2(y)}, \quad t<T,\\ \psi(T,y) &= 0,\end{aligned} \tag{47}$$

where $\mathcal{L}_y^0$ is the infinitesimal generator of the process (Y_t) under P^0 and is given by

$$\mathcal{L}_y^0 = \frac{1}{2}\,a^2(y)\frac{\partial^2}{\partial y^2} + \left(b(y) - \rho a(y)\frac{\mu}{\sigma(y)}\right)\frac{\partial}{\partial y}.$$

Performing the maximization in (47), we obtain

$$\begin{aligned}\psi_t + \mathcal{L}_y^0\psi + \frac{1}{2}(1-\rho^2)a^2(y)\psi_y^2 &= \frac{\mu^2}{2\sigma^2(y)}, \quad t<T,\\ \psi(T,y) &= 0,\end{aligned} \tag{48}$$

with the corresponding optimal control

$$\lambda_t^* = \rho' a(Y_t)\psi_y(t,Y_t). \tag{49}$$

The PDE in (48) can be linearized by a logarithmic transformation:

$$\psi(t,y) = \frac{1}{(1-\rho^2)}\log f(t,y).$$

Then f satisfies

$$\begin{aligned}f_t + \mathcal{L}_y^0 f &= (1-\rho^2)\frac{\mu^2}{2\sigma^2(y)}f, \quad t<T,\\ f(T,y) &= 1.\end{aligned} \tag{50}$$

Using the probabilistic representation of the solution of (50), we have

$$\psi(t,y) = \frac{1}{(1-\rho^2)}\log\mathbb{E}^{P^0}\left\{\exp\left(-\int_t^T\frac{\mu^2(1-\rho^2)}{2\sigma^2(Y_s)}\,\mathrm{d}s\right)\Big|Y_t=y\right\}. \tag{51}$$

Lemma 4.1 *Under Assumption 4.1, the density of the minimal entropy martingale measure is*

$$\frac{\mathrm{d}Q^0}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_t)}\mathrm{d}W_t^1 + \int_0^T \lambda^*(t,Y_t)\mathrm{d}W_t^2 - \frac{1}{2}\int_0^T \left(\frac{\mu^2}{\sigma^2(Y_t)} + (\lambda^*(t,Y_t))^2\right)\mathrm{d}t\right) \quad (52)$$

with λ^* and ψ given in (49), (51) respectively. The minimum relative entropy $H(Q^0|P)$ is equal to $-\psi(0,y)$.

Proof: From Theorem 2.9.10 in [28], under Assumption 4.1, f which is given by

$$f(t,y) = \mathbb{E}^{P^0}\left\{\exp\left(-\int_t^T \frac{\mu^2(1-\rho^2)}{2\sigma^2(Y_s)}\,\mathrm{d}s\right)\Big|Y_t = y\right\}$$

is in $C^{1,2}([0,T)\times\mathbb{R})$ and satisfies a polynomial growth condition in y. Moreover, f is the unique solution in this class of functions. As ψ is attained by logarithmic transformation of f which is strictly positive, it will satisfy the same conclusions. Then, the optimality of the solution can be concluded by Theorem IV.3.1 in Fleming and Soner [12]. Notice that $f(t,y)$ is bounded. Under Assumption 4.1, taking the derivative of (50) with respect to y and using the probabilistic representation of the solution, we conclude that $\psi_y(t,y)$ and hence $\lambda^*(t,y)$ are also bounded. Therefore, λ^* defined in (49) is an optimizer of (46). The final step that Q^0 is given by (52), in other words it was sufficient to consider $\lambda\in\mathcal{H}^2(P^\lambda)$ follows from Proposition 3.2 of [17]. We refer to [4] and [22] for detailed calculations. □

4.2 Indifference Pricing PDE

In similar fashion, we introduce the stochastic control problem

$$\nu(t,S,y) = \inf_\lambda \mathbb{E}^{P^\lambda}\left\{\alpha g(S_T,Y_T) + \frac{1}{2\gamma}\int_t^T \left(\frac{\mu^2}{\sigma^2(Y_s)} + \lambda_s^2\right)\mathrm{d}s\Big|S_t = S, Y_t = y\right\}. \quad (53)$$

Given the solution of this stochastic control problem, the indifference price is

$$h(\alpha G,\gamma) = \nu(0,S,y) + \frac{1}{\gamma}\psi(0,y).$$

The HJB equation associated with the stochastic control problem in (53) is as follows:

$$\begin{aligned} \nu_t + \mathcal{L}^0_{S,y}\nu + \inf_\lambda\left(\rho' a(y)\nu_y\lambda + \frac{1}{2\gamma}\lambda^2\right) &= -\frac{\mu^2}{2\gamma\sigma^2(y)}, \quad t<T, \\ \nu(T,S,y) &= \alpha g(S,y), \end{aligned} \quad (54)$$

where $\mathcal{L}^0_{S,y}$ is the generator of (S_t,Y_t) under P^0

$$\mathcal{L}^0_{S,y} = \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)a(y)S\frac{\partial^2}{\partial S\partial y} + \mathcal{L}^0_y.$$

Performing the minimization in (54), we have

$$\begin{aligned} \nu_t + \mathcal{L}^0_{S,y}\nu - \frac{1}{2}\gamma(1-\rho^2)a^2(y)\nu_y^2 &= -\frac{\mu^2}{2\gamma\sigma^2(y)}, \quad t<T, \\ \nu(T,S,y) &= \alpha g(S,y), \end{aligned} \tag{55}$$

with the corresponding optimal control

$$\lambda^\alpha(t,S,y) = -\gamma\rho' a(y)\nu_y(t,S,y).$$

For $h(t,S,y) = \nu(t,S,y) + \frac{1}{\gamma}\psi(t,y)$, it follows from (55) and (48) that $h(t,S,y)$ solves

$$\begin{aligned} h_t + \mathcal{L}^{Q^0}_{S,y}h - \frac{1}{2}\gamma(1-\rho^2)a^2(y)h_y^2 &= 0, \quad t<T, \\ h(T,S,y) &= \alpha g(S,y), \end{aligned} \tag{56}$$

where

$$\mathcal{L}^{Q^0}_{S,y} = \mathcal{L}^0_{S,y} - \rho'\, a(y)\psi_y(t,y)\frac{\partial}{\partial y}$$

is the generator of (S_t, Y_t) under Q^0. The PDE in (56) is the HJB equation associated with the stochastic control problem in (19).

If the claim is contingent on Y_T only ($G = g(Y_T)$), S vanishes from the PDE in (56), and by a logarithmic transformation, the nonlinear PDE in (56) can be linearized. Hence, in this case an explicit solution for the indifference price could be found (see ***article in this volume***),

$$h(t,y) = -\frac{1}{\gamma(1-\rho^2)}\log \mathbb{E}^{Q^0}\left\{ e^{-\gamma(1-\rho^2)g(Y_T)} \mid Y_t = y \right\}. \tag{57}$$

Such a situation arises when Y is not considered as a volatility driving process, but rather as a non-traded asset, where another correlated asset S is available for trading. If further σ is assumed to be independent of y, the minimal entropy martingale measure coincides with the minimal martingale measure (since ψ_y is zero). The canonical example is modelling the traded asset price process as a geometric Brownian motion (see, for example, [32]).

4.2.1 Regularity of the Value Function

The PDE in (55) does not admit an explicit solution and, in this section, we verify the existence and uniqueness of a solution. The traditional existence results for HJB equations require the feasible set of controls to be compact, which prevents us from direct use of these results. Therefore, we will first consider bounded subsets and study the limiting behavior of the value function as this bound is taken to infinity. A similar analysis was conducted by Pham [34] for power utility.

In this section, we will further impose that $a(y) = a$ is a constant. However, as Pham suggests in Remark 2.1 in [34], this assumption is not restrictive as the original model can be re-written in this form with the change of variable suggested in this remark. Then one needs to be careful in verifying the assumptions that guarantee existence and uniqueness of a solution to the new system.

Under P^0, the corresponding stochastic differential equations for the stock price process and the volatility driving process are

$$\begin{aligned} \mathrm{d}S_t &= \sigma(Y_t)S_t\,\mathrm{d}W_t^{0,1}, &(58)\\ \mathrm{d}Y_t &= \left(b(Y_t) - \rho a\frac{\mu}{\sigma(Y_t)}\right)\mathrm{d}t + a\left(\rho\,\mathrm{d}W_t^{0,1} + \rho'\,\mathrm{d}W_t^{0,2}\right), &(59)\end{aligned}$$

where $W^{0,1}$ and $W^{0,2}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, P^0)$. Let us re-write the PDE in (55) as

$$\begin{aligned} \nu_t + \mathcal{L}^0_{S,y}\nu + H(\nu_y) &= -\frac{\mu^2}{2\gamma\sigma^2(y)}, \quad t < T, &(60)\\ \nu(T,S,y) &= \alpha g(S,y), \end{aligned}$$

where

$$H(p) = -\frac{1}{2}\gamma(1-\rho^2)a^2p^2. \tag{61}$$

We also define L by

$$L(q) = \max_{p\in\mathbb{R}}\left[H(p) + qp\right], \tag{62}$$

which is the Fenchel-Legendre transform of H up to the sign of the qp term. The explicit form for L is found as

$$L(q) = \frac{q^2}{2\gamma(1-\rho^2)a^2}. \tag{63}$$

As H is concave in p, and we have the following duality relation

$$H(p) = \min_{q\in\mathbb{R}}\left[L(q) + qp\right].$$

Let us introduce the truncated functions

$$H^k(p) = \min_{q\in B_k}\left[L(q) + qp\right],$$

where B_k is the (compact) interval of length $2k$,

$$B_k = \{q \in \mathbb{R} : |q| \le k\}, \quad k > 0.$$

We consider the following differential equations

$$\begin{aligned} \nu^k_t + \mathcal{L}^0_{S,y}\nu + H^k(\nu^k_y) &= -\frac{\mu^2}{2\gamma\sigma^2(y)}, \quad t < T, &(64)\\ \nu^k(T,S,y) &= \alpha g(S,y), \end{aligned}$$

and assume the following:

Assumption 4.2 *Assume $g(S, y)$ is such that*

$$|\nu^k_y(t, S, y)| \le C,$$

for a positive constant C independent of k.

Theorem 4.2 *Under Assumption 4.1 and Assumption 4.2, equation* (60) *has a unique solution $\nu \in C^{1,2,2}_p([0, T) \times \mathbb{R}_+ \times \mathbb{R})$ with ν continuous in $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$.*

Proof: Under Assumption 4.1, the function $b(y) - \rho a\mu/\sigma(y)$ is C^2 with a bounded first derivative. By Theorem VI.6.2 in Fleming and Rishel [11], there exist unique solutions $\nu^k \in C^{1,2,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R})$ which satisfy polynomial growth conditions in S and y, and which are continuous on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, and which solve (64). Moreover, applying Theorem IV.3.1 in Fleming and Soner [12], these solutions have the following stochastic control representations

$$\nu^k(t, S, y) = \inf_{q \in B_k} \mathbb{E}^{Q^k} \left\{ \int_t^T \left(L(q_s) + \frac{\mu^2}{2\gamma\sigma^2(Y_s)} \right) \mathrm{d}s + \alpha g(S_T, Y_T) \Big| S_t = S, Y_t = y \right\}, \quad (65)$$

where the controlled dynamics under Q^k are given by

$$\mathrm{d}S_t = \sigma(Y_t) S_t \, \mathrm{d}W^{k,1}_t, \quad (66)$$

$$\mathrm{d}Y_t = \left(q_t + b(Y_t) - \rho a \frac{\mu}{\sigma(Y_t)} \right) \mathrm{d}t + a \left(\rho \, \mathrm{d}W^{k,1}_t + \rho' \, \mathrm{d}W^{k,2}_t \right), \quad (67)$$

with $W^{k,1}$ and $W^{k,2}$ being two independent Brownian motions under Q^k.

The function $q \to L(q) + q\nu^k_y$ attains its minimum in $\mathbb{R}$ at $\hat{q}^k(t, S, y) = -\gamma(1-\rho^2)a^2\nu^k_y(t, S, y)$. From Assumption 4.2, there exists a positive constant C independent of k such that

$$|\hat{q}^k(t, S, y)| \le C, \quad \text{for all } t \in [0, T],\ S \in \mathbb{R}_+,\ y \in \mathbb{R}.$$

For $k \ge C$,

$$\begin{aligned} H^k(\nu^k_y) &= \min_{q \in B_k} \left[L(q) + q\nu^k_y \right], \\ &= \min_{q \in \mathbb{R}} \left[L(q) + q\nu^k_y \right], \\ &= H(\nu^k_y), \end{aligned}$$

for all $(t, S, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}$. We deduce that ν^k is a solution to (60) with the desired smoothness conditions.

Assume $\nu^{(1)}$ and $\nu^{(2)}$ are two solutions to (60) that are in $C^{1,2,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R})$ and that satisfy a polynomial growth condition on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, and let $\zeta = \nu^{(1)} - \nu^{(2)}$. Then, ζ solves

$$\begin{aligned} \zeta_t + \mathcal{L}^0_{S,y}\zeta + \frac{1}{2}\gamma(1-\rho^2)a^2(\nu^{(1)}(t, S, y) + \nu^{(2)}(t, S, y))\zeta_y &= 0, \quad t < T, \\ \zeta(T, S, y) &= 0. \end{aligned}$$

The probabilistic representation of the solution indicates that $\zeta(t,S,y)$ is the conditional expectation of zero under the measure defined by the following Girsanov transformation

$$\begin{aligned}\frac{\mathrm{d}P^{\nu}}{\mathrm{d}P^{0}} &= \exp\left(\int_0^T \frac{1}{2}\gamma(1-\rho^2)a^2(\nu^{(1)}(t,S_t,Y_t)+\nu^{(2)}(t,S_t,Y_t))\,\mathrm{d}W_t\right. \\ &\left. -\frac{1}{4}\int_0^T \gamma(1-\rho^2)a^2(\nu^{(1)}(t,S_t,Y_t)+\nu^{(2)}(t,S_t,Y_t))\,\mathrm{d}t\right)\end{aligned}$$

and therefore is equal to zero. Notice that P^{ν} is well-defined under the assumptions on $\nu^{(1)}$ and $\nu^{(2)}$. This guarantees uniqueness of the solution and completes the proof. □

Corollary 4.3 *Let $\lambda^{\alpha}(t,S,y) = -\gamma\rho' a\nu_y(t,S,y)$ where ν is the unique solution to (60) in the class of $C^{1,2,2}([0,T)\times\mathbb{R}_+\times\mathbb{R})$ functions that satisfy a polynomial growth condition on $[0,T]\times\mathbb{R}_+\times\mathbb{R}$. Define Q^{α} as*

$$Q^{\alpha} = \arg\min_{Q\in\mathbb{P}_f}\left(\mathbb{E}^{Q}\{\alpha G\} + \frac{1}{\gamma}H(Q|P)\right), \tag{68}$$

in the stochastic volatility model given in (41) and (42) with $a(y)=a$. Then, the density of Q^{α} is given by

$$\frac{\mathrm{d}Q^{\alpha}}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_t)}\,\mathrm{d}W_t^1 + \int_0^T \lambda^{\alpha}(t,S_t,Y_t)\,\mathrm{d}W_t^2 - \frac{1}{2}\int_0^T\left(\frac{\mu^2}{\sigma^2(Y_t)} + (\lambda^{\alpha}(t,S_t,Y_t))^2\right)\mathrm{d}t\right). \tag{69}$$

Proof: Note that ν_y is bounded. Therefore, so is λ^{α}. From the Verification theorem (Theorem IV.3.1 and Corollary IV.3.1 in [12]), ν is the optimal solution to (53), and λ^{α} is an optimal Markov control policy. As in the case without claims in Section 4.1, the final step to show that the optimal measure Q^{α} defined in (68) is given by (69) again follows from Proposition 3.2 of [17]. We refer to [22] for details. □

Corollary 4.4 *Under Assumption 4.1 and Assumption 4.2, equation* (56) *has a unique solution $h\in C_p^{1,2,2}([0,T)\times\mathbb{R}_+\times\mathbb{R})$ with h continuous in* $([0,T]\times\mathbb{R}_+\times\mathbb{R})$.

Proof: Write $h(t,S,y)=\nu(t,S,y)+\frac{1}{\gamma}\psi(t,y)$, where the ν and ψ are the unique solutions of (60) and (48). The result follows trivially. □

4.3 Asymptotic Expansions

We end this section with two concave approximations of the indifference price as a function of quantity that can be used to compute the approximate optimal derivatives positions under appropriate assumptions. The first is based on a direct power series expansion in the quantity α and so is valid for small quantities; the second is based on the slow time-scale of fluctuation of an important factor of market volatility.

4.3.1 Small α Approximation

The dependence of the indifference price on α appears in the terminal condition of (56), and the indifference price is zero when α is zero. To gain an understanding of the price, we construct a power series expansion for small α:

$$h(t,S,y) = \alpha h^{(1)}(t,S,y) + \alpha^2 h^{(2)}(t,S,y) + \cdots .$$

Inserting this approximation into (56) and grouping order α terms, we deduce that $h^{(1)}$ satisfies

$$\begin{aligned} h^{(1)}_t + \mathcal{L}^{Q^0}_{S,y} h^{(1)} &= 0, \quad t<T, \quad S>0, \\ h^{(1)}(T,S,y) &= g(S,y), \end{aligned} \tag{70}$$

and the solution is given by

$$h^{(1)}(t,S,y) = \mathbb{E}^{Q^0}\{g(S_T,Y_T) \mid S_t = S, Y_t = y\}. \tag{71}$$

Notice that $h^{(1)}(t,S,y)$ is Davis' fair price of the claim [8].

Considering terms of order α^2, we deduce that $h^{(2)}$ satisfies

$$\begin{aligned} h^{(2)}_t + \mathcal{L}^{Q^0}_{S,y} h^{(2)} &= \frac{1}{2}\gamma(1-\rho^2)a^2(Y_t)\left(h^{(1)}_y\right)^2, \quad t<T, \quad S>0, \\ h^{(2)}(T,S,y) &= 0, \end{aligned} \tag{72}$$

and by the Feynman-Kac formula,

$$h^{(2)}(t,S,y) = \mathbb{E}^{Q^0}\left\{-\frac{1}{2}\gamma(1-\rho^2)\int_t^T a^2(Y_s)\left(h^{(1)}_y(s,S_s,Y_s)\right)^2 \mathrm{d}s \mid S_t = S, Y_t = y\right\}. \tag{73}$$

Notice that $h^{(2)}$ is negative reflecting the concavity of h. Considering only these first two terms, the optimal number α^* to hold is given by, approximately,

$$\alpha^* = \frac{p - h^{(1)}(t,S,y)}{2h^{(2)}(t,S,y)}.$$

For $G = g(Y_T)$, an explicit expression for $h^{(2)}(t,y)$ for the indifference price can be found as

$$h^{(2)}(t,y) = \frac{1}{2}\gamma(1-\rho^2)\left(\left(h^{(1)}(t,y)\right)^2 - \mathbb{E}^{Q^0}\left\{g(Y_T)^2\right\} \mid Y_t = y\right) = -\frac{1}{2}\gamma(1-\rho^2)\mathrm{var}^{Q^0}_{t,y}\left(g(Y_T)\right),$$

where $\mathrm{var}^{Q^0}_{t,y}$ denotes the conditional variance given $\{Y_t = y\}$, under the measure Q^0. Of course, in this case, it is easier to obtain the terms directly by a Taylor series expansion on (57). A similar expansion using Malliavin calculus is studied by Davis in [8], where the case with high-correlation between the two assets is considered.

4.3.2 Slow Volatility Approximation

There are a number of approaches for constructing stochastic volatility models that reflect historical and option price data in a parsimonious manner. For example, [1, 10] advocate one-factor stochastic volatility jump-diffusion models, while [2] employ Ornstein-Uhlenbeck Levy processes. The motivation for departing from 'traditional' one-dimensional diffusion models [19] is to bridge the seeming inconsistency between slow mean-reversion estimated from daily stock returns and pronounced implied volatility skews at short maturities.

Another way of capturing these observations is to allow for two-factor stochastic volatility models in which one factor is varying slowly and the other is fast mean-reverting. The advantage is in remaining within a diffusion framework (at the cost of increased dimensionality), where statistical, analytical and simulation tools are extremely convenient. In addition, many problems can be tackled by constructing asymptotic approximations, using singular perturbation techniques for the fast factor, and regular perturbation for the slow one. The asymptotic analysis with just the fast factor is studied for a variety of derivative pricing problems in [14], for partial hedging and utility maximization problems in [24, 23], for exotic options pricing (and in particular passport options with their embedded portfolio optimization problems) in [21], and for indifference prices in [38]. The joint asymptotics for *no arbitrage* European option pricing with both scales appears in [15].

Here, we shall ignore the fast factor and concentrate on the slow scale asymptotics. The assumption is that the time horizon of the investor's problem is long enough that the effect of the fast ergodic factor averages out. To this end, we introduce a small parameter $\delta > 0$ representing the slow scale and replace b and a in (42) by $\delta b(Y_t)$ and $\sqrt{\delta}a(Y_t)$ respectively. Therefore the dynamics of our volatility driving factor is given by

$$Y_t = \delta\, b(Y_t)\, \mathrm{d}t + \sqrt{\delta}\, a(Y_t) \left(\rho\, \mathrm{d}W_t^1 + \rho'\, \mathrm{d}W_t^2\right).$$

We first construct the expansion for the function $\psi(t,y)$, which is related to the value function of the plain Merton problem by $M(x,\gamma) = -e^{-\gamma x + \psi(0,y)}$. It is the solution of the PDE problem (48), which we re-write as

$$\psi_t + \delta \mathcal{M}_2 \psi - \sqrt{\delta}\, \frac{\rho\mu a(y)}{\sigma(y)} \psi_y + \frac{1}{2}\delta(1-\rho^2)\psi_y^2 = \frac{\mu^2}{2\sigma(y)^2}, \tag{74}$$

in $t < T$, with $\psi(T,y) = 0$. Here, we define

$$\mathcal{M}_2 = \frac{1}{2} a(y)^2 \frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}, \tag{75}$$

the infinitesimal generator of Y on the unit time-scale (that is, $\delta = 1$).

We look for a formal expansion

$$\psi = \psi^{(0)} + \sqrt{\delta}\psi^{(1)} + \delta\psi^{(2)} + \cdots, \tag{76}$$

which, for fixed (t,y), converges as $\delta \downarrow 0$. In fact, we want to construct the expansion of the indifference price h up to order δ (to obtain some concavity as a function of α), and so we shall only need the first two terms of the ψ expansion.

Inserting the expansion (76) into (74) and comparing powers of δ, we find that $\psi^{(0)}$ should be chosen to solve

$$\psi_t^{(0)} = \frac{\mu^2}{2\sigma(y)^2},$$

with zero terminal condition. This yields

$$\psi^{(0)}(t,y) = -(T-t)\frac{\mu^2}{2\sigma(y)^2}.$$

Moving to terms of order $\sqrt{\delta}$, we find that $\psi^{(1)}$ should be chosen to solve

$$\psi_t^{(1)} = \frac{\rho\mu a(y)}{\sigma(y)}\psi_y^{(0)},$$

again with zero terminal condition. The solution is

$$\psi^{(1)}(t,y) = (T-t)^2\frac{\rho\mu^3 a(y)}{4\sigma(y)}\frac{\partial}{\partial y}\sigma(y)^{-2}.$$

Now we construct an expansion in powers of $\sqrt{\delta}$ for the indifference pricing function h. We first re-write the PDE (56) as

$$h_t + \frac{1}{2}\sigma(y)^2 S^2 h_{SS} + \sqrt{\delta}\mathcal{M}_1 h + \delta\mathcal{M}_2' h - \frac{1}{2}\delta\gamma(1-\rho^2)a(y)^2 h_y^2 = 0, \tag{77}$$

with $h(T,S,y) = \alpha g(S)$. Here,

$$\mathcal{M}_1 = \rho\sigma(y)a(y)S\frac{\partial^2}{\partial S\partial y} + a(y)\left(\rho'\,\psi_y^{(0)} - \frac{\rho\mu}{\sigma(y)}\right)\frac{\partial}{\partial y} \tag{78}$$

$$\mathcal{M}_2' = \mathcal{M}_2 + \rho'\,a(y)\psi_y^{(1)}\frac{\partial}{\partial y}, \tag{79}$$

and we have substituted the expansion (76) for ψ.

We look for an expansion

$$h = h^{(0)} + \sqrt{\delta}h^{(1)} + \delta h^{(2)} + \cdots. \tag{80}$$

Inserting (80) into (77) and comparing order one terms yields that $h^{(0)}$ should be chosen to solve

$$\mathcal{L}_{BS}(\sigma(y))h^{(0)} = 0, \tag{81}$$

with $h^{(0)}(T,S,y) = \alpha g(S)$, and where

$$\mathcal{L}_{BS}(\sigma(y)) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma(y)^2 S^2\frac{\partial^2}{\partial S^2},$$

the Black-Scholes differential operator at volatility level $\sigma(y)$. Since (81) is simply the Black-Scholes PDE with volatility coefficient $\sigma(y)$, $h^{(0)}$ is the Black-Scholes price of the European contract with payoff αg, which we denote

$$h^{(0)}(t,S,y) = h_{BS}(t,S;\sigma(y)).$$

Comparing terms of order $\sqrt{\delta}$ yields that $h^{(1)}$ should be chosen to solve

$$\mathcal{L}_{BS}(\sigma(y))h^{(1)} = -\mathcal{M}_1 h^{(0)},$$

with zero terminal condition.

At this stage, it is convenient to introduce the notation

$$D_k = S^k \frac{\partial^k}{\partial S^k},$$

the kth logarithmic derivative. We will use shortly D for D_1. As used extensively in the singular perturbation analysis in [14], the solution of the PDE problem

$$\begin{aligned} \mathcal{L}_{BS}(\sigma(y))u_{k,\ell} &= -(T-t)^\ell D_k h_{BS} \\ u_{k,\ell}(T,S,y) &= 0, \end{aligned}$$

in $t < T$ is given by

$$u_{k,\ell}(t,S,y) = \frac{(T-t)^{\ell+1}}{\ell+1} D_k h_{BS}(t,S;\sigma(y)),$$

as can be verified by direct substitution.

Using that $h^{(0)}$ solves the Black-Scholes PDE and the relation

$$\frac{\partial}{\partial \sigma} h_{BS} = (T-t)\sigma S^2 \frac{\partial}{\partial S^2} h_{BS}$$

between the Greeks Vega and Gamma of Black-Scholes European option prices, we have

$$\mathcal{M}_1 h^{(0)} = \rho\sigma^2 a\sigma'(T-t)DD_2 h^{(0)} + (T-t)a\sigma\sigma'\left(\rho' c_0'(T-t) - \frac{\rho\mu}{\sigma}\right) D_2 h^{(0)},$$

where

$$c_0(y) = -\frac{\mu^2}{2\sigma(y)^2}.$$

Therefore,

$$h^{(1)} = \frac{1}{2}\rho a\sigma^2\sigma'(T-t)^2 DD_2 h^{(0)} + a\sigma\sigma'\left(\frac{1}{3}\rho' c_0'(T-t)^3 - \frac{1}{2}\frac{\rho\mu}{\sigma}(T-t)^2\right) D_2 h^{(0)}.$$

Note that, so far, the nonlinear term in (77) has played no role and the approximation $h^{(0)} + \sqrt{\delta}h^{(1)}$ is linear in α. To pick up concavity in α, we need to proceed to the next term in the expansion. Comparing terms of order δ in the PDE (77) with the substituted expansion (80) gives

$$\mathcal{L}_{BS}(\sigma(y))h^{(2)} = -\mathcal{M}_2' h^{(0)} - \mathcal{M}_1 h^{(1)} + \frac{1}{2}\gamma(1-\rho^2)a(y)^2(h_y^{(0)})^2,$$

with zero terminal condition. We write

$$h^{(2)} = h^{(2,1)} + h^{(2,2)} + \alpha^2 F(t,S,y),$$

where

$$\begin{aligned}
\mathcal{L}_{BS}(\sigma(y))h^{(2,1)} &= -\mathcal{M}_2' h^{(0)}, \\
\mathcal{L}_{BS}(\sigma(y))h^{(2,2)} &= -\mathcal{M}_1 h^{(1)}, \\
\mathcal{L}_{BS}(\sigma(y))F &= \frac{1}{2}\gamma(1-\rho^2)a(y)^2(h_y^{(0)})^2,
\end{aligned}$$

each with zero terminal condition.

Using that

$$\frac{\partial^2}{\partial\sigma^2}h_{BS} = (T-t)D_2h_{BS} + \sigma^2(T-t)^2D_2^2h_{BS},$$

and

$$h_{yy}^{(0)} = \sigma''\frac{\partial}{\partial\sigma}h_{BS} + (\sigma')^2\frac{\partial^2}{\partial\sigma^2}h_{BS},$$

we calculate

$$\mathcal{M}_2' h^{(0)} = A(T-t)D_2h_{BS} + \frac{1}{2}a^2\sigma^2(\sigma')^2(T-t)^2D_2^2h_{BS} + \rho' a c_1' \sigma\sigma'(T-t)^3D_2h_{BS},$$

where

$$\begin{aligned}
A(y) &= \frac{1}{2}a(y)^2(\sigma(y)\sigma''(y) + \sigma'(y)^2) + b(y)\sigma(y)\sigma'(y), \\
c_1(y) &= \frac{\rho\mu^3 a(y)}{4\sigma(y)}\frac{\partial}{\partial y}\sigma(y)^{-2}.
\end{aligned}$$

Then,

$$h^{(2,1)} = \frac{1}{2}A(T-t)^2D_2h_{BS} + \frac{1}{6}a^2\sigma^2(\sigma')^2(T-t)^3D_2^2h_{BS} + \frac{1}{4}ac_1'\sigma\sigma'(T-t)^4D_2h_{BS}.$$

Similarly, writing

$$h^{(1)} = c_2(y)(T-t)^2DD_2h^{(0)} + \left(c_3(y)(T-t)^3 + c_4(y)(T-t)^2\right)D_2h^{(0)},$$

and

$$\mathcal{M}_1 = (c_5(y)D + (c_6(y)(T-t) + c_7(y))I)\frac{\partial}{\partial y},$$

where

$$\begin{aligned}
c_2 &= \frac{1}{2}\rho a\sigma^2\sigma', \\
c_3 &= \frac{1}{3}a\sigma\sigma'\rho' c_0', \\
c_4 &= -\frac{1}{2}a\sigma'\rho\mu, \\
c_5 &= \rho\sigma a, \\
c_6 &= a\rho' c_0', \\
c_7 &= -\rho a\mu/\sigma,
\end{aligned}$$

and $I = D_0$ is the identity operator, we obtain

$$\mathcal{M}_1 h^{(1)} = (c_5(y)D + (c_6(y)(T-t) + c_7(y))I)\left(c_2'(T-t)DD_2h^{(0)}\right.$$
$$\left.+(c_3'(T-t)^3 + c_4'(T-t)^2)D_2h^{(0)} + \tilde{c}_2(T-t)^3DD_2^2h^{(0)} + (\tilde{c}_3(T-t)^4 + \tilde{c}_4(T-t)^3)D_2^2h^{(0)}\right),$$

where

$$\tilde{c}_i = c_i\sigma\sigma',$$

and therefore

$$\begin{aligned}
h^{(2,2)} &= \frac{1}{2}c_5c_2'(T-t)D^2D_2h^{(0)} + \frac{1}{4}c_5\tilde{c}_2(T-t)^4D^2D_2^2h^{(0)} \\
&+ \left(\frac{1}{4}c_5c_3'(T-t)^4 + \frac{1}{3}c_5c_4'(T-t)^3 + \frac{1}{3}c_6c_2'(T-t)^3 + \frac{1}{2}c_7c_2'(T-t)^2\right)DD_2h^{(0)} \\
&+ \left(\frac{1}{5}c_5\tilde{c}_3(T-t)^5 + \frac{1}{4}c_5\tilde{c}_4(T-t)^4 + \frac{1}{5}c_6\tilde{c}_2(T-t)^5 + \frac{1}{4}c_7\tilde{c}_2(T-t)^4\right)DD_2^2h^{(0)} \\
&+ \left(\frac{1}{6}c_6\tilde{c}_3(T-t)^6 + \frac{1}{5}c_6\tilde{c}_4(T-t)^5 + \frac{1}{5}c_7\tilde{c}_3(T-t)^5 + \frac{1}{4}c_7\tilde{c}_4(T-t)^4\right)D_2^2h^{(0)} \\
&+ \left(\frac{1}{5}c_6c_3'(T-t)^5 + \frac{1}{4}c_6c_4'(T-t)^4 + \frac{1}{4}c_7c_3'(T-t)^4 + \frac{1}{3}c_7c_4'(T-t)^3\right)D_2h^{(0)}.
\end{aligned}$$

Finally, F is given by

$$(\sigma'(y))^2\mathbb{E}^{P^0}\left\{\int_t^T \frac{1}{2}\gamma(1-\rho^2)a(y)^2\mathcal{V}^2(u,\widetilde{S}_u)\mathrm{d}u\right\}, \tag{82}$$

and the Vega $\mathcal{V} = \frac{\partial}{\partial\sigma}h_{BS}$. In (82), $\widetilde{S}$ is the solution of

$$\mathrm{d}\widetilde{S}_t = \sigma(y)\widetilde{S}_t\,\mathrm{d}\widetilde{W}_t,$$

with $\widetilde{W}$ is a standard Brownian motion in $(\Omega, \mathcal{F}, P^0)$. In other words, $\widetilde{S}$ is a geometric Brownian motion with constant volatility $\sigma(y)$.

For example, if the option $g(\widetilde{S}_T, Y_T)$ were a put option written on $\widetilde{S}$ with strike price K, the explicit form of $\mathcal{V}$ would be as follows:

$$\mathcal{V}(t,\widetilde{S}) = \frac{\widetilde{S}\sqrt{T-t}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_1^2(t,\widetilde{S})\right),$$

where

$$d_1(t,\widetilde{S}) = \frac{\log(\widetilde{S}/K)}{\sigma(y)\sqrt{T-t}} + \frac{1}{2}\sigma(y)\sqrt{T-t}.$$

In this case, the expectation in (82) reduces to

$$\int_t^T \frac{1}{2}\gamma(1-\rho^2)a(y)^2(\sigma_y(y))^2\frac{\exp\left(-\frac{(T-t)^2\sigma(y)^4/4+2\log(K/S)}{(T-2t+u)\sigma(y)^2}\right)}{2\pi\sqrt{T-2t+u}}(T-u)^{3/2}S^{\frac{T-t}{T-2t+u}}K^{\frac{T-3t+2u}{T-2t+u}}\mathrm{d}u,$$

which can be calculated very fast.

Given the market price p per unit of the derivative security with payoff $g(S_T)$, the optimal number of derivative α^* to hold is approximately given by the maximizer of

$$\alpha(\widetilde{h^{(0)}} + \sqrt{\delta}\widetilde{h^{(1)}} + \delta h^{(2,1)}) + \alpha^2 \delta F - \alpha p,$$

where $\widetilde{h^{(0)}} = h^{(0)}/\alpha$ is the α-independent Black-Scholes price of one contract, and similarly $\widetilde{h^{(1)}}$. Therefore

$$\alpha^* \approx \frac{p - (\widetilde{h^{(0)}} + \sqrt{\delta}\widetilde{h^{(1)}} + \delta h^{(2,1)})}{2\delta F}.$$

Clearly, this blows up as $\delta \downarrow 0$, unless p is the Black-Scholes price of the contract (with today's volatility). This is what we would expect in the convergence to a complete market, when derivatives become redundant, and induce arbitrage opportunities for infinite profit unless priced correctly by the market.

References

[1] G. Bakshi, C. Cao, and Z. Chen, *Empirical performance of alternative option pricing models*, J. Fin. **52** (1997), no. 5.

[2] O.E. Barndorff-Nielsen and N. Shephard, *Econometric analysis of realized volatility and its use in estimating stochastic volatility models*, Journal of the Royal Statistical Society, Series B **64** (2002), 253–280.

[3] D. Becherer, *Rational hedging and valuaton with utility-based preferences*, Ph.D. thesis, Technical University of Berlin, 2001.

[4] F. E. Benth and K. H. Karlsen, *Minimal entropy martingale measure and stochastic volatility*, Preprint (2003).

[5] P. Carr and D. Madan, *Optimal positioning in derivative securities*, Quantitative Finance **1** (2001), 19–37.

[6] J. Cvitanić, W. Schachermayer, and H. Wang, *Utility maximization in incomplete markets with random endowment*, Finance and Stochastics **5** (2001), no. 2, 259–272.

[7] M. Davis, *Option pricing in incomplete markets*, Mathematics of Derivative Securities (M.A.H. Dempster and S.R. Pliska, eds.), 1998.

[8] ———, *Optimal hedging with basis risk*, Preprint, Imperial College (2000).

[9] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker, *Exponential hedging and entropic penalties*, Mathematical Finance **12** (2002), no. 2, 99–123.

[10] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and option pricing for affine jump-diffusions*, Econometrica **68** (2000), 1343–1376.

[11] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*, Springer-Verlag, 1975.

[12] W. H. Fleming and H.M. Soner, *Controlled markov processes and viscosity solutions*, Springer-Verlag, 1993.

[13] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, Applied Stochastic Analysis (M.H.A. Davis and R.J. Elliott, eds.), Gordon and Breach, London, 1990, pp. 389–414.

[14] J.-P. Fouque, G. Papanicolaou, and R. Sircar, *Derivatives in financial markets with stochastic volatility*, Cambridge University Press, 2000.

[15] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, *Multiscale stochastic volatility asymptotics*, SIAM J. Multiscale Modeling and Simulation **2** (2003), no. 1, 22–42.

[16] M. Frittelli, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Mathematical Finance **10** (2000), no. 1, 39–52.

[17] P. Grandits and T. Rheinländer, *On the minimal entropy martingale measure*, The Annals of Probability **30** (2002), no. 3, 1003–1038.

[18] S.D. Hodges and A. Neuberger, *Optimal replication of contingent claims under transaction costs*, Review of Futures Markets **8** (1989), 222–239.

[19] J. Hull and A. White, *The pricing of options on assets with stochastic volatilities*, Journal of Finance **42** (1987), no. 2, 281–300.

[20] A. İlhan, M. Jonsson, and R. Sircar, *Optimal investment with derivative securities*, submitted (2004).

[21] ______, *Singular perturbations for boundary value problems arising from exotic options*, SIAM Journal on Applied Mathematics **64** (2004), no. 4, 1268–1293.

[22] A. İlhan and R. Sircar, *Optimal static-dynamic hedges for barrier options*, submitted (2003).

[23] M. Jonsson and R. Sircar, *Optimal investment problems and volatility homogenization approximations*, Modern Methods in Scientific Computing and Applications (A. Bourlioux, M. Gander, and G. Sabidussi, eds.), NATO Science Series II, vol. 75, Kluwer, 2002, pp. 255–281.

[24] ______, *Partial hedging in a stochastic volatility environment*, Mathematical Finance **12** (2002), no. 4, 375–409.

[25] Y. M. Kabanov and C. Stricker, *On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper*, Mathematical Finance **12** (2002), no. 2, 125–34.

[26] I. Karatzas and S. Shreve, *Methods of mathematical finance*, Springer-Verlag, 1998.

[27] D. Kramkov and W. Schachermayer, *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*, Annals of Applied Probability **9** (1999), no. 3, 904–950.

[28] N. V. Krylov, *Controlled diffusion processes*, Springer-Verlag, 1980.

[29] J. Liu and J. Pan, *Dynamic derivatives strategies*, Journal of Financial Economics **69** (2003), 401–430.

[30] R. C. Merton, *Lifetime portfolio selection under uncertainty: the continous-time case*, Rev. Econom. Statist. **51** (1969), 247–257.

[31] ———, *Optimum consumption and portfolio rules in a continuous-time model*, J. Economic Theory **3** (1971), no. 1/2, 373–413.

[32] M. Musiela and T. Zariphopoulou, *An example of indifference prices under exponential preferences*, Finance and Stochastics **8** (2004), no. 2, 229–239.

[33] M. P. Owen, *Utility based optimal hedging in incomplete markets*, The Annals of Applied Probability **12** (2002), no. 2, 691–709.

[34] H. Pham, *Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints*, Applied Mathematics and Optimization **46** (2002), 55–78.

[35] R. Rouge and N. El Karoui, *Pricing via utility maximization and entropy*, Mathematical Finance **10** (2000), no. 2, 259–76.

[36] W. Schachermayer, *Optimal investment in incomplete markets when wealth may become negative*, Annals of Applied Probability **11** (2001), 694–734.

[37] ———, *Introduction to the mathematics of financial markets*, Lectures on Probability Theory and Statistics, Saint-Fleur summer school 2000 (Pierre Bernard, ed.), Lecture Notes in Mathematics, no. 1816, Springer Verlag, 2003, pp. 111–177.

[38] R. Sircar and T. Zariphopoulou, *Bounds and asymptotic approximations for utility prices when volatility is random*, SIAM J. Control & Optimization (2004), To appear.