Optimal Static-Dynamic Hedges for Exotic Options under Convex Risk Measures

Aytaç İlhan * Mattias Jonsson † Ronnie Sircar ‡

April 8, 2008; revised February 6, 2009 and June 15, 2009

Abstract

We study the problem of optimally hedging exotic derivatives positions using a combination of dynamic trading strategies in underlying stocks and static positions in vanilla options when the performance is quantified by a convex risk measure. We establish conditions for the existence of an optimal static position for general convex risk measures, and then analyze in detail the case of shortfall risk with a power loss function. Here we find conditions for uniqueness of the static hedge. We illustrate the computational challenge of computing the market-adjusted risk measure in a simple diffusion model for an option on a non-traded asset.

1 Introduction

Many recent papers have analyzed the stochastic control problem of portfolio optimization under exponential utility:

$$\sup_{\theta} \mathbb{E}\left[-e^{-\gamma(V_T-G)}\right].$$

Here, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, G is the bounded $\mathcal{F}_T$-measurable payoff of a derivative security, $V_T = \int_0^T \theta_t \, \mathrm{d}S_t$ is the terminal value of following a trading strategy θ in some underlying stocks S, and $\gamma > 0$ is a risk-aversion coefficient. Typically, this problem is an intermediate step in finding the (seller's) indifference price of the claim G. We refer, for instance, to [8, 24] and the collection [6].

Recast as a hedging problem

$$\inf_{\theta} \frac{1}{\gamma} \log \mathbb{E}\left[e^{-\gamma(V_T-G)}\right],$$

this can be viewed as to optimally hedge with respect to the so-called *entropic risk measure*

$$e_\gamma(X) = \frac{1}{\gamma}\log\left(\mathbb{E}\left[e^{-\gamma X}\right]\right), \quad X \in L^\infty(P). \tag{1.1}$$

*Mathematical and Computational Finance Group, 24-29 St Giles', Oxford, OX1 3LB, UK.

†Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109. (mattiasj@umich.edu).

‡(Corresponding author): Department of Operations Research & Financial Engineering, Princeton University, Sherrerd Hall, Princeton, NJ 08544 (sircar@princeton.edu). Tel: +1 609 258 2841; Fax: +1 609 258 4363. Work partially supported by NSF grants DMS-0456195 and DMS-0807440.

Some nice properties of this functional $e_\gamma : L^\infty(P) \to \mathbb{R}$, namely monotonicity, translation invariance and convexity have been adopted as axioms for the class of *convex risk measures* proposed by Föllmer & Schied [13] (see Definition 2.1 below). In moving away from the entropic risk measure to this more general class, while the axiomatic properties are kept, other convenient features may be sacrificed, especially in terms of analytical and computational tractability.

Our goal here is to analyze a specific hedging problem (static-dynamic hedging of exotic options), which we found to be quite tractable under the entropic risk measure [20, 18, 19], under other convex risk measures, and specifically the shortfall risk measure.

Static-Dynamic Hedging of Exotic Options

Exotic options are non-standard options, which may be variations of standard (vanilla) calls and puts, like barrier options, or tailored according to clients' needs. These options are mostly traded in over-the-counter (OTC) markets. It is common to think of hedging strategies as trading the underlying stock and bank account appropriately. In continuous-time models, the hedging portfolio is re-balanced at every instant, and this type of hedging is called dynamic hedging. There is an alternative approach to hedge exotic options, which is less known, called static hedging. The idea of static hedging was introduced in [5] and it involves trading other liquid options. Trading occurs at some start time and the initial position is held throughout, which is why these hedges are called static.

In our approach, the investor, who assesses the risks associated with his financial position by a convex risk measure, chooses an optimal combined strategy. The static hedging component is buying or selling standard options at initiation, and the dynamic hedging component is following a stock-bank account trading strategy which is re-balanced continuously during the life of the options. We allow the investor to trade the standard options only statically, whereas she can trade the underlying stock and bank account continuously because of i) the higher transaction costs associated with option trading compared to stock and bond trading, and ii) lesser liquidity in derivatives markets, but we do not explicitly model either of these frictions here.

We will assume that the market is incomplete, therefore not all the risks are hedgeable through trading the underlying stock. If the market were complete, given sufficient initial capital, all claims could be replicated by trading the stock dynamically, and any position in the standard option could be synthesized with such a trading strategy. Static derivatives hedges do not add anything to dynamic hedges in complete markets, but of course they are very valuable tools in realistic incomplete market models, where there may be risk factors that cannot be eliminated just by dynamic trading of the underlying stock. Exotic options can be vulnerable to these risk factors, for example volatility risk. As standard options, in general, will also be exposed to similar risk factors, they can be exploited to hedge these risks. By incorporating static hedges, we enlarge the set of feasible hedging strategies that the investor can choose from and allow for a better hedging performance.

In Section 2, we set up the problem and give sufficient conditions on an abstract convex risk measure for existence of an optimal static-dynamic hedge of the exotic options position. Uniqueness is related to strict convexity of a certain value function and it is not simple to give useful conditions for it in general; rather, we focus on a sub-class of convex risk measures. For practical purposes, it seems there are two concrete classes of convex-but-not-coherent risk measures: the entropic risk measure related to exponential utility, and shortfall risk. (There are also risk measures with more abstract definitions in terms of, for example, their penalty functions, or acceptance sets, or drivers of BSDEs; see [3, 22]). In Section 3, we analyze the problem under shortfall risk measures, which are of the form

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \mid \mathbb{E}\left[\frac{1}{p}((-m - X)^+)^p \right] \leq \xi \right\}, \quad X \in L^\infty$$

for some $\xi > 0$ and $p > 1$. We establish a sufficient condition for uniqueness of an optimal static hedge, depending on whether a dual optimization problem is solved by an *equivalent* (martingale) measure. A simple one-period example suggests the condition is not necessary.

In Section 5, we look at the computational problem in the case of dynamic hedging of an option on a non-traded asset in a diffusion model, and under a shortfall risk measure. In this case, passing to the conjugate in the threshold level makes the problem amenable to dynamic programming methods (Section 4), and we give a numerical solution of the associated HJB equation. We conclude in Section 6.

2 Problem Formulation & Analysis

We consider an investor who has x dollars along with a short position in an exotic option, with payoff G^e, that matures at time T. The investor tries to minimize the risks due to this option position by trading the underlying stock and bonds dynamically, and vanilla options available in the market statically. We denote the stock price process $(S_t)_{0 \leq t \leq T}$, and the payoffs of the vanilla options by $G = (G_1, \dots, G_n)$, and we assume that G and G^e are bounded. A possible combined trading strategy is identified by $((\theta_t)_{0 \leq t \leq T}, \lambda)$ where θ_t is the number of underlying assets held at time t and $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the number of options sold initially. The investor assesses the risk of a given trading strategy and initial wealth by

$$\rho\left(-\lambda \cdot G - G^e + \int_0^T \theta \, dS + x + \lambda \cdot g \right), \tag{2.1}$$

where ρ is a convex risk measure, defined below, $g = (g_1, \cdots, g_n)$ is the market price vector of G, and "$\cdot$" denotes the the usual inner product on $\mathbb{R}^n$. Here, we assume for simplicity that the vanilla options also mature at time T and that the market uses a linear pricing rule. Note that λ takes values in $\mathbb{R}^n$ and its components can be negative to imply long positions in G. Our problem is the minimization of (2.1) over (θ, λ).

2.1 Convex Risk Measures

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions, and a locally bounded semimartingale, $S = (S_t)_{0 \leq t \leq T}$, that models the price process of the underlying asset.

Following the axiomatic framework for coherent risk measures introduced in [1], the theory of convex risk measures is developed in [13] over the linear space of all bounded functions. In this generality, no prior probability measure is assumed and duality results are with respect to the set of finitely additive and non-negative set functions. We shall restrict ourselves to the case where there is a prior probability measure P, and risk measures are defined for bounded random variables in $L^\infty = L^\infty(P)$. Under a further continuity assumption on ρ, the duality formulas are then in terms of sets of probability measures. We denote by $\mathbb{P}_a = \mathbb{P}_a(P)$ the set of probability measures that are *absolutely continuous* with respect to P.

We assume zero interest rates and consider the riskiness in terms of values at time T.

Definition 2.1. A mapping $\rho : L^\infty \mapsto \mathbb{R}$ is called a *convex risk measure* if it satisfies the following for all $X, Y \in L^\infty$:

- Monotonicity: If $X \leq Y$, $\rho(X) \geq \rho(Y)$.
- Translation Invariance: If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \leq \lambda \leq 1$.

A convex risk measure is *coherent* if it also satisfies:

- Positive Homogeneity: If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

A classical example of a convex (but not coherent) risk measure is related to exponential utility (also called the entropic risk measure) as given in (1.1). Another example, shortfall risk, is analyzed in detail in Section 3.

Assumption 2.2. We shall assume throughout that our primary convex risk measure ρ is *continuous from below* and *law-invariant*. In other words,

$$X_n \nearrow X \Rightarrow \rho(X_n) \searrow \rho(X),$$

and

$$\rho(X) = \rho(Y), \qquad \text{if } X = Y \ P-\text{a.s.}$$

The dual representation for coherent risk measures goes back to [1]. In the case of convex risk measures it is given in [13], and recalled in the following theorem.

Theorem 2.3. *(From Theorem 4.15, Propositions 4.14 & 4.21, and Lemma 4.30 in [13]) Any convex risk measure ρ on L^∞ that satisfies Assumption 2.2 has the dual representation*

$$\rho(X) = \max_{Q \in \mathbb{P}_a} \left(\mathbb{E}^Q[-X] - \alpha(Q) \right), \quad \forall X \in L^\infty, \tag{2.2}$$

where the minimal convex penalty function $\alpha : \mathbb{P}_a \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\alpha(Q) = \sup_{X \in L^\infty} \left(\mathbb{E}^Q[-X] - \rho(X) \right), \quad \forall Q \in \mathbb{P}_a. \tag{2.3}$$

The risk measure ρ is coherent if and only if the penalty function takes the form

$$\alpha(Q) = \begin{cases} 0 & \text{if } Q \in \mathcal{Q} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

for some convex subset $\mathcal{Q}$ of $\mathbb{P}_a$.

Proof. This result is given in Theorem 4.15 and Proposition 4.15 of [13] for a larger set, the set of finitely additive set functions, instead of $\mathbb{P}_a$. Under Assumption 2.2, Proposition 4.21 in [13] concludes that any Q with finite penalty is σ-additive, and is a probability measure. Therefore any additive set function which is not σ-additive but only finitely additive cannot attain the maximum. When there is a prior measure P, under Assumption 2.2, Lemma 4.30 in [13] states that measures that are not absolutely continuous with respect to P have infinite penalty, hence cannot attain the maximum. □

Remark 2.4. Working on L^∞ may be a little restrictive from a practical point of view, especially when we wish to consider unbounded derivatives positions such as call options. Many authors have investigated extension to L^p-spaces, and we refer for instance to [10, 26, 28] for recent results in this direction. Typically, if the risk measure is not real-valued, such as the entropic risk measure, an extra assumption of lower semicontinuity is required in order to obtain a nice dual representation. The implications for some of the dynamic hedging problems considered in the current paper are investigated under L^p convex risk measures in [29].

2.2 Dynamic Hedging

We call a predictable and S-integrable process $(\theta_t)_{0\leq t\leq T}$ *admissible* if the process $\left(\int_0^t \theta_u \,\mathrm{d}S\right)$ is uniformly bounded from below by a constant. We denote the corresponding set of terminal values by

$$\mathcal{H} = \left\{ \int_0^T \theta \,\mathrm{d}S \,|\, \theta \text{ admissible} \right\},$$

and the set of almost surely bounded, super-hedgeable claims by

$$\mathcal{C} = (\mathcal{H} - L_0^+) \cap L^\infty. \tag{2.5}$$

Definition 2.5. For $X \in L^\infty$, we define

$$u(X) = \inf_{Y\in\mathcal{C}} \rho(-X+Y). \tag{2.6}$$

It turns out that u has a very convenient dual representation. Let $\mathcal{M}^a \subset \mathbb{P}_a$ (resp. $\mathcal{M}^e \subset \mathbb{P}_e$) be the set of measures absolutely continuous (resp. equivalent) to P under which S is a local martingale. We note that, as in [27, Proposition 5.1], $\mathcal{M}^e$ is dense in $\mathcal{M}^a$ in the norm topology of $L^1(P)$.

Definition 2.6. We define

$$\mathcal{M}_f^a = \{Q \in \mathcal{M}^a \,|\, \alpha(Q) < \infty\}, \qquad \mathcal{M}_f^e = \{Q \in \mathcal{M}^e \,|\, \alpha(Q) < \infty\}.$$

Assumption 2.7. Given a convex risk measure satisfying Assumption 2.2, we assume that $\mathcal{M}_f^e$ is non empty.

In general, it is difficult to find useful conditions guaranteeing this except in relatively trivial cases such as finite probability spaces.

Proposition 2.8. *Under Assumptions 2.2 and 2.7, u has the dual representation*

$$u(X) = \max_{Q\in\mathcal{M}_f^a} \left(\mathbb{E}^Q[X] - \alpha(Q)\right), \quad \forall X \in L^\infty. \tag{2.7}$$

Proof. Consider the coherent risk measure $\nu^{-\mathcal{C}}$ associated with the convex set $-\mathcal{C}$ and defined by

$$\nu^{-\mathcal{C}}(-X) = \inf\{m \in \mathbb{R} \,|\, \exists V \in -\mathcal{C}, m + X \geq V,\ P - \text{a.s.}\}.$$

By point 3 following [3, Definition 1.5], the minimal penalty function $\alpha^{-\mathcal{C}}(Q)$ is $+\infty$ if Q does not belong to the polar cone

$$\{Q \in \mathbb{P}_a \mid \mathbb{E}^Q[H] \leq 0, \forall H \in \mathcal{C}\}, \tag{2.8}$$

and zero otherwise. Since the set in (2.8) is well-known to be $\mathcal{M}^a$ (see for example [27, Proposition 5.1]), we have

$$\alpha^{-\mathcal{C}}(Q) = \infty \mathbf{1}_{(\mathcal{M}^a)^c}(Q),$$

where $0 \times \infty = 0$. From [3, Corollary 3.6], the function $X \mapsto u(-X) = \inf_{Y \in \mathcal{C}} \rho(X + Y)$, the inf-convolution of ρ and $\nu^{-\mathcal{C}}$, is a new convex risk measure which inherits continuity from below from ρ. The penalty function after inf-convolution is the sum of the penalty functions α and $\alpha^{-\mathcal{C}}$. Arguing as in the proof of Theorem 2.3 for the convex risk measure $u(-X)$, leads to (2.7). □

For a given λ, the value function of the optimal dynamic hedging problem is

$$\inf_{Y \in \mathcal{C}} \rho(x + \lambda \cdot g + Y - \lambda \cdot G - G^e) = u(\lambda \cdot G + G^e) - \lambda \cdot g - x. \tag{2.9}$$

The problem of the investor is thus to minimize the right hand side in (2.9) over static derivatives positions λ. This problem is evidently independent of x, so we shall take $x = 0$ in the sequel.

Therefore, we need to find the Fenchel-Legendre transform at the level g of the function $\lambda \mapsto u(\lambda \cdot G + G^e)$. Alternatively, one could define the *indifference price* $h(X)$ of a claim $X \in L^\infty(\mathcal{F}_T)$ as the smallest compensation at time zero for an investor to undertake an obligation of paying X at maturity, such that he or she is indifferent in terms of the risk. By translation invariance, this is just the additional capital requirement to equate $u(X)$ and $u(0)$, so $h(X) = u(X) - u(0)$. The static part of the optimal combined hedge is equivalent therefore to minimizing $h(G^e + \lambda \cdot G) - \lambda \cdot g$, that is, to find the Fenchel-Legendre transform of the indifference price as function of quantity λ at the market price g.

Existence and uniquness of an optimal static hedge λ^* reduce therefore to studying the convexity, strict convexity and large $|\lambda|$ slope asymptotics of the indifference price, or equivalently $u(\lambda \cdot G + G^e)$, as a function of the static position $\lambda \in \mathbb{R}^n$.

2.3 Existence of an Optimal Static Hedging Position

Our first step establishes convexity of u as a function of the static holding λ.

Proposition 2.9. *Under Assumptions 2.2 and 2.7, the function $\lambda \mapsto u(\lambda \cdot G + G^e)$ is convex.*

Proof. Note that

$$\lambda \mapsto \mathbb{E}^Q[G^e + \lambda \cdot G] - \alpha(Q)$$

is affine in λ, and u being a supremum of affine functions on $\mathbb{R}^n$ over absolutely continuous probability measures, we conclude the convexity of $\lambda \mapsto u(\lambda \cdot G + G^e)$. □

For risk measures with strictly convex penalty function, we can now establish differentiability and a condition on the market price vector g for existence of an optimal static hedge.

Assumption 2.10. Assume that α is strictly convex on $\mathcal{M}_f^a$.

Note that under this assumption, the maximizer in Theorem 2.3 is unique.

Theorem 2.11. *Under Assumptions 2.2, 2.7 and 2.10, the function $\lambda \mapsto u(G^e + \lambda \cdot G)$ is continuously differentiable on $\mathbb{R}^n$ and its gradient is*

$$\nabla u(G^e + \lambda \cdot G) = \mathbb{E}^{Q^\lambda}[G], \tag{2.10}$$

where Q^λ is the unique maximizer of $\mathbb{E}^Q[G^e + \lambda \cdot G] - \alpha(Q)$ over $\mathcal{M}_f^a$. Moreover, the function $\phi(t) := u(G^e + t\lambda \cdot G)$ is convex, differentiable and

$$\lim_{t\to+\infty} \phi'(t) = \lim_{t\to+\infty} \frac{\phi(t)}{t} = \sup_{Q\in\mathcal{M}_f^e} \mathbb{E}^Q[\lambda \cdot G] \tag{2.11}$$

$$\lim_{t\to-\infty} \phi'(t) = \lim_{t\to-\infty} \frac{\phi(t)}{t} = \inf_{Q\in\mathcal{M}_f^e} \mathbb{E}^Q[\lambda \cdot G]. \tag{2.12}$$

In preparation for the proof of Theorem 2.11, we first recall the following definition and proposition in [2].

Definition 2.12. For a risk tolerance coefficient $\gamma > 0$, let ρ_γ denote the dilated risk measure associated with ρ, defined by

$$\rho_\gamma(X) = \gamma\rho\left(\frac{1}{\gamma}X\right), \quad \forall X \in L^\infty.$$

Proposition 2.13. *(From Proposition 3.10 in [2]) Suppose that $\rho(0) = 0$. Then $\rho_\infty := \lim_{\gamma\to\infty} \rho_\gamma$ is a coherent risk measure and*

$$\rho_\infty(X) = \sup_{\{Q\in\mathbb{P}_a \,|\, \alpha(Q)=0\}} \mathbb{E}^Q[-X].$$

On the other hand, $\rho_0 = \lim_{\gamma\to 0} \rho_\gamma$ is simply the "super-pricing rule" of $-X$:

$$\rho_0(X) = \sup_{\{Q\in\mathbb{P}_a \,|\, \alpha(Q)<\infty\}} \mathbb{E}^Q[-X].$$

The following definition and lemma will also be useful.

Definition 2.14. Given a convex risk measure ρ satisfying Assumption 2.2, and $Y \in L^\infty$ with $\rho(Y) < \infty$, we define the mapping $\rho^Y : L^\infty \mapsto \mathbb{R}$ as

$$\rho^Y(X) = \rho(X + Y + \rho(Y)), \qquad \text{for all } X \in L^\infty. \tag{2.13}$$

Lemma 2.15. *For a given convex risk measure ρ, ρ^Y as defined above is a convex risk measure, and is normalized such that $\rho^Y(0) = 0$. Moreover, the minimal penalty function α^Y associated with ρ^Y, is related to the minimal penalty function α associated with ρ, via*

$$\alpha^Y(Q) = \alpha(Q) + \mathbb{E}^Q[Y] + \rho(Y). \tag{2.14}$$

Proof. It is simple to show that ρ^Y satisfies the assertions in Definition 2.1. From (2.3), the minimal penalty function associated with ρ^Y, α^Y, is given by

$$\begin{aligned}
\alpha^Y(Q) &= \sup_{X\in L^\infty}\left(\mathbb{E}^Q[-X]-\rho^Y(X)\right), \\
&= \sup_{X\in L^\infty}\left(\mathbb{E}^Q[-X]-\rho(X+Y)\right)+\rho(Y), \\
&= \sup_{Z\in L^\infty}\left(\mathbb{E}^Q[-Z]-\rho(Z)\right)+\mathbb{E}^Q[Y]+\rho(Y), \\
&= \alpha(Q)+\mathbb{E}^Q[Y]+\rho(Y), \quad \forall Q\in\mathbb{P}_a,
\end{aligned}$$

which establishes (2.14). □

Proof of Theorem 2.11. For $\lambda\in\mathbb{R}^n$ define

$$\tilde{\rho}_\lambda(X) := u(\lambda\cdot G+G^e-X)-u(\lambda\cdot G+G^e), \quad X\in L^\infty.$$

Using Lemma 2.15, we easily verify that $\tilde{\rho}_\lambda$ is a convex risk measure, normalized such that $\tilde{\rho}_\lambda(0)=0$, and whose penalty function is

$$\tilde{\alpha}_\lambda(Q)=\alpha(Q)-\mathbb{E}^Q[\lambda\cdot G+G^e]+u(\lambda\cdot G+G^e) \quad \text{for } Q\in\mathcal{M}_f^a$$

and $\tilde{\alpha}_\lambda(Q)=+\infty$ when $Q\notin\mathcal{M}_f^a$. Notice that $\tilde{\alpha}_\lambda(Q)\geq 0$ for all Q. As α, and hence $\tilde{\alpha}_\lambda$, is strictly convex, equality holds for a *unique* measure $Q^\lambda\in\mathcal{M}_f^a$, which is then also the unique maximizer of $\mathbb{E}^Q[G^e+\lambda\cdot G]-\alpha(Q)$.

Now fix $\mu\in\mathbb{R}^n$ and $\epsilon\in\mathbb{R}\backslash\{0\}$. We can write

$$\frac{u((\lambda+\epsilon\mu)\cdot G+G^e)-u(\lambda\cdot G+G^e)}{\epsilon}=\frac{1}{\epsilon}\tilde{\rho}_\lambda(-\epsilon\mu G). \tag{2.15}$$

As ϵ decreases to zero, Proposition 2.13 applied to the risk measure $\tilde{\rho}_\lambda$ and $\gamma=1/\epsilon$ shows that (2.15) converges to

$$\sup\{\mathbb{E}^Q[\mu\cdot G]\mid Q\in\mathcal{M}_f^e,\tilde{\alpha}_\lambda(Q)=0\}=\mathbb{E}^{Q^\lambda}[\mu\cdot G].$$

As ϵ increases to zero, the same Proposition applied to $\gamma=-1/\epsilon$ instead shows that (2.15) converges to

$$-\sup\{\mathbb{E}^Q[-\mu\cdot G]\mid Q\in\mathcal{M}_f^e,\tilde{\alpha}_\lambda(Q)=0\}=\mathbb{E}^{Q^\lambda}[\mu\cdot G].$$

This proves the first part of the proposition.

As for the second part, it follows from the first part that ϕ is convex and continuously differentiable. The first equality in (2.11) holds by convexity. The second equality is obtained by setting $\mu=\lambda$, letting $\epsilon\to+\infty$ in (2.15) and applying Proposition 2.13 to $\gamma=\epsilon$ and $X=\lambda\cdot G+G^e$. Indeed, $\tilde{\alpha}_\lambda(Q)<\infty$ iff $Q\in\mathcal{M}_f^a$. Thus (2.11) holds and (2.12) is proved in the same way. □

Corollary 2.16. *Under Assumptions 2.2, 2.7, and 2.10, the minimizer of*

$$\lambda\mapsto u(\lambda\cdot G+G^e)-\lambda\cdot g$$

exists if

$$\inf_{Q\in\mathcal{M}_f^e}\mathbb{E}^Q[\lambda\cdot G]<\lambda\cdot g<\sup_{Q\in\mathcal{M}_f^e}\mathbb{E}^Q[\lambda\cdot G] \quad \forall\lambda\in\mathbb{R}^n\setminus\{0\}. \tag{2.16}$$

The condition (2.16) on the market price vector g is *sufficient* to guarantee no static arbitrage opportunities among the hedging instruments G. In the case of the risk measure associated with exponential utility (equation (1.1)), the existence of a minimizer whenever g is a *no arbitrage price vector* (that is, condition (2.16) with the inf and sup taken over the set $\mathcal{M}^e$) follows from the $L^1(P)$-denseness of $\{\frac{dQ}{dP} \mid Q \in \mathcal{M}^e_f\}$ in $\{\frac{dQ}{dP} \mid Q \in \mathcal{M}^e\}$, for the particular case that $\mathcal{M}^e_f$ is the set of equivalent local martingale measures with finite relative entropy. Uniqueness of the minimizer in that case follows from the *strict* convexity of u. We refer to [18] for details and references. In the remainder of the paper we analyze specifically a family of convex-but-not-coherent risk measures, namely shortfall risk with power loss function.

3 Shortfall Risk

We consider the shortfall risk at level $\xi > 0$, with power loss, which is defined as

$$\rho(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}[\ell(-m - X)] \le \xi\}, \quad X \in L^\infty \tag{3.1}$$

where

$$\ell(x) = \begin{cases} \frac{1}{p}x^p & x \ge 0, \\ 0 & x < 0, \end{cases} \tag{3.2}$$

with $p > 1$.

Remark 3.1. The choice $\ell(x) = e^{\gamma x}$ for some $\gamma > 0$ in (3.1) leads to

$$\rho(X) = \frac{1}{\gamma}\log\left(\mathbb{E}\left[e^{-\gamma X}\right]\right) - \frac{1}{\gamma}\log\xi,$$

which is the entropic risk measure $e_\gamma(X)$ defined in (1.1), up to a constant depending on ξ. The analog of the main result of this section, Theorem 3.2 below, in the case of the entropic risk measure, is Theorem 5.1 in [18].

3.1 Uniqueness of the Static Hedging Position

Clearly ρ is law invariant and, by Proposition 4.104 of [13], it is continuous from below, and so satisfies Assumption 2.2. Its dual representation is

$$\rho(X) = \max_{Q\in\mathbb{P}_a}\left(\mathbb{E}^Q[-X] - (p\xi)^{1/p}H^q(Q|P)\right), \quad X \in L^\infty,$$

where $q > 1$ is the conjugate exponent: $\frac{1}{p} + \frac{1}{q} = 1$, and

$$H^q(Q|P) := \left(\mathbb{E}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^q\right]\right)^{1/q}, \tag{3.3}$$

the q-distance between Q and P: see Example 4.109 in [13].

We define

$$\mathcal{M}^a_q = \left\{Q \in \mathcal{M}^a \mid \frac{\mathrm{d}Q}{\mathrm{d}P} \in L^q(P)\right\},$$

and similarly $\mathcal{M}^e_q$ for L^q-integrable *equivalent* local martingale measures. We will throughout assume that $\mathcal{M}^e_q$ is nonempty. It follows from Minkowski's inequality (see e.g. [14], Exercise

3.2.7) that H^q is strictly convex on $\mathcal{M}_q^a$, hence Assumption 2.10 is satisfied. In view of Proposition 2.8, we have

$$u(X) = \max_{Q \in \mathcal{M}_q^a} U(Q,X) \quad \text{where} \quad U(Q,X) = \mathbb{E}^Q[X] - (p\xi)^{1/p} H^q(Q|P), \tag{3.4}$$

and the maximizer is unique.

Our next result gives a condition for the uniqueness of the optimal static hedging position.

Theorem 3.2. *Assume that S is continuous, and that for all $\lambda \in \mathbb{R}^n \setminus \{0\}$,*

$$\inf_{Q \in \mathcal{M}_q^e} \mathbb{E}^Q[\lambda \cdot G] < \sup_{Q \in \mathcal{M}_q^e} \mathbb{E}^Q[\lambda \cdot G]. \tag{3.5}$$

Then the map $\lambda \mapsto u(G^e + \lambda \cdot G)$ is differentiable. Furthermore if, for $\lambda^* \in \mathbb{R}^n$, the maximizing measure $Q^* \in \mathcal{M}_q^a$ in (3.4) with $X = G^e + \lambda^* \cdot G$, is in fact an equivalent measure ($Q^* \in \mathcal{M}_q^e$), then $\lambda \mapsto u(G^e + \lambda \cdot G)$ is strictly convex at λ^*.

It follows that, given an optimal static hedge λ^*, it is unique if the associated maximizing measure $Q^* \in \mathcal{M}_q^e$.

Remark 3.3. The maximizing measure Q^* is not automatically in $\mathcal{M}_q^e$ (see the example in Section 3.3). This is in contrast with the case of the entropic risk measure, where the infinite slope of the entropy function $x \log x$ at $x = 0$ forces the maximizing measure to be equivalent.

Remark 3.4. Moreover, even if $Q^* \notin \mathcal{M}_q^e$, $\lambda \mapsto u(G^e + \lambda \cdot G)$ may still be strictly convex at λ^*. Again, see the example in Section 3.3.

To proceed, we need the following two lemmas whose proofs are given below.

Lemma 3.5. *Given $X \in L^\infty$ and $Q \in \mathcal{M}_q^a$ define $Z = Z(Q,X) \in L^q$ by*

$$Z(Q,X) = X - (p\xi)^{\frac{1}{p}} \mathbb{E}\left[\left(\frac{dQ}{dP}\right)^q\right]^{-\frac{1}{p}} \left(\frac{dQ}{dP}\right)^{q-1}. \tag{3.6}$$

Then the following hold:

(a) $\mathbb{E}^Q[Z(Q,X)] = U(Q,X)$ for all $Q \in \mathcal{M}_q^a$, where $U(Q,X)$ is given by (3.4);

(b) a given measure $Q^* \in \mathcal{M}_q^a$ is the unique maximizer of $U(Q,X)$ over $Q \in \mathcal{M}_q^a$ iff $\mathbb{E}^Q[Z(Q^*,X)] \leq \mathbb{E}^{Q^*}[Z(Q^*,X)]$ for all $Q \in \mathcal{M}_q^a$.

The lemma says that we can linearize the optimization problem in (3.4). See Figure 1.

Lemma 3.6. *Let $H^1, H^2 \in L^p$, with $H^1 - H^2$ bounded, and suppose there exists $Q^* \in \mathcal{M}_q^e$ such that*

$$\mathbb{E}^Q[H^i] \leq \mathbb{E}^{Q^*}[H^i] = 0 \quad \text{for } i = 1,2 \text{ and all } Q \in \mathcal{M}_q^a.$$

Then $\mathbb{E}^Q[H^1] = \mathbb{E}^Q[H^2]$ for all $Q \in \mathcal{M}_q^a$.

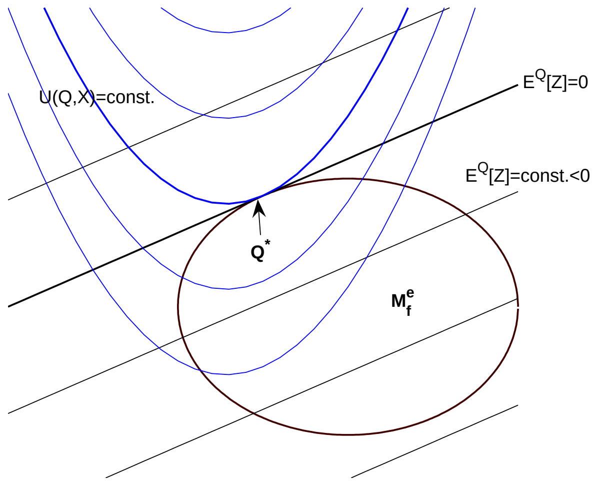

Figure 1: *Illustration of Lemma 3.5(b). Here,* $Z = Z(Q^*, X)$.

Proof of Theorem 3.2. We have already seen that Assumptions 2.2, 2.7 and 2.10 are satisfied, so Theorem 2.11 implies that $\lambda \mapsto u(\lambda \cdot G + G^e)$ is convex and differentiable on $\mathbb{R}^n$. Its gradient is $\nabla u(G^e + \lambda \cdot G) = \mathbb{E}^{Q^\lambda}[G]$, where $Q^\lambda \in \mathcal{M}_q^a$ is the unique maximizer of

$$\psi(Q; \lambda) := \mathbb{E}^Q[G^e + \lambda \cdot G] - (p\xi)^{1/p} H^q(Q|P) \tag{3.7}$$

over $\mathcal{M}_q^a$.

To prove strict convexity of $\lambda \mapsto u(\lambda \cdot G + G^e)$ at λ^*, we argue by contradiction: assume there exist $\lambda^1 \neq \lambda^2$ such that λ^* lies in the interior of the line segment between λ^1 and λ^2, and that $\lambda \mapsto u(G^e + \lambda \cdot G)$ is affine on this segment. Denote by $Q^1 = Q^{\lambda^1}$ and $Q^2 = Q^{\lambda^2}$ the unique maximizers over $\mathcal{M}_q^a$ of $\psi(\cdot; \lambda^1)$ and $\psi(\cdot; \lambda^2)$ respectively. It follows from (3.7) that

$$\left(\psi(\lambda^1; Q^1) - \psi(\lambda^1; Q^2)\right) + \left(\psi(\lambda^2; Q^2) - \psi(\lambda^2; Q^1)\right) = (\lambda^1 - \lambda^2) \left(\mathbb{E}^{Q^1}[G] - \mathbb{E}^{Q^2}[G]\right).$$

The right hand side is equal to zero by the linearity assumption and the gradient formula above. The left hand side is the sum of two nonnegative terms since $\psi(\lambda^i; \cdot)$ attains its maximum at Q^i. These maxima being unique then implies that $Q^1 = Q^2 = Q^*$.

Next, we define

$$H^i = Z(Q^*, G^e + \lambda^i \cdot G) - u(G^e + \lambda^i \cdot G), \quad i = 1, 2,$$

with $Z(Q, X)$ as in (3.6). As Q^* maximizes $U(Q, G^e + \lambda^i \cdot G)$, with $u(G^e + \lambda^i \cdot G)$ being the maximum value, Lemma 3.5 yields $\mathbb{E}^Q[H^i] \leq \mathbb{E}^{Q^*}[H^i] = 0$ for all $Q \in \mathcal{M}_q^a$. As $H^1 - H^2$ is bounded, Lemma 3.6 shows that $\mathbb{E}^Q[H^1 - H^2] = 0$ for all $Q \in \mathcal{M}_q^a$. But this means precisely that $E^Q[(\lambda^1 - \lambda^2) \cdot G]$ does not depend on $Q \in \mathcal{M}_q^e$, which contradicts (3.5) and completes the proof. □

3.2 Proofs of Lemma 3.5 and Lemma 3.6

We now prove the two lemmas.

Proof of Lemma 3.5. The proof of (a) is a simple computation. To prove (b) we fix $Q^*, Q \in \mathcal{M}_q^a$ and set $\phi(t) = U((1-t)Q^* + tQ, X)$ for $0 \le t \le 1$. Then ϕ is concave on $[0,1]$, hence admits one-sided derivatives everywhere. A direct computation shows that

$$\phi'(0+) = E^Q[Z(Q^*, X)] - E^{Q^*}[Z(Q^*, X)].$$

On the one hand, if Q^* is the maximizer of $U(\cdot, X)$, then $\phi'(0+) \le 0$ for any choice of Q. On the other hand, if Q^* is not the maximizer, then we may pick Q such that $\phi(1) > \phi(0)$. The concavity of ϕ then gives $\phi'(0+) > 0$. □

Lemma 3.6 is taken from Theorem 1.2 in [9]. It is re-written here in a modified form: the theorem in [9] concerns attainable claims, but the argument works for "approximately super- and sub-hedgeable claims", which is what we need. We give the modified proof for completeness. First we need a definition and lemma, cf. [9, p.747].

Definition 3.7. We call a predictable process a simple p-admissible integrand for S if it is a linear combination of processes of the form

$$\theta = f \mathbf{1}_{]T_1, T_2]},$$

where T_1 and T_2 are finite stopping times dominated by some U_n, where $(U_n)_{n=1}^\infty$ is a localizing sequence for S; and $f \in \mathrm{L}^\infty(\Omega, \mathcal{F}_{T_1}, P)$. We define

$$K_p^s = \left\{ \int_0^T \theta_u \, \mathrm{d}S_u \,|\, \theta \text{ simple } p\text{-admissible} \right\} \subset L^p(P).$$

Lemma 3.8.

$$H \in \overline{K_p^s - L_+^p(P)}^{L^p(P)} \iff \mathbb{E}^Q[H] \le 0, \text{ for all } Q \in \mathcal{M}_q^a;$$
$$H \in \overline{K_p^s + L_+^p(P)}^{L^p(P)} \iff \mathbb{E}^Q[H] \ge 0, \text{ for all } Q \in \mathcal{M}_q^a.$$

Proof. Follows from a simple modification of the proofs of $(i) \iff (iii')$ in [9, Theorem 1.2]. □

Proof of Lemma 3.6. Adapting the approach in the proof of [9, Theorem 1.2], we proceed in three steps, first to show that $H^i \in \overline{K_p^s - L_+^p(P)}^{L^p(P)}$, then that $H^i \in \overline{K_p^s}^{L^1(Q^*)}$, and finally that $H^1 - H^2 \in \overline{K_p^s}^{L^p(P)}$. The first step follows from Lemma 3.8. Since $L^p(P) \subset L^1(Q^*)$ (by Hölder's inequality), we have that $H^i \in \overline{K_p^s - L_+^1(Q^*)}^{L^1(Q^*)}$, and there exist two sequences $(H_n^i)_{n=1}^\infty \in K_p^s$, $i = 1, 2$ such that

$$\lim_{n\to\infty} \mathbb{E}^{Q^*}(H^i - H_n^i)_- = 0, \quad i = 1, 2.$$

From the facts that elements of K_p^s have Q^*-expectation zero, and that $\mathbb{E}^{Q^*}[H^i] = 0$, we deduce

$$\lim_{n\to\infty} \mathbb{E}^{Q^*}|H^i - H_n^i| = 0, \quad i = 1, 2,$$

in other words the H^i are in the $L^1(Q^*)$ closure of K_p^s. Then $\tilde{H} := H^1 - H^2$ is also in the $L^1(Q^*)$ closure of K_p^s, and is bounded by hypothesis.

For the last step, we may identify $\tilde{H}$ with a uniformly integrable martingale $(h_t)_{t\geq 0}$ by letting $h_t = \mathbb{E}^{Q^*}[\tilde{H}\,|\mathcal{F}_t]$, and applying Corollary 2.5.2 in [30] to exhibit a predictable integrand φ such that $\int_0^T \varphi_u \,\mathrm{d}S_u = \tilde{H}$. Note that $\int_0^t \varphi_u \,\mathrm{d}S_u = \mathbb{E}^{Q^*}[\tilde{H}|\mathcal{F}_t]$ is bounded in absolute value by $||\tilde{H}||_\infty$. Since we assumed that S is continuous, we can apply Lemma 2.3 in [9] to find a sequence $(\varphi^n)_{n=1}^\infty$ of ∞-admissible simple integrands such that $\int_0^T \varphi_u^n \,\mathrm{d}S_u$ converges to $\int_0^T \varphi_u \,\mathrm{d}S_u = \tilde{H}$ in $L^p(P)$. Therefore $\tilde{H}$ is also in $\overline{K_p^s}^{L^p(P)}$. But then Lemma 3.8 implies that $\mathbb{E}^Q[\tilde{H}] = 0$ for all $Q \in \mathcal{M}_q^a$, and the conclusion of the lemma follows. □

3.3 A Simple Example

We present a simple one-period quadrinomial tree example that demonstrates that even if the maximizing measure Q^* in (3.4) with $X = G^e + \lambda^* \cdot G$ is in $\mathcal{M}_q^a$, but *not* in $\mathcal{M}_q^e$, then $\lambda \mapsto u(G^e + \lambda \cdot G)$ may still be strictly convex at λ^*. In other words, the condition in Theorem 3.2 that the maximizing measure is not only absolutely continuous (with respect to P) but also equivalent, is sufficient, but not necessary for strict convexity.

The probability space has four elements: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The current stock price is $S_0 = 4$, and at the end of the single period, $S_T(\Omega) = \{7, 5, 3, 1\}$, with historical probabilities

$$P(\{\omega_1\}) = \frac{1}{2}, P(\{\omega_2\}) = \frac{1}{4}, P(\{\omega_3\}) = \frac{1}{8}, P(\{\omega_4\}) = \frac{1}{8}.$$

The exotic option G^e and the single static hedging instrument G have payoffs

$$G^e(\Omega) = \{-40, -20, -20, -40\}, \qquad G(\Omega) = \{3, -1, 1, 3\}.$$

The absolutely continuous martingale measures $Q \in \mathcal{M}_q^a$ are parametrized by $(q_1, q_2, q_3, q_4) \in [0, 1]^4$, with the (probability and martingale) constraints

$$q_1 + q_2 + q_3 + q_4 = 1, \qquad 7q_1 + 5q_2 + 3q_3 + q_4 = 4.$$

This is conveniently represented as $(q_2, q_3) \in \Delta$, where Δ is the convex subset of $[0, 1]^2$ shown in Figure 2.

We choose $p = q = 2$ and the shortfall threshold level ξ such that the optimization problem (3.4) is

$$w(\lambda) := u(G^e + \lambda G) = \max_{Q \in \mathcal{M}_q^a} \mathbb{E}^Q[G^e + \lambda G] - H^2(Q \mid P),$$

which, in the quadrinomial model, becomes of the form

$$w(\lambda) = \max_{(q_2,q_3)\in\Delta} \mathcal{L}(q_2, q_3) - \sqrt{\mathcal{Q}(q_2, q_3)},$$

for some affine function $\mathcal{L}$, and quadratic function $\mathcal{Q}$.

It is straightforward, but tedious, to see that for any $\lambda \in \mathbb{R}$, the optimizing measure is always attained on the boundary of Δ, and so is absolutely continuous, but not equivalent. In particular, there exist finite $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ such that the optimizer (q_2^*, q_3^*) is either on the edges σ_1 and σ_2 in Figure 2, or at the vertices $(\frac{3}{4}, 0)$, $(\frac{1}{2}, \frac{1}{2})$ or $(0, \frac{3}{4})$, depending on where λ lies. In particular, when the optimizer is stuck at a vertex while λ increases, $w(\lambda)$ is affine, and while λ moves between vertices, $w(\lambda)$ is strictly convex. This is summarized in the following table and Figure 3.

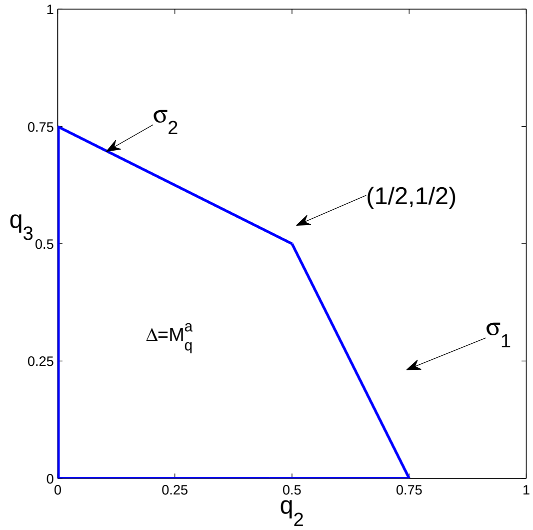

Figure 2: $\mathcal{M}_q^a$ *in the quadrinomial model. The interior of the polygon represents* $\mathcal{M}_q^e$.

λ	(q_2^*, q_3^*)	$w(\lambda)$	$w'(\lambda)$
$-\infty < \lambda \leq \lambda_1$	at $(\frac{3}{4}, 0)$	affine	$= \text{SubH}$
$\lambda_1 < \lambda < \lambda_2$	on σ_1	strictly convex	$\in (\text{SubH}, 0)$
$\lambda_2 \leq \lambda \leq \lambda_3$	at $(\frac{1}{2}, \frac{1}{2})$	affine	0
$\lambda_3 < \lambda < \lambda_4$	on σ_2	strictly convex	$\in (0, \text{SupH})$
$\lambda_4 \leq \lambda < \infty$	at $(0, \frac{3}{4})$	affine	$= \text{SupH}$

The sub- and super-hedging prices of G are given by

$$\text{SubH} = -1.5, \qquad \text{SupH} = 1.5.$$

A numerical computation yields

$$\lambda_1 \approx -4.62, \quad \lambda_2 \approx -1.25, \quad \lambda_3 \approx 3.93, \quad \lambda_4 \approx 4.69.$$

3.4 A Comparison to Utility Maximization and Partial Hedging

The problem of dynamically hedging a derivatives position, say G, using the underlying securities (*i.e.* the stocks) so as to minimize an expected loss was studied in [7, 12], among others. Specifically, defining $V_t = v + \int_0^t \theta_u \, dS_u$, where θ is an admissible strategy, and $v > 0$ the initial wealth (the hedging cost), the basic partial hedging problem is to find

$$\inf_{\theta} \mathbb{E}[\ell(G - V_T)],$$

under the constraint $V_t \geq 0$ for all $t \in [0, T]$, for some given decreasing convex loss function ℓ. This is closely related to utility maximization problems with a random endowment, for example

$$\mathcal{U}(v, \lambda) = \sup_{\theta} \mathbb{E}[U(V_T + \lambda G)].$$

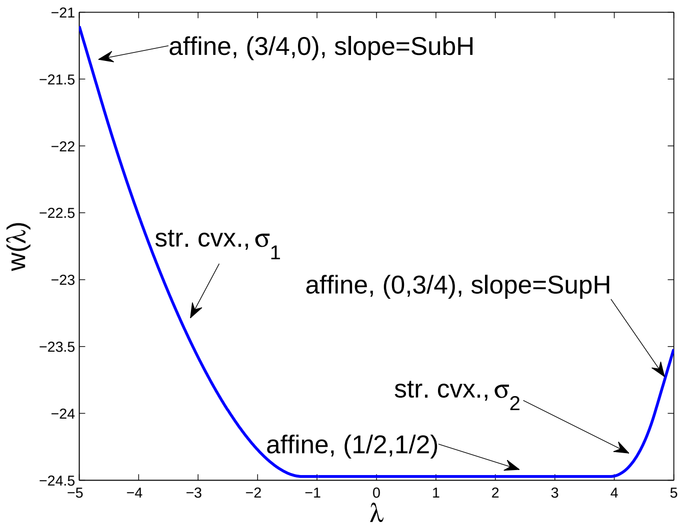

Figure 3: *Graph $w(\lambda)$ showing regions where it is affine and where it is strictly convex, where the maximum is attained on Δ, and where the slope attains the sub- and super-hedging prices.*

Here, twice differentiability of $\mathcal{U}$ (and associated utility prices) as a function of the quantity λ, and as $\lambda \downarrow 0$, are studied in [23] for quite general utility functions U. When U is strictly concave, strict concavity of $\mathcal{U}$ follows directly. For convex risk measures, this is not so clear, as can be seen from the example of the entropic risk measure (1.1), where the log is a concave function of the expectation of a convex loss function.

For the case of a one-sided loss function such as in (3.2), only shortfalls are considered, and there is no utility for overshooting the target. We refer to [7, 12], and also [21] for analysis and asymptotic approximations under diffusion stochastic volatility models, [4] for the duality theory addressing the non-smoothness of ℓ, and [26] for generalizations when expected shortfall is replaced by a convex risk measure of the shortfall.

In general, the solution to the partial hedging problem depends on the initial hedging capital v. In this paper, we choose to minimize the hedging error under a convex risk measure ρ constructed on $L^\infty(P)$. The optimization is first over dynamic trading strategies whose terminal values $V_T \in \mathcal{C}$, the set of almost surely bounded, super-hedgeable claims, defined in (2.5), without the restriction that the wealth process V remains positive; and then over static positions $\lambda \in \mathbb{R}^n$ in G. In terms of the associated acceptance set of the convex risk measure ρ, namely $\mathcal{A} = \{X \in L^\infty(P) \mid \rho(X) \le 0\}$, we seek to minimize the amount of money that needs to be added to the position for the risk to be acceptable:

$$\inf_{\lambda\in\mathbb{R}^n} \inf_{V_T\in\mathcal{C}} \inf\{m \in \mathbb{R} \mid V_T - (G^e + \lambda\cdot(G-g)) \in \mathcal{A}\}.$$

The optimal hedging strategy θ (if it exists) is then independent of the initial capital v: any remaining required start-up cost is simply borrowed at initiation of the hedge.

Finally, one could consider minimization of risk measures induced by the hedging instruments (other than cash), for example

$$\inf_{\lambda\in\mathbb{R}^n} \inf\{V_T \in \mathcal{C} \mid V_T-(G^e+\lambda\cdot(G-g)) \in \mathcal{A}\}, \quad \text{or} \quad \inf_{V_T\in\mathcal{C}} \inf\{\lambda \in \mathbb{R}^n \mid V_T-(G^e+\lambda\cdot(G-g)) \in \mathcal{A}\}.$$

The first is the minimization of a $\mathcal{C}$-valued risk measure over static hedges, the latter the minimization of a vector valued risk measure over dynamic hedges. Formulation of the problem of course requires defining the notion of infimum for set-valued risk measures. We refer to [16] for work in this direction.

4 Varying the shortfall threshold

With a view towards numerical computations (Section 5), we study the properties of u as a function of the threshold level $\xi > 0$, which we introduce as an argument in the notation, denoting U and u in (3.4) as $U(\xi, Q, X)$ and $u(\xi, X)$, respectively.

As $1 < q < \infty$, U is strictly convex in ξ. Let us introduce its Fenchel-Legendre transform:

$$\begin{aligned} \hat{U}(z, Q, X) &= \inf_{\xi > 0} (U(\xi, Q, X) + \xi z), \quad z \geq 0, \\ &= \mathbb{E}^Q[X] - \frac{1}{q} z^{1-q} \mathbb{E}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^q\right], \end{aligned} \tag{4.1}$$

and the conjugate optimization problem

$$\hat{u}(z, X) = \sup_{Q \in \mathcal{M}_q^a} \hat{U}(z, Q, X). \tag{4.2}$$

Note that $\hat{u}(z, -X)$ is another market modified convex risk measure with penalty function $\frac{1}{q} z^{1-q} \mathbb{E}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^q\right]$, which is finite and strictly convex on $\mathcal{M}_q^a$. The important difference with the dual representation (3.4) of u is that the "outside power" $1/q$ is missing compared with (3.3), and the objective function $\hat{U}$ is therefore an expectation of a function of the Radon-Nikodym derivative dQ/dP. This is exploited when we use dynamic programming for a numerical computation in Section 5.

Theorem 4.1. *For $X \in L^\infty$, we have*

$$u(\xi, X) = \sup_{z > 0} \left(\hat{u}(z, X) - \xi z\right), \tag{4.3}$$

and

$$\hat{u}(z, X) = \inf_{\xi > 0} \left(u(\xi, X) + \xi z\right). \tag{4.4}$$

We will make use of the following analog of Lemma 3.5 part (b). The proof, being almost identical, is omitted.

Lemma 4.2. *Given $z > 0$, $X \in L^\infty$ and $Q \in \mathcal{M}_q^a$ define $W = W(z, Q, X) \in L^q$ by*

$$W(z, Q, X) = X - z^{1-q} \left(\frac{dQ}{dP}\right)^{q-1}. \tag{4.5}$$

Then a given measure $Q^* \in \mathcal{M}_q^a$ is the unique maximizer of $\hat{U}(z, Q, X)$ over $Q \in \mathcal{M}_q^a$ iff $\mathbb{E}^Q[W(z, Q^*, X)] \leq \mathbb{E}^{Q^*}[W(z, Q^*, X)]$ for all $Q \in \mathcal{M}_q^a$.

Proof of Theorem 4.1. The functions $U(\xi, Q, X)$ and $\hat{U}(z, Q, X)$ being conjugate, we have

$$U(\xi, Q, X) = \sup_{z>0} \Big(\hat{U}(z, Q, X) - \xi z \Big) \quad \text{for } Q \in \mathcal{M}_q^a.$$

Taking supremum over Q in both sides, and changing the order of maximization problems on the right hand side, we arrive at (4.3).

To prove (4.4) we fix z and let $Q^* \in \mathcal{M}_q^a$ be (uniquely) defined by $\hat{u}(z, X) = \hat{U}(z, Q^*, X)$. By Lemma 4.2, we have $E^Q[W(z, Q^*, X)] \leq E^{Q^*}[W(z, Q^*, X)]$ for all $Q \in \mathcal{M}_q^a$.

Next, we set $\xi^* = p^{-1} z^{-q} \mathbb{E}[(\frac{dQ^*}{dP})^q]$. A straightforward calculation shows that $W(z, Q^*, X) = Z(\xi^*, Q^*, X)$ where the right hand side is defined as in Lemma 3.5, using this same measure Q^*. Applying Lemma 4.2, we find that

$$\mathbb{E}^{Q^*}[Z(\xi^*, Q^*, X)] = \mathbb{E}^{Q^*}[W(z, Q^*, X)] \geq \mathbb{E}^{Q}[W(z, Q^*, X)] = \mathbb{E}^{Q}[Z(\xi^*, Q^*, X)],$$

for all $Q \in \mathcal{M}_q^a$. Lemma 3.5 part (b) then implies that Q^* maximizes $U(\xi^*, Q, X)$, and so we have $u(\xi^*, X) = U(\xi^*, Q^*, X)$. A further direct computation reveals that $\hat{U}(z, Q^*, X) = U(\xi^*, Q^*, X) + \xi^* z$, which yields

$$\hat{u}(z, X) = \hat{U}(z, Q^*, X) = U(\xi^*, Q^*, X) + \xi^* z = u(\xi^*, X) + \xi^* z \geq \inf_{\xi>0} (u(\xi, X) + \xi z).$$

But (4.3) implies $\hat{u}(x, X) \leq \inf_{\xi>0}(u(\xi, X) + \xi z)$, and hence (4.4) holds. □

5 Computation of Shortfall Risk in the Nontraded Asset Model

In this section, we address computation of the optimal hedge within a dynamic Brownian motion based financial model. Our goal is to provide a comparison in a case where the entropic risk measure (or, equivalently, exponential utility) has been enormously successful, namely the problem of hedging (or indifference pricing) of an option on a non-traded asset, using a correlated tradeable asset. In the canonical set-up, the price processes of the traded asset S and the non-traded asset Y are described by the stochastic differential equations

$$\mathrm{d}S_t = \mu(Y_t) S_t \,\mathrm{d}t + \sigma(Y_t) S_t \,\mathrm{d}W_t^1, \quad S_0 = S, \tag{5.1}$$

$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + a(Y_t)(\nu \,\mathrm{d}W_t^1 + \nu' \,\mathrm{d}W_t^2), \quad Y_0 = y. \tag{5.2}$$

Here W^1 and W^2 are independent standard Brownian motions on our probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the standard filtration generated by them. The constant $\nu \in (-1, 1)$ is a correlation coefficient, and $\nu' = \sqrt{1 - \nu^2}$. We assume sufficient regularity on the coefficients of the SDEs to guarantee existence of a unique strong solution. Specifically, we assume that a and σ are bounded above and below away from zero, and smooth with bounded derivatives. We also assume that μ and b are smooth with bounded derivatives. The object of interest is a European derivative contract written on Y.

5.1 Dynamic Programming Equation

For our hedging problem, the option payoffs that we need to work with will be path dependent in general, but to ease the representation, in this section we will assume that $G^e + \lambda G = h(Y_T)$, that is, European. The extension to path dependent payoffs would introduce additional boundary conditions, and/or extra dimensions in the resulting Hamilton-Jacobi-Bellman (HJB) equations we will use for analysis of the optimization problems.

One approach would be to deal with the primal problem (2.9). In this case, to apply dynamic programming techniques, we introduce the wealth process V_t corresponding to holding, at time t, π_t dollars in the traded asset S. We will assume throughout that the interest rate is zero, and so the wealth process evolves according to

$$\mathrm{d}V_t = \mu(Y_t)\pi_t\,\mathrm{d}t + \sigma(Y_t)\pi_t\,\mathrm{d}W_t^1, \quad V_0 = v. \tag{5.3}$$

The value function of the dynamic hedging problem is then defined as

$$H(t,v,y) = \inf_{\pi}\mathbb{E}\left[\ell((h(Y_T) - V_T))|V_t = v, Y_t = y\right], \tag{5.4}$$

and the shortfall risk w at level ξ is found (at time zero) by solving

$$H(0, v + w, y) = \xi.$$

It might be natural here to pass to the HJB equation for H, but for the loss function (3.2), we know that $H \equiv 0$ for v large enough, particularly $v \geq v_{\mathrm{sup}}$, the superhedging price (among admissible strategies that trade only S) of the claim h. Therefore, H may not have sufficient smoothness for the HJB equation to apply for all $v \in \mathbb{R}$, and we pass to the study of the dual problem (4.2).

From Girsanov's theorem, the set of equivalent local martingale measures is characterized in the model (5.1)-(5.2) by

$$\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu(Y_t)}{\sigma(Y_t)}\,\mathrm{d}W_t^1 - \int_0^T \gamma_t\,\mathrm{d}W_t^2 - \frac{1}{2}\int_0^T\left(\frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \gamma_t^2\right)\mathrm{d}t\right),$$

for some adapted process γ with $\int_0^T \gamma_t^2\,\mathrm{d}t < \infty$ a.s. We denote by $\mathcal{N}$ the space of adapted processes γ that satisfy the Novikov condition: $E[\exp(\frac{1}{2}\int_0^T \gamma_t^2\,dt)] < \infty$. For $\gamma \in \mathcal{N}$, Q^γ is then an equivalent martingale measure, and by Jensen's inequality, the Novikov condition implies $E[\int_0^T \gamma_t^2\,dt] < \infty$.

Remark 5.1. The q-distance of Q^γ with respect to P is

$$H^q(Q^\gamma \mid P) = \mathbb{E}\left[\exp\left(-\frac{1}{2}q\int_0^T\left(\frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \gamma_t^2\right)\mathrm{d}t - q\int_0^T \frac{\mu(Y_t)}{\sigma(Y_t)}\,\mathrm{d}W_t^1 - q\int_0^T \gamma_t\,\mathrm{d}W_t^2\right)\right]^{1/q}.$$

The choice $\gamma \equiv 0$ gives the well-known minimal martingale measure Q^0. By the assumptions on the coefficients, $H^q(Q^0 \mid P) < \infty$, and so $\mathcal{M}_q^e$ is non-empty, and Assumption 2.7 is satisfied.

For $\gamma \in \mathcal{N}$, we define

$$W_t^{\gamma,1} = W_t^1 + \int_0^t \frac{\mu(Y_s)}{\sigma(Y_s)}\,\mathrm{d}s, \quad W_t^{\gamma,2} = W_t^2 + \int_0^t \gamma_s\,\mathrm{d}s$$

and the process (Z_t) by

$$\mathrm{d}Z_t = Z_t\left(\frac{\mu(Y_t)}{\sigma(Y_t)}\mathrm{d}\,W_t^{\gamma,1} + \gamma_t\mathrm{d}\,W_t^{\gamma,2}\right), \quad Z_0 = z.$$

By Girsanov's theorem, $W^{\gamma,1}$ and $W^{\gamma,2}$ are Q^γ-Brownian motions. Moreover, $Z_T = z\frac{dP}{dQ^\gamma}$ and (Z_t) is a Q^γ-martingale.

We are interested in computing $\hat{u}(z, h(Y_T))$ of equation (4.2). A priori we have to optimize over all absolutely continuous local martingale measures of finite q-distance, that is, $Q \in \mathcal{M}_q^a$. The supremum does not change (but may no longer be attained) if we optimize over only equivalent local martingale measures, that is, $Q \in \mathcal{M}_q^e$.

Assumption 5.2. Assume that we only need to optimize over measures of the form Q^γ, where γ satisfies the Novikov condition:

$$\hat{u}(z, h(Y_T)) = \sup_{\gamma\in\mathcal{N}} \left(\mathbb{E}^{Q^\gamma}[h(Y_T)] - \frac{1}{q} z^{1-q} \mathbb{E}\left[\left(\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P}\right)^q\right]\right). \tag{5.5}$$

Re-writing (5.5) as

$$\hat{u}(z, h(Y_T)) = \sup_{\gamma\in\mathcal{N}} \left(\mathbb{E}^{Q^\gamma}[h(Y_T)] - \frac{1}{q} z^{1-q} \mathbb{E}^{Q^\gamma}\left[\left(\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P}\right)^{q-1}\right]\right),$$

leads us to consider the value function

$$\hat{u}(t, y, z) = \sup_{\gamma\in\mathcal{N}} \mathbb{E}^{Q^\gamma}\left[h(Y_T) - \frac{1}{q} Z_T^{1-q} \mid Y_t = y, Z_t = z\right], \tag{5.6}$$

which we have also labeled $\hat{u}$ in a slight abuse of notation.

Proposition 5.3. *Suppose i) Assumption 5.2 holds; ii) the value function $\hat{u}(t, y, z)$ is continuously differentiable in t and twice continuously differentiable in y and z, and is strictly concave in z; and iii) that γ_t^* defined by*

$$\gamma_t^* = -\sqrt{1-\nu^2}\, a(Y_t) \frac{(Z_t\, \hat{u}_{zy}(t, Y_t, Z_t) - \hat{u}_y(t, Y_t, Z_t))}{Z_t^2\, \hat{u}_{zz}(t, Y_t, Z_t)}$$

satisfies Novikov's condition. Then $\hat{u}(t, y, z)$ satisfies the HJB equation

$$\hat{u}_t + \mathcal{L}_y \hat{u} + \frac{\nu\mu(y)a(y)}{\sigma(y)}(z\hat{u}_{zy} - \hat{u}_y) + \frac{\mu^2(y)}{2\sigma^2(y)} z^2 \hat{u}_{zz} - \frac{1}{2} a^2(y)(1-\nu^2)\frac{(z\hat{u}_{zy} - \hat{u}_y)^2}{z^2 \hat{u}_{zz}} = 0, \tag{5.7}$$

with the terminal condition

$$\hat{u}(T, y, z) = h(y) - \frac{1}{q} z^{1-q}, \tag{5.8}$$

where

$$\mathcal{L}_y = \frac{1}{2} a^2(y) \frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}. \tag{5.9}$$

The optimum in (5.6) is attained by (γ_t^*).

Proof. Clearly for $\gamma \in \mathcal{N}$,

$$\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P} = z Z_T^{-1},$$

and, under Q^γ,

$$\mathrm{d}Y_t = \left(b(Y_t) - \nu\frac{\mu(Y_t)}{\sigma(Y_t)} a(Y_t) - \nu' a(Y_t)\gamma_t \right) \mathrm{d}t + a(Y_t)(\nu\, \mathrm{d}W_t^{\gamma,1} + \nu' \mathrm{d}W_t^{\gamma,2}), \quad Y_0 = y.$$

Given the strong regularity assumptions, the results follow from standard verification arguments [11]. $\square$

An alternative derivation at the level of value functions, obtained from the HJB equation for H in (5.4) associated with the primal problem, is given in Appendix A.

Remark 5.4. In the case of the exponential loss function $\ell(x) = e^{\gamma x}$, the analysis and the resulting PDEs are the same, only the terminal conditions change. In particular, (5.8) becomes $\hat{u}(T, y, z) = h(y) + \gamma^{-1}(1 + \log(\gamma z))$. Then the solution to (5.7) is additively separable in y and z, and is given by

$$\hat{u}(t, y, z) = K(t, y) + L(z), \tag{5.10}$$

where $L(z) = \gamma^{-1} \log(\gamma z)$, and

$$K(t, y) = \frac{1}{\gamma} + \frac{1}{(1-\nu^2)} \log \mathbb{E}^{Q^0}\left[\exp\left(-\int_t^T \frac{\mu^2(Y_s)(1-\nu^2)}{2\sigma^2(Y_s)}\,\mathrm{d}s + (1-\nu^2)h(Y_T)\right) \mid Y_t = y\right].$$

We refer to [24]. This simplification is particular to the exponential loss function, and of course can be exploited in the dual problem itself without passing to the conjugate.

In general, the PDE problem (5.7) is not analytically tractable, but for a very special case as when the terminal condition comes from the exponential loss function, as discussed in Remark 5.4. For the terminal condition (5.8) coming from our power loss function, the solution is not separable as (5.10), even if L is allowed to depend on t as well. However, in the case of no claim ($h \equiv 0$), the dual problem is to find the q-optimal measure that minimizes

$$\mathbb{E}\left[\left(\frac{\mathrm{d}Q^\gamma}{\mathrm{d}P}\right)^q\right].$$

This problem is considered in some generality in [15], and for stochastic volatility models in [17, 25]. Similarly, conditions for verifying the optimality of a candidate measure which involve only that measure are available in the case of the entropic risk measure [15, Proposition 3.2], and in the problem of finding the q-optimal measure when there is no claim [15, Proposition 4.2], but we are not aware of a similar result in the latter case when there is a claim, and verification remains an open problem.

5.2 Numerical Solution

To illustrate the market-adjusted shortfall risk measure of a derivatives position, we present a numerical solution of the PDE for the conjugate of the dual problem, which is then Legendre-transformed to return the risk measure. Specifically, we suppose that the claim on Y is a put option with strike K:

$$h(Y_T) = (K - Y_T)^+,$$

where Y is a geometric Brownian motion:

$$dY_t = bY_t\,\mathrm{d}t + aY_t\,(\nu\,\mathrm{d}W_t^1 + \nu'\,\mathrm{d}W_t^2), \tag{5.11}$$

and we want to compute the risk

$$u = \inf_\pi \rho\left(h(Y_T) - V_T\right),$$

where ρ is the shortfall risk measure with *quadratic* power loss function and thereshold ξ, defined in (3.1)-(3.2), with $p = 2$, and V_T is the terminal value of the hedging portfolio, defined in (5.3).

We do not tackle here the problem of hedging exotic options with the underlying and other vanilla options. In the case of the exponential loss function, numerical solutions for the full static-dynamic hedging of barrier options are given in [20], but we leave for a future work extension of this to the power loss shortfall case.

Since the initial wealth level v merely reduces the risk by subtraction, we take $v = 0$ without loss of generality. Then, by Theorem 4.1, given the solution $\hat{u}(0, Y_0, z)$, the risk under this measure of the short put position, mitigated by trading optimally in the correlated asset S, is given by

$$u = \sup_{z>0} \left(\hat{u}(0, Y_0, z) - \xi z\right). \tag{5.12}$$

We employ an explicit finite-difference solution of (5.7) (after a transformation $\eta = \log z$), with terminal condition (5.8). In Figure 4, we show $\hat{u}$ as a function of y and $\log z$ for some example parameters. For fixed $Y_0 = y = K$ (an at-the-money option), we numerically compute

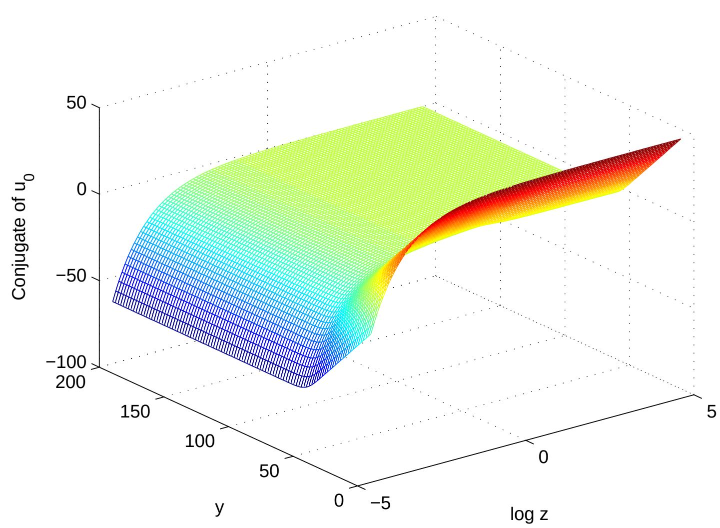

Figure 4: *Numerical solution of conjugate function $\hat{u}(0, y, z)$ with parameters $a = 0.2, b = 0.08, \mu = 0.1, \sigma = 0.1, \rho = 0.5$ and put option strike $K = 50$, maturity $T = 0.25$ years.*

the Legendre transform (5.12) to determine the risk as a function of the threshold level ξ. This is shown in Figure 5. Note the risk is the amount of cash needed to reduce the error in hedging the put position by optimally trading the correlated asset S to below the threshold level ξ. The Black-Scholes price of the put if the asset Y could be traded is shown for comparison. Note also the limit as $\xi \downarrow 0$ indicates that the superhedging price is approximately double the Black-Scholes price in this case.

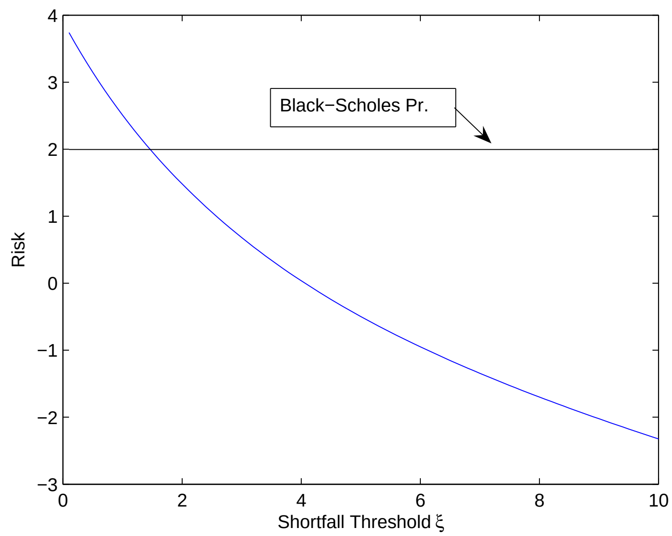

Figure 5: *Market-modified shortfall risk of the put option position, with parameters as in Figure 4.*

6 Conclusions

In this paper, we have investigated conditions under which there is a unique optimal static hedging position in the problem of hedging exotic derivatives using a static-dynamic combination of continuous trading in underlying stocks, and buy-and-hold hedges in vanilla options. The performance is judged by a convex risk measure.

The first step is to analyze the residual function u after the optimal dynamic hedge. This leads to a sufficient condition in general for existence of the optimal static hedge. To investigate uniqueness, we study the shortfall family of risk measures, which provide an example of directly defined convex, but not coherent, risk measures other than the entropic risk measure. With power loss functions, we are able to find a characterization of the optimal solution of the dual problem, and so analyze conditions for strict convexity of u and from there, uniqueness.

Computational issues are illustrated in a simple incomplete market model for optimal dynamic hedging of an option on a non-traded asset. While reasonably tractable in this simple case, using dynamic programming, we conclude that a lot of the flexibility of the entropic risk measure (exponential utility) is lost in passing to other convex risk measures, and computation remains a major challenge.

A Dualization & Conjugation at the PDE Level

We give a formal derivation of the HJB equation (5.7) for the conjugate $\hat{u}$ in the case of the non-traded asset model, obtained directly from the primal problem. Throughout, we assume the necessary smoothness of the value functions for the following calculations to apply.

We start with the value function of the dynamic hedging problem, $H(t, v, y)$, defined in

(5.4). Its associated HJB equation is

$$H_t + \mathcal{L}_y H + \inf_{\pi}\left(\frac{1}{2}\pi^2\sigma^2(y)H_{vv} + \pi(\mu(y)H_v + \rho\sigma(y)a(y)H_{vy})\right) = 0 \qquad \text{(A.1)}$$

with

$$H(T,v,y) = \frac{1}{p}\left((h(y)-v)^+\right)^p,$$

where $\mathcal{L}_y$, defined in (5.9), is the infinitesimal generator of Y. Evaluating the internal minimum supposing that $H_{vv} > 0$ in $t < T$ gives

$$H_t + \mathcal{L}_y H - \frac{(\mu(y)H_v + \rho\sigma(y)a(y)H_{vy})^2}{2\sigma^2(y)H_{vv}} = 0.$$

For v less than the superhedging price of the claim, we need to find the "inverse" of H, namely the solution w of

$$H(t, v + w(t,v,y,\xi), y) = \xi.$$

Then it follows easily that $w = -v + u(t,y,\xi)$, for some function u, which is in fact the total capital needed to reduce the expected shortfall to level ξ. (The additional capital w is found by simply reducing u by the initial capital v). By successive differentiation of the identity $H(t,u(t,y,\xi),y) = \xi$, we obtain the following PDE problem for u:

$$\begin{aligned} u_t + \frac{1}{2}a^2(y)\left(u_{yy} - 2\frac{u_{\xi y}u_y}{u_\xi} + \frac{u_{\xi\xi}u_y^2}{u_\xi^2}\right) + b(y)u_y & \\ -\frac{(\mu(y)u_\xi - \rho\sigma(y)a(y)u_{y\xi} + \rho a(y)\sigma(y)\frac{u_y u_{\xi\xi}}{u_\xi})^2}{2\sigma^2(y)u_{\xi\xi}} &= 0 \end{aligned}$$

with

$$u(T,y,\xi) = h(y) - (p\xi)^{1/p}.$$

Note that we need to treat ξ as a variable in order to have a self-contained equation for u.

Next, we introduce the Legendre transform of u

$$\hat{u}(t,y,z) = \inf_{\xi>0}\left(u(t,y,\xi) + \xi z\right),$$

and the optimizer $\chi(t,y,z)$ that solves

$$u_\xi(t,y,\chi) = -z.$$

Then, successively differentiating and manipulating this expression, we substitute partial derivatives of u in terms of $\hat{u}$ and obtain

$$\hat{u}_t + \mathcal{L}_y\hat{u} + \frac{\rho\mu a(y)}{\sigma}(z\hat{u}_{zy} - \hat{u}_y) + \frac{\mu^2}{2\sigma^2}z^2\hat{u}_{zz} - \frac{1}{2}a^2(y)(1-\rho^2)\frac{(z\hat{u}_{zy} - \hat{u}_y)^2}{z^2u_{zz}} = 0,$$

which is exactly (5.7).

References

[1] P. Artzner, F. Delbaen, J. M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9:203–228, 1999.

[2] P. Barrieu and N. El Karoui. Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics*, 9(2):269–298, 2005.

[3] P. Barrieu and N. El Karoui. Pricing, hedging and optimally designing derivatives via minimization of risk measures. In R. Carmona, editor, *Indifference Pricing*. Princeton University Press, 2008.

[4] B. Bouchard, N. Touzi, and A. Zeghal. Dual formulation of the utility maximization problem: The case of nonsmooth utility. *The Annals of Applied Probability*, 14(2):678–717, 2004.

[5] J. Bowie and P. Carr. Static simplicity. *Risk*, 7:45–49, 1994.

[6] R. Carmona, editor. *Indifference Pricing*. Princeton University Press, 2008.

[7] J. Cvitanić and I. Karatzas. On dynamic measures of risk. *Finance and Stochastics*, 3(4), 1999.

[8] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, 2002.

[9] F. Delbaen and W. Schachermayer. Attainable claims with p'th moments. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(6):743–763, 1996.

[10] D. Filipović and G. Svindland. The canonical model space for law-invariant convex risk measures is L^1. *Mathematical Finance*, 2009. To appear.

[11] W. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer, 2nd edition, 2005.

[12] H. Föllmer and P. Leukert. Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 4(2):117–146, 2000.

[13] H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. De Gruyter Studies in Mathematics. Walter de Gruyter, 2nd edition, 2004.

[14] Avner Friedman. *Foundations of Modern Analysis*. Dover Publications Inc, 1970.

[15] P. Grandits and T. Rheinländer. On the minimal entropy martingale measure. *The Annals of Probability*, 30(3):1003–1038, 2002.

[16] A. Hamel, F. Heyde, and M. Höhne. Set-valued measures of risk. Technical report, Princeton University, 2007.

[17] D. Hobson. Stochastic volatility models, correlation and the q-optimal measure. *Mathematical Finance*, 14(4):537–556, 2004.

[18] A. İlhan, M. Jonsson, and R. Sircar. Optimal investment with derivative securities. *Finance & Stochastics*, 9:585–595, 2005.

[19] A. İlhan, M. Jonsson, and R. Sircar. Portfolio optimization with derivatives and indifference pricing. In R. Carmona, editor, *Indifference Pricing*. Princeton University Press, 2008.

[20] A. İlhan and R. Sircar. Optimal static-dynamic hedges for barrier options. *Mathematical Finance*, 16:359–385, 2006.

[21] M. Jonsson and R. Sircar. Partial hedging in a stochastic volatility environment. *Mathematical Finance*, 12(4):375–409, October 2002.

[22] S. Klöppel and M. Schweizer. Dynamic utility indifference valuation via convex measures. Technical report, NCCR FINRISK, ETH Zurich, 2005. Shorter version in Mathematical Finance vol. 17, pages 599-627, 2007.

[23] D. Kramkov and M. Sirbu. Sensitivity analysis of utility based prices and risk-tolerance wealth processes. *Annals of Applied Probability*, 16(4):2140–2194, 2006.

[24] M. Musiela and T. Zariphopoulou. An example of indifference prices under exponential preferences. *Finance and Stochastics*, 8(2):229–239, 2004.

[25] T. Rheinländer. An entropy approach to the Stein and Stein model with correlation. *Finance and Stochastics*, 9(3):399–413, 2005.

[26] B. Rudloff. Convex hedging in incomplete markets. *Applied Mathematical Finance*, 14(5):437 – 452, 2007.

[27] W. Schachermayer. Introduction to the mathematics of financial markets. In Pierre Bernard, editor, *Lectures on Probability Theory and Statistics, Saint-Fleur summer school 2000*, number 1816 in Lecture Notes in Mathematics, pages 111–177. Springer Verlag, 2003.

[28] A. Toussaint. Hedging under L^2 convex risk measures. Technical report, Princeton University, 2006.

[29] A. Toussaint and R. Sircar. A framework for dynamic hedging under convex risk measures. In R. Dalang, M. Dozzi, and F. Russo, editors, *Proceedings of the Fifth Seminar on Stochastic Analysis, Random Fields and Applications*, Progress in Probability. Birkhauser Verlag, 2008. To appear.

[30] M. Yor. Sous-espaces denses dans L^1 ou H^1 et representation des martingales. In *Séminaire de Probabilités XII*, number 649 in Lecture Notes in Mathematics, pages 205–309. Springer, 1978.