OPTIMAL INVESTMENT WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY PART II: FINITE HORIZON

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Abstract. In this companion paper to "Optimal Investment with Transaction Costs and Stochastic Volatility Part I: Infinite Horizon", we give an accuracy proof for the finite time optimal investment and consumption problem under fast mean-reverting stochastic volatility of a joint asymptotic expansion in a time scale parameter and the small transaction cost.

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1. Introduction. In Part I of this work [4], we derived formal asymptotic expansions for the *infinite horizon* problem of optimal investment when the risky asset has fast or slowly fluctuating stochastic volatility, where the expansions are in the time scale parameter. Part I also includes a historical overview and references. In this companion paper, we study the finite time problem and include consumption also. That said, this methodology is also very interesting in itself as we provide a rigorous asymptotic expansion simultaneously in two variables, without imposing any relationship between them, as is done e.g. in [6] or [2].

In Sections 3 and 4, we give formal derivations of a joint expansion in the fast volatility time scale parameter and the small transaction cost regime, which can be computed explicitly. Section 5 is devoted the the formulation and proof of our main accuracy result, which is given in Theorem 5.4.

2. A Class of Stochastic Volatility Models with Transaction Costs. An investor can dynamically allocate capital between a risky stock with price S and a risk-free money market account with constant rate of interest r, that evolves according to the following *fast mean reverting* stochastic volatility model:

$$\frac{dS_t}{S_t} = (\mu + r)\,dt + f(Z_t)\,dB^1_t, \tag{2.1}$$
$$dZ_t = \frac{1}{\varepsilon}\alpha(Z_t)\,dt + \frac{1}{\sqrt{\varepsilon}}\beta(Z_t)\,dB^2_t.$$

Here, B^1, B^2 are Brownian motions, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$, with constant correlation coefficient $\rho \in (-1,1)$: $d\left\langle B^1, B^2\right\rangle_t = \rho\,dt$, and where $0 < T < \infty$ is the investment horizon. We recall our assumptions that f is a smooth, and strictly positive function, and that the stochastic volatility factor Z_t is a fast mean-reverting process, meaning that the parameter $\varepsilon > 0$ is small, and that Z is an ergodic process with a unique invariant distribution Φ that is independent of ε. Additionally μ and r are positive constants, and α, β are smooth functions. The assumptions are made precise in Section 5.

2.1. Investment Problem. The wealth X invested in the money market account and the wealth Y invested in the stock follow

$$dX_t = rX_t\,dt - C_t dt - (1+\lambda)\,dL_t + (1-\lambda)\,dM_t,$$
$$dY_t = (\mu + r)Y_t\,dt + f(Z_t)Y_t\,dB^1_t + dL_t - dM_t,$$

where the investor controls L and M that are nondecreasing and right-continuous processes with left limits, and $L_{0-} = M_{0-} = 0$. The control L_t represents the cumulative dollar value of stock purchased up to time t, while M_t is the cumulative dollar value of stock sold. Different from [4], we also allow for consumption at rate $C \ge 0$. The constant $\lambda \in (0,1)$ represents the proportional transaction costs for selling the stock.

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Next, define the liquidation function

$$\mathrm{Liq}(x,y) := x + y - \lambda\,|y|\,,$$

together with the *solvency region*

$$\mathcal{S} := \{(x,y)\colon \mathrm{Liq}(x,y) > 0\}\,, \tag{2.2}$$

which is the set of all positions, such that if the investor were forced to liquidate immediately, he would not be bankrupt. This leads to a definition that a policy $(C_s, L_s, M_s)_{t\le s\le T}$ is *admissible* for the initial position $(X_{t-}, Y_{t-}) = (x,y)$ starting at time $t-$ and $Z_t = z$, if (X_s, Y_s) is in the closure of the solvency region, $\overline{\mathcal{S}}$, for $s \in [t,T]$. (Since the investor may choose to immediately rebalance his position, we have denoted the initial time $t-$.)

We will utilize the process

$$\xi_t := \frac{Y_t}{X_t + Y_t}, \tag{2.3}$$

and we restrict an admissible strategy so that either $X_t = Y_t = 0$, or $\xi_t \in \mathcal{K}$, for some compact $\mathcal{K} \subset \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right)$. Let $\mathcal{A}(t,x,y,z)$ be the set of all such policies. Clearly, if $(x,y) \in \overline{\mathcal{S}}$, then we can always liquidate the position and hold the resulting cash position in the risk-free money market account; see [10] for a rigorous proof that $\mathcal{A}(t,x,y,z) \neq \emptyset$ if and only if $(t,x,y,z) \in [0,T] \times \overline{\mathcal{S}} \times \mathbb{R}$.

We work with CRRA or power utility functions $\mathcal{U}(w)$ defined on $\mathbb{R}_+$ as: $\mathcal{U}(w) := \frac{w^{1-\gamma}}{1-\gamma},\ \gamma > 0,\ \gamma \neq 1$, where γ is the constant relative risk aversion parameter. We are interested in maximizing:

$$\sup_{(C,L,M)\in\mathcal{A}} \mathbb{E}_t^{x,y,z}\left[\int_t^T \mathrm{e}^{-\nu(s-t)}\,\mathcal{U}(C_s)\,ds + \mathrm{e}^{-\nu(T-t)}\,\mathcal{U}\left(\mathrm{Liq}(X_T, Y_T)\right)\right], \tag{2.4}$$

for $(t,x,y,z) \in [0,T] \times \overline{\mathcal{S}} \times \mathbb{R}$, where $\mathbb{E}_t^{x,y,z}[\cdot] := \mathbb{E}[\cdot\,|X_{t-} = x, Y_{t-} = y, Z_t = z]$, and $\nu \geq 0$ is the rate of discounting utility over time.

This is a problem of optimizing the terminal wealth at time T, as well as the consumption rate. Note, that it does not matter if the optimization problem is stated as (2.4), or if we change the terminal wealth to be $X_T + Y_T$, that is a liquidation of the stock position is not required. The reason is that we will be looking for an asymptotic expansion, up to $O\left(\lambda^{\frac{2}{3}}\right)$, so one trade of order λ at the terminal time does not make a difference.

2.2. HJB Equation. Consider the value function for our terminal wealth maximization:

$$\widehat{V}(t,x,y,z) = \sup_{(C,L,M)\in\mathcal{A}(t,x,y,z)} \mathbb{E}_t^{x,y,z}\left[\int_t^T \mathrm{e}^{-\nu(s-t)}\,\mathcal{U}(C_s)ds + \mathrm{e}^{-\nu(T-t)}\,\mathcal{U}\left(\mathrm{Liq}(X_T, Y_T)\right)\right]. \tag{2.5}$$

Assume for a moment that $\widehat{V}$ is smooth enough to apply Itô's formula, from which follows that

$$\begin{aligned}
d\left(\mathrm{e}^{-\nu t}\,\widehat{V}(t,X_t,Y_t,Z_t)\right) &= \mathrm{e}^{-\nu t}\left(-\nu\widehat{V} + \widehat{V}_t + rX_t\widehat{V}_x + (\mu + r)\,Y_t\widehat{V}_y + \frac{1}{2}f^2(Z_t)Y_t^2\widehat{V}_{yy}\right)dt \\
&\quad + \mathrm{e}^{-\nu t}\left(\frac{1}{\varepsilon}\left(\alpha(Z_t)\widehat{V}_z + \frac{1}{2}\beta^2(Z_t)\widehat{V}_{zz}^2\right)dt + \left(\mathcal{U}(C_t) - C_t\widehat{V}_x\right)dt + \frac{1}{\sqrt{\varepsilon}}\rho f(Z_t)\beta(Z_t)Y_t\widehat{V}_{yz}dt\right) \\
&\quad + \mathrm{e}^{-\nu t}\left(f(Z_t)Y_t\widehat{V}_y\,dB_t^1 + \frac{1}{\sqrt{\varepsilon}}\beta(Z_t)V_z\,dB_t^2 + \left(\widehat{V}_y - (1+\lambda)\widehat{V}_x\right)dL_t + \left((1-\lambda)\widehat{V}_x - \widehat{V}_y\right)dM_t\right).
\end{aligned}$$

Since $\widehat{V}$ must be a supermartingale, the dt, dL_t and dM_t terms must be nonpositive. It follows that $\widehat{V}_y - (1+\lambda)\widehat{V}_x \leq 0$ and $(1-\lambda)\widehat{V}_x - \widehat{V}_y \leq 0$. Alternatively,

$$\frac{1}{1+\lambda} \leq \frac{\widehat{V}_x}{\widehat{V}_y} \leq \frac{1}{1-\lambda}. \tag{2.6}$$

We will define the no-trade ($\widehat{\mathrm{NT}}$) region, associated with $\widehat{V}$, to be the region where both of these inequalities are strict. Moreover, for the optimal strategy, $\widehat{V}$ is a martingale, and so the dt term above must be zero inside the $\widehat{\mathrm{NT}}$ region.

Additionally, note that the optimal consumption C_t is also easy to find, once the value function is known. Maximizing the dt terms that involve consumption: $\max_{C_t\geq 0}\mathcal{U}(C_t) - C_t\widehat{V}_x = \tilde{\mathcal{U}}(V_x)$, where the convex conjugate function $\tilde{\mathcal{U}}$ is defined as

$$\tilde{\mathcal{U}}(\tilde{w}) := \sup_{w\geq 0}\left(\mathcal{U}(w) - w\tilde{w}\right) = \frac{\gamma}{1-\gamma}\tilde{w}^{-\frac{1-\gamma}{\gamma}}, \quad \tilde{w} > 0, \tag{2.7}$$

we conclude that the optimal consumption is given by $C_t^{\lambda,\varepsilon} = (\mathcal{U}')^{-1}(\widehat{V}_x)$.

Then $\widehat{V}$ will satisfy the HJB equation

$$\begin{aligned} &\min\left\{-(\partial_t + \mathcal{D}^\varepsilon)\widehat{V} - \tilde{\mathcal{U}}(\widehat{V}_x), ((1+\lambda)\,\partial_x - \partial_y)\widehat{V}, (\partial_y - (1-\lambda)\,\partial_x)\,\widehat{V}\right\} = 0,\\ &\widehat{V}(T,x,y,z) = \mathcal{U}(x+y-\lambda\,|y|), \end{aligned} \tag{2.8}$$

where $\mathcal{D}^\varepsilon = -\nu I + rx\partial_x + (\mu + r)\,y\partial_y + \frac{1}{2}f^2(z)y^2\partial^2_{yy} + \frac{1}{\sqrt{\varepsilon}}\rho f(z)\beta(z)y\partial^2_{yz} + \frac{1}{\varepsilon}\left(\alpha(z)\partial_z + \frac{1}{2}\beta^2(z)\partial^2_{zz}\right)$, where I is the identity operator.

Remark 2.1. The verification theorem that the value function $\hat{V}$ from (2.5) is the unique viscosity solution to the HJB equation (2.8) is an extension to the classical theorems of [10], as shown in Lemmas 1.1, 1.3 and 1.4, in the supplemental document [5]. In our case, the connection between the value function and the HJB equation is only needed for the heuristic derivation of Section 4, and is not used in the rigorous proof of Section 5. The result is presented for completeness only.

Next, we look for a solution of the HJB equation (2.8) of the form

$$\widehat{V}(t,x,y,z) = (x+y)^{1-\gamma}v^{\lambda,\varepsilon}\left(t,\xi,z\right), \qquad \xi = \frac{y}{x+y}, \tag{2.9}$$

where the function $v^{\lambda,\varepsilon}$ remains to be found. Note that the solvency region in the new variables becomes

$$\mathcal{S}_\xi = \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right). \tag{2.10}$$

It is also convenient to introduce

$$D_k = (1-\xi)^k\frac{\partial^k}{\partial\xi^k},\ k = 1,2,\cdots, \qquad \mathcal{L}_{\mathcal{U}} := (1-\gamma)I - \xi\partial_\xi, \qquad \Gamma = \gamma(1-\gamma). \tag{2.11}$$

Inserting the transformation (2.9) into (2.8) leads to the following equation for $v^{\lambda,\varepsilon}$:

$$\begin{aligned} &\max\left\{\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)v^{\lambda,\varepsilon} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,\varepsilon}\right),\ \mathcal{B}v^{\lambda,\varepsilon},\ \mathcal{S}v^{\lambda,\varepsilon}\right\} = 0,\\ &v^{\lambda,\varepsilon}(T,\xi,z) = \mathcal{U}\left(1-\lambda\,|\xi|\right), \end{aligned} \tag{2.12}$$

where we define the operators

$$\begin{aligned} &\mathcal{L}_0 := \frac{1}{2}\beta^2(z)\partial^2_{zz} + \alpha(z)\partial_z, \quad \mathcal{L}_1 = \rho f(z)\beta(z)\xi\partial_z\left((1-\gamma)I + D_1\right),\\ &\mathcal{L}_2 = \partial_t + \mu\xi D_1 + \left((1-\gamma)\left(r+\mu\xi\right) - \nu\right)I + \frac{\xi^2}{2}f^2(z)\left(D_2 - 2\gamma D_1 - \Gamma I\right), \end{aligned}$$

and the buy and sell operators by

$$\mathcal{B} := \left(1+\lambda\xi\right)\partial_\xi - \lambda(1-\gamma)I, \qquad \mathcal{S} := -\left(1-\lambda\xi\right)\partial_\xi - \lambda(1-\gamma)I,$$

respectively. For future reference, we also define their derivatives

$$\mathcal{B}' = \partial_\xi \mathcal{B} = (1+\lambda\xi)\,\partial_{\xi\xi} + \lambda\gamma\partial_\xi\,, \qquad \mathcal{S}' = \partial_\xi \mathcal{S} = -\,(1-\lambda\xi)\,\partial_{\xi\xi} + \lambda\gamma\partial_\xi\,.$$

A calculation shows that $\mathcal{L}_2$ can be also written as

$$\mathcal{L}_2 = \partial_t - \gamma f^2(z)\xi\,(\xi - \theta(f(z)))\,D_1 + \frac{f^2(z)}{2}\xi^2 D_2 + (1-\gamma)\left[A(f(z)) - \frac{\gamma f^2(z)}{2}\,(\xi-\theta(f(z)))^2\right] I, \tag{2.13}$$

where we define

$$A(\sigma) := r - \frac{\nu}{1-\gamma} + \frac{1}{2}\frac{\mu^2}{\gamma\sigma^2}, \qquad \theta(\sigma) := \frac{\mu}{\gamma\sigma^2}. \tag{2.14}$$

2.3. Free Boundary Formulation. We will look for a solution to the variational inequality (2.12) in the following free-boundary form. The NT region for $v^{\lambda,\varepsilon}$ is defined by strict inequalities in (2.6). Using the transformation (2.9), this translates to $-\lambda < \frac{v_\xi^{\lambda,\varepsilon}}{(1-\gamma)v^{\lambda,\varepsilon} - \xi v_\xi^{\lambda,\varepsilon}} < \lambda$ for $v^{\lambda,\varepsilon}(\xi,z)$. Similar to the case with constant volatility, we assume that inside the NT region $\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)v^{\lambda,\varepsilon} = 0$, and its boundaries are $\ell^\varepsilon(t,z)$ and $u^\varepsilon(t,z)$. We write this region as

$$\min\{\ell^\varepsilon(t,z), u^\varepsilon(t,z)\} < \xi < \max\{\ell^\varepsilon(t,z), u^\varepsilon(t,z)\},$$

where $\ell^\varepsilon(t,z)$ and $u^\varepsilon(t,z)$ are free boundaries to be found. In typical parameter regimes, we will have $0 < \ell^\varepsilon(t,z) < u^\varepsilon(t,z)$, so we can think of them as lower and upper boundaries respectively, with ℓ^ε being the buy boundary, and u^ε the sell boundary. (The other two possibilities are that $\ell^\varepsilon < u^\varepsilon < 0$ with ℓ^ε being the buy boundary, and u^ε the sell boundary, or that $\ell^\varepsilon < u^\varepsilon < 0$ with ℓ^ε being the sell boundary, and u^ε the buy boundary).

Inside this region we have from the HJB equation (2.12) that

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)v^{\lambda,\varepsilon} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}} v^{\lambda,\varepsilon}\right) = 0, \qquad \xi \in (\ell^\varepsilon(t,z), u^\varepsilon(t,z)).$$

The free boundaries ℓ^ε and u^ε are determined by continuity of the first and second derivatives of $v^{\lambda,\varepsilon}$ with respect to ξ, that is looking for a C^2 solution, which leads to

$$\mathcal{B}v^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) := (1+\lambda\ell^\varepsilon(t,z))\,v_\xi^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) - \lambda(1-\gamma)v^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) = 0, \tag{2.15}$$

$$\mathcal{B}'v^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) := (1+\lambda\ell^\varepsilon(t,z))\,v_{\xi\xi}^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) + \lambda\gamma v_\xi^{\lambda,\varepsilon}(t,\ell^\varepsilon(t,z),z) = 0. \tag{2.16}$$

at the buy boundary, and

$$\mathcal{S}v^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) = -\,(1-\lambda u^\varepsilon(t,z))\,v_\xi^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) - \lambda(1-\gamma)v^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) = 0, \tag{2.17}$$

$$\mathcal{S}'v^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) := -\,(1-\lambda u^\varepsilon(t,z))\,v_{\xi\xi}^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) + \lambda\gamma v_\xi^{\lambda,\varepsilon}(t,u^\varepsilon(t,z),z) = 0. \tag{2.18}$$

at the sell boundary.

3. Fast-scale Asymptotic Analysis.

We look for an expansion for the value function

$$v^{\lambda,\varepsilon} = v^{\lambda,0} + \sqrt{\varepsilon}\,v^{\lambda,1} + \varepsilon v^{\lambda,2} + \cdots = \sum_{i,j=0}^{\infty} \lambda^{i/3}\varepsilon^{j/2} v^{i,j} \tag{3.1}$$

as well as for the free boundaries

$$\ell^\varepsilon = \ell_0 + \sqrt{\varepsilon}\,\ell_1 + \varepsilon\ell_2 + \cdots = \sum_{i,j=0}^{\infty} \lambda^{i/3}\varepsilon^{j/2}\ell_{i,j}, \qquad u^\varepsilon = u_0 + \sqrt{\varepsilon}\,u_1 + \varepsilon u_2 + \cdots = \sum_{i,j=0}^{\infty} \lambda^{i/3}\varepsilon^{j/2}u_{i,j}, \tag{3.2}$$

which are asymptotic as $\varepsilon, \lambda \downarrow 0$.

We will also use extensively the Taylor expansion of the conjugate utility function in (2.7):

$$\tilde{\mathcal{U}}\left(x+h\right)=\tilde{\mathcal{U}}(x)-x^{-\frac{1}{\gamma}}h+\frac{1}{2\gamma}x^{-\frac{1+\gamma}{\gamma}}h^2+O\left(h^3\right). \tag{3.3}$$

Central to this analysis is the Fredholm alternative (or centering condition). We will use the notation $\langle\cdot\rangle$ to denote the expectation with respect to the invariant distribution Φ of the process Z, namely $\langle g\rangle:=\int g(z)\Phi(dz)$. The Fredholm alternative tells us that a Poisson equation of the form $\mathcal{L}_0 v+\chi=0$ has a solution v only if the solvability condition $\langle\chi\rangle=0$ is satisfied, and we refer for instance to [9, Section 3.2] for technical details.

In the following, a key role will be played by the square-averaged volatility $\bar{\sigma}$ defined by

$$\bar{\sigma}^2=\left\langle f^2\right\rangle. \tag{3.4}$$

The principal terms in the expansions will be related to the *constant volatility* transaction costs problem, and we define the operator $\mathcal{L}_{\rm NT}(\sigma)$ that acts in the no trade region by

$$\mathcal{L}_{\rm NT}(\sigma):=\partial_t-\gamma\sigma^2\xi(\xi-\theta(\sigma))D_1+\frac{\sigma^2}{2}\xi^2D_2+(1-\gamma)\left(A(\sigma)-\frac{\gamma\sigma^2}{2}\left(\xi-\theta(\sigma)\right)^2\right)I\ , \tag{3.5}$$

and it is written as a function of the parameter σ. Note that, from (2.13), we have $\mathcal{L}_2=\mathcal{L}_{\rm NT}(f(z))$.

3.1. Power expansion inside the NT region. In this subsection we will concentrate on constructing the expansion inside the NT region $\xi\in(\ell^\varepsilon(t,z),u^\varepsilon(t,z))$, where (2.12) holds. We now insert the expansion (3.1) and match powers of ε.

The terms of order ε^{-1} lead to $\mathcal{L}_0v^{\lambda,0}=0$. Since the $\mathcal{L}_0$ operator takes derivatives in z, we seek a solution of the form $v^{\lambda,0}=v^{\lambda,0}(t,\xi)$, independent of z.

At order $\varepsilon^{-1/2}$, we have $\mathcal{L}_1v^{\lambda,0}+\mathcal{L}_0v^{\lambda,1}=0$. But since $\mathcal{L}_1$ takes a derivative in z, $\mathcal{L}_1v^{\lambda,0}=0$, and so $\mathcal{L}_0v^{\lambda,1}=0$. Again, we seek a solution of the form $v^{\lambda,1}=v^{\lambda,1}(t,\xi)$ that is independent of z.

The terms of order one give $\mathcal{L}_2v^{\lambda,0}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)+\mathcal{L}_1v^{\lambda,1}+\mathcal{L}_0v^{\lambda,2}=0$. Since $\mathcal{L}_1$ takes derivatives in z, and $v^{\lambda,1}$ is independent of z, we have that

$$\mathcal{L}_2v^{\lambda,0}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)+\mathcal{L}_0v^{\lambda,2}=0. \tag{3.6}$$

This is a Poisson equation for $v^{\lambda,2}$ whose solvability condition implies that $\left\langle\mathcal{L}_2+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)\right\rangle=\langle\mathcal{L}_2\rangle v^{\lambda,0}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)=0$. We observe that $\langle\mathcal{L}_2\rangle=\mathcal{L}_{\rm NT}(\bar{\sigma})$, where $\bar{\sigma}$ is the square-averaged volatility defined in (3.4), and $\mathcal{L}_{\rm NT}$ is the constant volatility no trade operator defined in (3.5). Then we have

$$\mathcal{L}_{\rm NT}(\bar{\sigma})v^{\lambda,0}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)=0, \tag{3.7}$$

which, along with boundary conditions we will find in the next subsection, will determine $v^{\lambda,0}$.

To find the equation for the next term $v^{\lambda,1}$ in the approximation, we proceed as follows. We write the first two terms of (3.6) as

$$\mathcal{L}_2v^{\lambda,0}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)=\left(\mathcal{L}_2-\mathcal{L}_{\rm NT}(\bar{\sigma})\right)v^{\lambda,0}=\frac{\xi^2}{2}\left(f^2(z)-\bar{\sigma}^2\right)\left(D_2-\Gamma I-2\gamma D_1\right)v^{\lambda,0},$$

where the constant Γ was defined in (2.11). Then solutions of (3.6) are given by

$$v^{\lambda,2}=-\frac{\xi^2}{2}\ \left(\phi(z)+c(t,\xi)\right)\left(-\Gamma I-2\gamma D_1+D_2\right)v^{\lambda,0}, \tag{3.8}$$

where $c(t,\xi)$ is independent of z, and $\phi(z)$ is a solution to the Poisson equation

$$\mathcal{L}_0\phi(z)=f^2(z)-\bar{\sigma}^2, \tag{3.9}$$

Next, from (3.3), we find that

$$\begin{aligned}\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,\varepsilon}\right)&=\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)-\sqrt{\varepsilon}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}v^{\lambda,1}\\&\quad+\varepsilon\left(\frac{1}{2\gamma}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)^{-\frac{1+\gamma}{\gamma}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,1}\right)^2-\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)^{-\frac{1}{\gamma}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,2}\right)\right)+O\left(\varepsilon^{\frac{3}{2}}\right).\end{aligned}$$

Using this and continuing to the order $\sqrt{\varepsilon}$ terms, we obtain $\mathcal{L}_2 v^{\lambda,1} - \left(\mathcal{L}_{\mathcal{U}} v^{\lambda,0}\right)^{-\frac{1}{\gamma}} \mathcal{L}_{\mathcal{U}} v^{\lambda,1} + \mathcal{L}_1 v^{\lambda,2} + \mathcal{L}_0 v^{\lambda,3} = 0$. Once again, this is a Poisson equation for $v^{\lambda,3}$ whose centering condition implies that $\langle \mathcal{L}_2 \rangle v^{\lambda,1} - \left(\mathcal{L}_{\mathcal{U}} v^{\lambda,0}\right)^{-\frac{1}{\gamma}} \mathcal{L}_{\mathcal{U}} v^{\lambda,1} + \left\langle \mathcal{L}_1 v^{\lambda,2} \right\rangle = 0$. From (3.8), it follows that

$$
\begin{aligned}
(3.10)\qquad \mathcal{L}_{\rm NT}(\bar{\sigma}) v^{\lambda,1} - \left(\mathcal{L}_{\mathcal{U}} v^{\lambda,0}\right)^{-\frac{1}{\gamma}} \mathcal{L}_{\mathcal{U}} v^{\lambda,1} &= - \left\langle \mathcal{L}_1 v^{\lambda,2} \right\rangle \\
&= \bar{V}_3 \xi (1-\gamma) \frac{\xi^2}{2} \left(-\gamma(1-\gamma) v^{\lambda,0} - 2\gamma(1-\xi) v^{\lambda,0}_{\xi} + (1-\xi)^2 v^{\lambda,0}_{\xi\xi} \right) \\
&\quad - \bar{V}_3 \xi (1-\xi)(1-\gamma)\gamma \left(\frac{1}{2}\xi^2 v^{\lambda,0}_{\xi} + \xi v^{\lambda,0} \right) \\
&\quad - \bar{V}_3 \xi (1-\xi)\gamma \left((1-\xi)\xi^2 v^{\lambda,0}_{\xi\xi} - \xi^2 v^{\lambda,0}_{\xi} + 2(1-\xi)\xi v^{\lambda,0}_{\xi} \right) \\
&\quad + \bar{V}_3 \xi (1-\xi) \left(\frac{1}{2}\xi^2 (1-\xi)^2 v^{\lambda,0}_{\xi\xi\xi} - \xi^2(1-\xi) v^{\lambda,0}_{\xi\xi} + (1-\xi)^2 \xi v^{\lambda,0}_{\xi\xi} \right) \\
&= \frac{\xi^2}{2} \rho \langle \beta f \phi' \rangle \Big(\{\xi(1+\gamma) - 2\}\Gamma I - \gamma\{4 - 3(1+\gamma)\xi\} D_1 + \{2 - 3(1+\gamma)\xi\} D_2 + \xi D_3 \Big) v^{\lambda,0}.
\end{aligned}
$$

We define

$$
(3.11)\qquad \bar{V}_3 = \frac{1}{2} \rho \langle \beta f \phi' \rangle .
$$

Then we write equation (3.10) as

$$
\begin{aligned}
(3.12)\qquad & \mathcal{L}_{\rm NT}(\bar{\sigma}) v^{\lambda,1} - \left(\mathcal{L}_{\mathcal{U}} v^{\lambda,0}\right)^{-\frac{1}{\gamma}} \mathcal{L}_{\mathcal{U}} v^{\lambda,1} \\
&= \xi^2 \bar{V}_3 \Big(\{\xi(1+\gamma) - 2\}\Gamma I - \gamma\{4 - 3(1+\gamma)\xi\} D_1 + \{2 - 3(1+\gamma)\xi\} D_2 + \xi D_3 \Big) v^{\lambda,0}.
\end{aligned}
$$

Though it is possible to continue this calculation and express $v^{\lambda,3}$ using various derivatives of $v^{\lambda,0}$, just as it was done in (3.8), we are not going to do this, as it turns out that this is not necessary for our proof.

3.2. Boundary Conditions. So far we have concentrated on the PDE (2.12) in the NT region. We now insert the expansions (3.1) and (3.2) into the boundary conditions (2.15)–(2.18). The terms of order one from (2.15) and (2.16) give

$$
(3.13)\qquad \left(\mathcal{B} v^{\lambda,0}\right)(t, \ell_0(t,z)) = 0, \qquad \text{and} \qquad \left(\mathcal{B}' v^{\lambda,0}\right)(t, \ell_0(t,z)) = 0,
$$

while the terms of order one from (2.15) and (2.16) give

$$
(3.14)\qquad \left(\mathcal{S} v^{\lambda,0}\right)(t, u_0(t,z)) = 0, \qquad \text{and} \qquad \left(\mathcal{S}' v^{\lambda,0}\right)(t, u_0(t,z)) = 0.
$$

Since $v^{\lambda,0}$ is independent of z, these equations imply that ℓ_0 and u_0 are also independent of z (they are functions of time t only).

Taking the order $\sqrt{\varepsilon}$ terms in (2.15) gives

$$
\begin{aligned}
&(1 + \lambda \ell_0(t)) \left(v^{\lambda,1}_{\xi}(t, \ell_0(t)) + \ell_1(t,z) v^{\lambda,0}_{\xi\xi}(t, \ell_0(t)) \right) + \lambda \ell_1(t,z) v^{\lambda,0}_{\xi}(t, \ell_0(t)) \\
&\quad - \lambda(1-\gamma) \left(v^{\lambda,1}(t, \ell_0(t)) + \ell_1(t,z) v^{\lambda,0}_{\xi}(t, \ell_0(t)) \right) = 0.
\end{aligned}
$$

Using the fact that $\mathcal{B}' v^{\lambda,0}(t, \ell_0(t)) = 0$, we see the terms in ℓ_1 cancel, and we obtain

$$
(3.15)\qquad \mathcal{B} v^{\lambda,1}(t, \ell_0(t)) = 0,
$$

which is a mixed-type boundary condition for $v^{\lambda,1}$ at the boundary ℓ_0.

From the order $\sqrt{\varepsilon}$ terms in (2.16), we obtain

$$
\begin{aligned}
&\ell_1(t,z) \left(\lambda v^{\lambda,0}_{\xi\xi}(t, \ell_0(t)) + (1 + \lambda \ell_0(t))\, v^{\lambda,0}_{\xi\xi\xi}(t, \ell_0(t)) + \lambda\gamma v^{\lambda,0}_{\xi\xi}(t, \ell_0(t)) \right) \\
&\quad + \left[(1 + \lambda \ell_0(t))\, v^{\lambda,1}_{\xi\xi}(t, \ell_0(t)) + \lambda\gamma v^{\lambda,1}_{\xi}(t, \ell_0(t)) \right] = 0,
\end{aligned}
$$

and so, as $v^{\lambda,1}$ does not depend on z, $\ell_1(t)$ is also a function of time t only (independent of z) given by

$$\ell_1(t) = -\left(\frac{\left(\mathcal{B}'v^{\lambda,1}\right)(t,\ell_0(t)}{(1+\lambda\ell_0(t))\,v^{\lambda,0}_{\xi\xi\xi}(t,\ell_0(t)) + \lambda(1+\gamma)v^{\lambda,0}_{\xi\xi}(t,\ell_0(t))}\right). \tag{3.16}$$

Similar calculations can be performed on the (right) sell boundary $u^\varepsilon \approx u_0 + \sqrt{\varepsilon}u_1$, where $\mathcal{S}v^{\lambda,\varepsilon} = 0$. The analogous equations to (3.15) and (3.16) are

$$\left(\mathcal{S}v^{\lambda,1}\right)(t,u_0(t)) = 0, \tag{3.17}$$

$$u_1(t) = -\left(\frac{\left(\mathcal{S}'v^{\lambda,1}\right)(t,u_0(t))}{-(1-\lambda u_0(t))\,v^{\lambda,0}_{\xi\xi\xi}(t,u_0(t)) + \lambda(1+\gamma)v^{\lambda,0}_{\xi\xi}(t,u_0(t))}\right). \tag{3.18}$$

Note that (3.17) is a mixed-type boundary condition for $v^{\lambda,1}$ at the boundary u_0, and (3.18) determines the correction term u_1 to the sell boundary.

To summarize, we have sought the zeroth and first order terms in the expansions (3.1) and (3.2) for $(v^{\lambda,\varepsilon},\ell^\varepsilon,u^\varepsilon)$. The principal terms are found from the PDE (3.7), with boundary and free boundary conditions (3.13)-(3.14). The next terms in the asymptotic expansion of the boundaries of the NT region, ℓ_1, u_1 are given by (3.16) and (3.18), and $v^{\lambda,1}$ for $\ell_0(t) < \xi < u_0(t)$ solves the PDE (3.12).

4. Small Transaction Costs.

In the previous section we have established that $v^{\lambda,0}$ solves the *constant volatility Merton problem with transaction costs*, but using the averaged volatility $\bar\sigma$, where $\bar\sigma^2 = \langle f^2\rangle$. In this section, we construct expansions for $v^{\lambda,0}, v^{\lambda,1}$ in small transaction costs λ.

4.1. Expansion For $v^{\lambda,0}$.

The exact solution for $v^{\lambda,0}$ is not known, so instead we will find its asymptotic expansion

$$v^{\lambda,0} = v^{0,0} + \lambda^{\frac13}v^{1,0} + \lambda^{\frac23}v^{2,0} + \cdots, \tag{4.1}$$

and the asymptotic expansion of the boundaries $\ell_0 = \ell_{0,0} + \lambda^{\frac13}\ell_{1,0} + \lambda^{\frac23}\ell_{2,0} + \cdots$, $\quad u_0 = u_{0,0} + \lambda^{\frac13}u_{1,0} + \lambda^{\frac23}u_{2,0} + \cdots$. Set $\theta := \theta(\bar\sigma)$, $A := A(\bar\sigma)$, $\tilde A = \frac{1-\gamma}{\gamma}A$. Additionally, we will also assume that $v^{\lambda,0}$ has an asymptotic expansion (4.1), and that the no-trade region $[\ell_0(t), u_0(t)]_{t\in[0,T]}$ is $O\left(\lambda^{\frac13}\right)$ wide and contains the Merton proportion θ. Specifically, we say that

$$\begin{aligned} v^{\lambda,0}(t,\xi) &= \gamma_0(t) - \gamma_1(t)\,\lambda^{\frac13} - \gamma_2(t)\,\lambda^{\frac23} - \gamma_3(t)\,\lambda - \gamma_{40}(t)\,\lambda^{\frac43} - \gamma_{41}(t)\,(\xi-\theta)\lambda \\ &\quad - \gamma_{42}(t)\,(\xi-\theta)^2\lambda^{\frac23} - \gamma_{43}(t)\,(\xi-\theta)^3\lambda^{\frac13} - \gamma_{44}(t)\,(\xi-\theta)^4 + O\big(\lambda^{\frac53}\big). \end{aligned}$$

Note that the terms $(\xi-\theta)$ and $(\xi-\theta)^2$ have been omitted since their coefficients are zero. This follows from the boundary conditions (3.13) and (3.14), from which one can see that at the boundaries, $\partial_\xi v^{\lambda,0}\mid_{(t,\ell_0(t)}$ $,\partial_\xi v^{\lambda,0}\mid_{(t,u_0(t)},\partial^2_{\xi\xi}v^{\lambda,0}\mid_{(t,\ell_0(t)},\partial^2_{\xi\xi}v^{\lambda,0}\mid_{(t,u_0(t)}= O(\lambda)$. For example, the coefficient of the term $(\xi-\theta)$ has to be zero, because otherwise it would violate the boundary condition that $\partial_\xi v^{\lambda,0}\mid_{(t,\ell_0(t)}= O(\lambda) = \partial_\xi v^{\lambda,0}\mid_{(t,u_0(t)}$, and similarly a non-zero coefficient of the term $(\xi-\theta)^2$ would be a violation of the boundary condition that at the boundary $\partial^2_{\xi\xi}v^{\lambda,0}\mid_{(t,\ell_0(t)}= O(\lambda) = \partial^2_{\xi\xi}v^{\lambda,0}\mid_{(t,u_0(t)}$.

Using (3.3), it follows that

$$\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right) = \frac{\gamma}{1-\gamma}\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1-\gamma}{\gamma}} + (1-\gamma)\gamma_1(t)\left((1-\gamma)\gamma_0(t)\right)^{-\frac1\gamma}\lambda^{\frac13} + O\left(\lambda^{\frac23}\right), \tag{4.2}$$

and so, from the no-trade region PDE (3.7), we obtain:

$$\begin{aligned} \mathcal{L}_{\rm NT}(\bar\sigma)v^{\lambda,0} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right) &= \dot\gamma_0(t) + (1-\gamma)A\gamma_0(t) + \gamma(1-\gamma)^{-\frac1\gamma}\left(\gamma_0(t)\right)^{-\frac{1-\gamma}{\gamma}} \\ &\quad - \lambda^{\frac13}\left(\dot\gamma_1(t) + (1-\gamma)A\gamma_1(t) - (1-\gamma)\left((1-\gamma)\gamma_0(t)\right)^{-\frac1\gamma}\gamma_1(t)\right) + O\left(\lambda^{\frac23}\right) = 0. \end{aligned} \tag{4.3}$$

The terminal time conditions for $v^{\lambda,0}$, which follow from the terminal conditions of $\widehat V$ in (2.8) and from the change of variables (2.9) are:

$$v^{0,0}(T,\xi) = \frac{1}{1-\gamma}, \qquad v^{1,0}(T,\xi) = v^{2,0}(T,\xi) = 0. \tag{4.4}$$

It now follows from the $O(1)$ terms in (4.3) that $\gamma_0(t)$ is given by

$$\gamma_0(t) := \frac{1}{1-\gamma}\left(-\frac{\gamma}{(1-\gamma)A} + \left(1+\frac{\gamma}{A(1-\gamma)}\right)\mathrm{e}^{\frac{1-\gamma}{\gamma}A(T-t)}\right)^{\gamma},$$

and, from the $O\left(\lambda^{\frac{1}{3}}\right)$ terms in (4.3) , that $\gamma_1 = 0$. Using this and (3.3), we can refine the approximation (4.2) to be: $\tilde{\mathcal{U}}\left((1-\gamma)v^{\lambda,0} - \xi v^{\lambda,0}_{\xi}\right) = \frac{\gamma}{1-\gamma}\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1-\gamma}{\gamma}} + (1-\gamma)\gamma_2(t)\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}}\lambda^{\frac{2}{3}} + O\left(\lambda^{\frac{4}{3}}\right)$. Now, we can calculate additional terms in the expansion (4.3). Denote $\tilde{\xi} = \frac{\xi-\theta}{\lambda^{\frac{1}{3}}}$. Note that $\tilde{\xi} = O(1)$ inside the no-trade region. With this notation we obtain

$$\begin{aligned}
\text{(4.5)}\qquad \mathcal{L}_{\text{NT}}(\bar{\sigma})v^{\lambda,0} &= \dot{\gamma}_0(t) + (1-\gamma)A\gamma_0(t) + \gamma(1-\gamma)^{-\frac{1}{\gamma}}\left(\gamma_0(t)\right)^{-\frac{1-\gamma}{\gamma}} \\
&- \lambda^{\frac{1}{3}}\left[\dot{\gamma}_1(t) + (1-\gamma)A\gamma_1(t) - (1-\gamma)\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}}\gamma_1(t)\right] \\
&- \lambda^{\frac{2}{3}}\left[(1-\gamma)\frac{\gamma\bar{\sigma}^2}{2}\tilde{\xi}^2\gamma_0(t) + \frac{\bar{\sigma}^2}{2}\theta^2(\theta-1)^2\left(2\gamma_{42}(t) + 6\gamma_{43}(t)\tilde{\xi} + 12\gamma_{44}(t)\tilde{\xi}^2\right)\right. \\
&\left. + (1-\gamma)A\gamma_2(t) + \dot{\gamma}_2(t) - (1-\gamma)^{-\frac{1}{\gamma}+1}\left(\gamma_0(t)\right)^{-\frac{1}{\gamma}}\gamma_2(t)\right] + O(\lambda).
\end{aligned}$$

We now turn to the boundary conditions. Note that from our assumption that the no-trade region is of width $O\left(\lambda^{\frac{1}{3}}\right)$, it follows that $\ell_{0,0} = u_{0,0} = \theta$, and that on the buy boundary $\ell_0(t) - \theta = \ell_0(t) - \ell_{0,0} = \ell_{1,0}(t)\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})$, and similarly on the sell boundary $u_0(t) - \theta = u_{1,0}(t)\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})$. Next, we calculate that

$$\text{(4.6)}\quad \mathcal{B}v^{\lambda,0}(t,\ell_0(t)) = -\left[\gamma_{41}(t) + 2\gamma_{42}(t)\,\ell_{1,0}(t) + 3\gamma_{43}(t)\ell_{1,0}^2(t) + 4\gamma_{44}(t)\ell_{1,0}^3(t) + (1-\gamma)\gamma_0(t)\right]\lambda = O(\lambda^{\frac{4}{3}}),$$

$$\text{(4.7)}\quad \mathcal{B}'v^{\lambda,0}(t,\ell_0(t)) = -\left[2\gamma_{42}(t) + 6\gamma_{43}(t)\ell_{1,0}(t) + 12\gamma_{44}(t)\ell_{1,0}^2(t)\right]\lambda^{\frac{2}{3}} + O(\lambda) = 0,$$

and also

$$\text{(4.8)}\quad \mathcal{S}v^{\lambda,0}(t,u_0(t)) = \left[\gamma_{41}(t) + 2\gamma_{42}(t)\,u_{1,0}(t) + 3\gamma_{43}(t)u_{1,0}^2(t) + 4\gamma_{44}(t)u_{1,0}^3(t) - (1-\gamma)\gamma_0(t)\right]\lambda = O(\lambda^{\frac{4}{3}}),$$

$$\text{(4.9)}\quad \mathcal{S}'v^{\lambda,0}(t,u_0(t)) = \left[2\gamma_{42}(t) + 6\gamma_{43}(t)u_{1,0}(t) + 12\gamma_{44}(t)u_{1,0}^2(t)\right]\lambda^{\frac{2}{3}} + O(\lambda) = 0.$$

We now want to equate (the leading terms of) (4.5), (4.6), (4.7), (4.8) and (4.9) to zero, in order to find $\gamma_2, \gamma_{4,0}, \gamma_{4,1}, \gamma_{4,2}, \gamma_{4,3}, \gamma_{4,4}$ and $\ell_{1,0}, u_{1,0}$. First, note that we have already seen that the $O(1)$ and $O\left(\lambda^{\frac{1}{3}}\right)$ terms in (4.5) are zero. Hence, setting $\tilde{\xi} = 0$, and $\tilde{\xi} = \ell_{0,1}, \tilde{\xi} = u_{0,1}$ in (4.5), we obtain at the leading order of $O\left(\lambda^{\frac{2}{3}}\right)$ respectively:

$$\text{(4.10)}\qquad \bar{\sigma}^2\theta^2(\theta-1)^2\gamma_{42}(t) = (1-\gamma)\left(\left(\gamma_0(t)(1-\gamma)\right)^{-\frac{1}{\gamma}} - A\right)\gamma_2(t) - \dot{\gamma}_2(t),$$

$$\text{(4.11)}\qquad \frac{\gamma\bar{\sigma}^2}{2}\ell_{0,1}^2\gamma_0(t) = \left(\left(\gamma_0(t)(1-\gamma)\right)^{-\frac{1}{\gamma}} - A\right)\gamma_2(t) - \frac{\dot{\gamma}_2(t)}{1-\gamma},$$

$$\text{(4.12)}\qquad \frac{\gamma\bar{\sigma}^2}{2}u_{0,1}^2\gamma_0(t) = \left(\left(\gamma_0(t)(1-\gamma)\right)^{-\frac{1}{\gamma}} - A\right)\gamma_2(t) - \frac{\dot{\gamma}_2(t)}{1-\gamma},$$

where we have used (4.7) and (4.9) to obtain (4.11) and (4.12). We conclude that

$$\gamma_{42}(t) = \frac{(1-\gamma)\left(\left(\gamma_0(t)(1-\gamma)\right)^{-\frac{1}{\gamma}} - A\right)\gamma_2(t) - \dot{\gamma}_2(t)}{\bar{\sigma}^2\theta^2(\theta-1)^2},$$

$$u_{1,0}(t) = -\ell_{1,0}(t) = \sqrt{2\frac{\left(\gamma_0(t)(1-\gamma)\right)^{-\frac{1}{\gamma}} - A}{\gamma\bar{\sigma}^2\gamma_0(t)}\gamma_2(t) - 2\frac{\dot{\gamma}_2(t)}{(1-\gamma)\gamma\bar{\sigma}^2\gamma_0(t)}},$$

where the last equation follows from the fact that we no-trade region should not degenerate, and our convention that we are in the case that $\ell^{\varepsilon} < u^{\varepsilon}$.

Next, using the fact that $u_{1,0} = -\ell_{1,0}$ and (4.7), (4.9), we conclude that $\gamma_{43}(t) = 0$, and similarly from (4.6), (4.8), that $\gamma_{41}(t) = 0$. Finally, we conclude from (4.7) that

$$\gamma_{44}(t) = -\frac{\gamma_{42}(t)}{6\ell_{1,0}^2(t)} = -\frac{\Gamma}{12\theta^2(\theta-1)^2}\gamma_0(t), \tag{4.13}$$

where the second equality follows by dividing (4.10) by (4.11). Substituting (4.13) into (4.6) it follows that $-8\ell_{1,0}^3(t)\gamma_{44}(t) = -(1-\gamma)\gamma_0(t)$, so that

$$u_{1,0} = -\ell_{1,0} := \left(\frac{3}{2\gamma}\theta^2(\theta-1)^2\right)^{1/3}. \tag{4.14}$$

Note, that the boundaries turn out to be constants, independent of time t. Moreover, substituting (4.13) and into (4.6), we calculate that $\frac{4}{3}\gamma_{42}(t)\ell_{1,0} = -(1-\gamma)\gamma_0(t)$, and by substituting (4.14), we obtain that $\gamma_{42}(t) = (1-\gamma)\left(\frac{9\gamma}{32(\theta-1)^2\theta^2}\right)^{1/3}\gamma_0(t)$. Additionally from (4.14), we see that (4.11) becomes:

$$\dot\gamma_2(t) - (1-\gamma)\left((\gamma_0(t)(1-\gamma))^{-\frac{1}{\gamma}} - A\right)\gamma_2(t) + (1-\gamma)\frac{\gamma\bar\sigma^2}{2}\left(\frac{3(\theta-1)^2\theta^2}{2\gamma}\right)^{2/3}\gamma_0(t) = 0.$$

Recall that $\tilde A = \frac{1-\gamma}{\gamma}A$, then the solution to this ODE with terminal condition $\gamma_2(T) = 0$ as follows from (4.4) can be shown to be

$$\gamma_2(t) := \frac{\ell_{1,0}^2\bar\sigma^2}{2(-A(1-\gamma))^{\gamma+1}}\left(\gamma - (A(1-\gamma)+\gamma)\,\mathrm{e}^{\tilde A(T-t)}\right)^{\gamma-1}$$
$$\times\left(A(1-\gamma)(T-t)\,\mathrm{e}^{\tilde A(T-t)}(A(1-\gamma)+\gamma) + \gamma^2\left(1-\mathrm{e}^{\tilde A(T-t)}\right)\right).$$

We summarize our finding: $v^{\lambda,0}$ admits the following representation:

$$v^{\lambda,0}(t,\xi) = \gamma_0(t) - \gamma_2(t)\lambda^{\frac{2}{3}} - (1-\gamma)\left(\frac{9\gamma}{32(\theta-1)^2\theta^2}\right)^{1/3}\gamma_0(t)(\xi-\theta)^2\lambda^{\frac{2}{3}}$$
$$+\frac{\Gamma}{12\theta^2(\theta-1)^2}\gamma_0(t)(\xi-\theta)^4 + O(\lambda), \tag{4.15}$$

and the expansion of the NT region is given as

$$\ell_0(t) = \theta - \lambda^{\frac{1}{3}}\left(\frac{3}{2\gamma}\theta^2(\theta-1)^2\right)^{1/3} + O(\lambda^{\frac{2}{3}}), \qquad u_0(t) = \theta + \lambda^{\frac{1}{3}}\left(\frac{3}{2\gamma}\theta^2(\theta-1)^2\right)^{1/3} + O(\lambda^{\frac{2}{3}}). \tag{4.16}$$

4.2. Finding $v^{\lambda,1}$. In the following section we find the approximation for $v^{\lambda,1}$. Recall that we have obtained the expansion (4.15) for $v^{\lambda,0}$. It follows that the source term in the equation (3.12) for $v^{\lambda,1}$ is given by

$$\xi^2\Big(\{\xi(1+\gamma)-2\}\Gamma I - \gamma\{4-3(1+\gamma)\xi\}D_1 + \{2-3(1+\gamma)\xi\}D_2 + \xi D_3\Big)v^{\lambda,0}$$
$$= -\theta^2\Gamma(2-\theta(1+\gamma))\gamma_0(t) + \Gamma\left(3\theta^2(1+\gamma) - 2\theta(1+\theta)\right)(\xi-\theta)\gamma_0(t) + O\left(\lambda^{\frac{2}{3}}\right).$$

Next, our goal is to solve (3.12). We will use the same idea as before, and assume that $v^{\lambda,1}$ has the asymptotic expansion

$$v^{\lambda,1}(t,\xi) = \tilde\gamma_0(t) - \tilde\gamma_1(t)\lambda^{\frac{1}{3}} - \tilde\gamma_{30}(t)(\xi-\theta)^3 - \tilde\gamma_{31}(t)(\xi-\theta)^2\lambda^{\frac{1}{3}} - \tilde\gamma_{32}(t)(\xi-\theta)\lambda^{\frac{2}{3}} + O\big(\lambda^{\frac{2}{3}}\big). \tag{4.17}$$

As opposed to before, we only expand now to capture the $O\left(\lambda^{\frac{1}{3}}\right)$ terms, and similar to before, we want to capture these terms, whose first and second derivatives are also $O\left(\lambda^{\frac{1}{3}}\right)$, and hence the higher order terms in the expansion (4.17) above. Also, note that the coefficients of the terms $(\xi-\theta), (\xi-\theta)^2$, and $(\xi-\theta)\lambda^{\frac{1}{3}}$ are all zero, because from the boundary condition (3.15), we find that $\partial_\xi v^{\lambda,1}\mid_{(t,\ell_0(t))} = -\partial_\xi v^{\lambda,1}\mid_{(t,u_0(t))} = \lambda(1-\gamma)\tilde{\gamma}_0(t)$. Hence these terms were omitted from the expansion (4.17) above. Note however, that not all the terms can be omitted this way, specifically we obtain from these boundary conditions that

$$
\begin{aligned}
&3\tilde{\gamma}_{30}\ell_{1,0}^2 + 2\tilde{\gamma}_{31}\ell_{10} + \tilde{\gamma}_{32} = 0, \\
&3\tilde{\gamma}_{30}u_{1,0}^2 + 2\tilde{\gamma}_{31}u_{10} + \tilde{\gamma}_{32} = 0.
\end{aligned}
\tag{4.18}
$$

Using the fact that $\ell_{1,0} = -u_{1,0}$ we can conclude that $\tilde{\gamma}_{31} = 0$.

With this, we calculate that

$$
\begin{aligned}
\mathcal{L}_{\mathrm{NT}}(\bar{\sigma})v^{\lambda,1} - \left(\mathcal{L}_{\mathcal{U}}v^{\lambda,0}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}v^{\lambda,1} &= \dot{\tilde{\gamma}}_0(t) + (1-\gamma)A\tilde{\gamma}_0(t) - (1-\gamma)\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}}\tilde{\gamma}_0(t) \\
&- \lambda^{\frac{1}{3}}\left(\dot{\tilde{\gamma}}_1(t) + (1-\gamma)A\tilde{\gamma}_1(t) - (1-\gamma)\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}}\tilde{\gamma}_1(t) + 3\bar{\sigma}^2\theta^2(1-\theta)^2\tilde{\gamma}_{30}\tilde{\xi}\right) + O\left(\lambda^{\frac{2}{3}}\right).
\end{aligned}
$$

It follows from (3.12) at $O(1)$ that we have $\dot{\tilde{\gamma}}_0(t) - (1-\gamma)\left(\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}} - A\right)\tilde{\gamma}_0(t) + (1-\gamma)\bar{V}_3\theta^2\gamma(2-\theta(1+\gamma))\gamma_0(t) = 0$. Since by (4.4), the terminal condition $\tilde{\gamma}_0(T) = 0$, we find that

$$
\tilde{\gamma}_0(t) = \frac{2\bar{V}_3\theta^2(2-\theta(1+\gamma))}{\bar{\sigma}^2}\left(\frac{3(\theta-1)^2\theta^2}{2\gamma}\right)^{-2/3}\gamma_2(t).
$$

At $O\left(\lambda^{\frac{1}{3}}\right)$, equation (3.12) becomes

$$
\begin{aligned}
&\dot{\tilde{\gamma}}_1(t) + (1-\gamma)A\tilde{\gamma}_1(t) + (1-\gamma)\left((1-\gamma)\gamma_0(t)\right)^{-\frac{1}{\gamma}}\tilde{\gamma}_1(t) \\
&= -\left(3\bar{\sigma}^2\theta^2(1-\theta)^2\tilde{\gamma}_{30} + \bar{V}_3\Gamma\left(3\theta^2(1+\gamma) - 2\theta - 2\theta^2\right)\gamma_0(t)\right)\tilde{\xi},
\end{aligned}
\tag{4.19}
$$

where again, from (4.4), $\tilde{\gamma}_1(T) = 0$. Since $\tilde{\gamma}_1$ is independent of ξ, we must have

$$
\tilde{\gamma}_{30}(t) = -\frac{\bar{V}_3\theta\Gamma\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{3\bar{\sigma}^2\theta^2(1-\theta)^2}\gamma_0(t),
\tag{4.20}
$$

and thus the right hand side of (4.19) is zero. This allows us to conclude that $\tilde{\gamma}_1(t) = 0$. Finally, we conclude from (3.16) and (3.18) that $\ell_{0,1}, u_{0,1}$ are given by

$$
u_{0,1} = -\ell_{0,1} = \frac{6\tilde{\gamma}_{30}\ell_{1,0}(t)}{24\gamma_{44}(t)\ell_{1,0}(t)} = \frac{\bar{V}_3\theta\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{\bar{\sigma}^2}.
$$

Note that $\ell_{0,1}$ and $u_{0,1}$ are constants, independent of time. Finally, from (4.18), (4.14) and (4.20), we conclude that

$$
\tilde{\gamma}_{32}(t) = -3\tilde{\gamma}_{30}(t)\ell_{1,0}^2 = \left(\frac{9\gamma}{4\theta^2(1-\theta)^2}\right)^{1/3}\frac{\bar{V}_3\theta(1-\gamma)\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{\bar{\sigma}^2}\gamma_0(t).
$$

We summarize our findings of the expansions:

$$
\begin{aligned}
v^{\lambda,1}(t,\xi) = &\frac{2\bar{V}_3\theta^2(2-\theta(1+\gamma))}{\bar{\sigma}^2}\left(\frac{3(\theta-1)^2\theta^2}{2\gamma}\right)^{-2/3}\gamma_2(t)\left(1 - \frac{1-\gamma}{2\ell_{10}}(\xi-\theta)^2\lambda^{\frac{2}{3}}\right) \\
&+ \frac{\bar{V}_3\Gamma\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{3\bar{\sigma}^2\theta(1-\theta)^2}\gamma_0(t)\left((\xi-\theta)^3 - 3(\xi-\theta)\ell_{1,0}^2\right) + O(\lambda^{\frac{2}{3}}),
\end{aligned}
$$

and the first term in the $O\left(\sqrt{\varepsilon}\right)$ of the asymptotic expansion of the NT region is given by

$$
\ell_1(t) = -\frac{\bar{V}_3\theta\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{\bar{\sigma}^2} + O(\lambda^{\frac{1}{3}}), \quad u_1(t) = \frac{\bar{V}_3\theta\left(3\theta(1+\gamma) - 2 - 2\theta\right)}{\bar{\sigma}^2} + O(\lambda^{\frac{1}{3}}).
\tag{4.21}
$$

Remark 4.1. An extensive discussion of the implications of these formulas appear in the companion paper [4]. For example the key observation that as fast-scale mean reversion increases the NT region tightens. As explained there in equity markets it is typical to assume a negative correlation between the volatility factor and the stock price. Hence, if the stock price goes up, the instantaneous volatility tends to be lower, because of the negative correlation. This results in an increase of the the buy boundary, as risk aversion requires the position to mirror the Merton proportion more closely, as a result of the increase of the volatility of the stock. Similar logic shows the the sell boundary will also increase, as with higher stock price, the volatility of the stock decreases, and risk aversion allows bigger deviations from the Merton proportion. Together, this results in an upward shift of the NT region.

Remark 4.2. The corrections to the boundaries, ℓ_1 and u_1 in (4.21), are proportional to $\bar{V}_3 = \frac{1}{2}\rho\left\langle \beta f \phi' \right\rangle$ from (3.11), which in turn is proportional to stock-volatility correlation ρ. $\bar{V}_3$ also depends on an average of the model coefficient functions β, f and α (through ϕ). Given a fully specified parametric model of these functions, this average, and therefore $\bar{V}_3$, may be computed explicitly, for instance in the exponential Ornstein-Uhlenbeck model (see [8]). Moreover, it turns out that the same constant $\bar{V}_3$ appears in fast mean-reverting stochastic volatility approximations for option prices. Therefore it can be calibrated from options prices or implied volatilities on the same stock. This has been done, for example with S&P 500 options, in [9, Section 5.3.5].

5. Convergence Theorem and Proof. In the rest of the paper we will concentrate on proving convergence of the expansions of $\widehat{V}$ and $v^{\lambda,\varepsilon}$. To simplify notation, we may assume without loss of generality that the initial time $t = 0$. Our remaining assumptions are as follows.

Assumption 5.1. (i) Our assumptions on the stock's growth rate μ, its square-averaged volatility $\bar{\sigma}$, and the investor's risk aversion γ are as follows. For simplicity, we will assume that $0 < \gamma < 1$, though the proof can be generalized for the case $\gamma > 1$, and that $\theta = \theta(\bar{\sigma}) = \frac{\mu}{\gamma\bar{\sigma}^2}$ in (2.14) satisfies $0 < \theta < \min\left\{1, \frac{4}{3(1+\gamma)}\right\}$.

(ii) The utility discounting rate ν satisfies $\nu \geq \gamma + (1-\gamma)\left(r + \frac{\mu^2}{2\gamma\bar{\sigma}^2}\right)$. As a consequence, we have that A, defined in (2.14), satisfies $A = A(\bar{\sigma}) \leq -\left(\frac{\gamma}{1-\gamma}\right) < 0$.

(iii) We assume $\bar{V}_3 \leq 0$, where $\bar{V}_3$ was given in (3.11), and is statistic of the stochastic volatility model, depending on the volatility function f, the correlation ρ and the volatility of volatility β, all given in the model (2.1), and the solution ϕ to the Poisson equation (3.9).

(iv) We assume that Z is ergodic and has a unique invariant distribution with density.

(v) The process Z admits moments of any order uniformly bounded in $t \leq T$, and also any moments scaled by its volatility squared $\sup_{t\leq T}\mathbb{E}_0\left[|Z_t|^p\right] \leq C$, $\sup_{t\leq T}\mathbb{E}_0\left[|Z_t|^p\beta^2(Z_t)\right] \leq C$, where the constant C is allowed to depend on the power p. Note, that CIR and OU processes fit this and the previous assumption.

(vi) The volatility function $f \in C^\infty(\mathbb{R})$ is strictly positive, bounded, has a polynomial growth, and moreover, the solution to Poisson equation (3.9) has at most polynomial growth.

Remark 5.2. Overall the assumptions are fairly broad, so that stochastic volatility models such as in [7] satisfy the requirements. Additionally, while, we are not aware of a standard model, that fits all the assumptions above, a small modification to the the Jacobi Stochastic Volatility Model of [1] will satisfy all our conditions.

5.1. Construction of sub- and super solutions. The first step is to define the NT region. There, we will define

$$\tilde{v}^{\lambda,0,\pm}(t,\xi) := \gamma_0(t) - \gamma_2(t)\lambda^{\frac{2}{3}} - (1-\gamma)\left(\frac{9\gamma}{32(\theta-1)^2\theta^2}\right)^{\frac{1}{3}}\gamma_0(t)(\xi-\theta)^2\lambda^{\frac{2}{3}} + \frac{\Gamma}{12\theta^2(\theta-1)^2}\gamma_0(t)(\xi-\theta)^4 \pm M_0\lambda, \tag{5.1}$$

$$\tilde{v}^{\lambda,1,\pm}(t,\xi) := \tilde{\gamma}_0(t) - \tilde{\gamma}_2(t)\lambda^{\frac{2}{3}} - \tilde{\gamma}_{30}(t)(\xi-\theta)^3 - \tilde{\gamma}_{32}(t)(\xi-\theta)\lambda^{\frac{2}{3}} - \tilde{\gamma}_{42}(t)(\xi-\theta)^2\lambda^{\frac{2}{3}} \pm M_1\lambda^{\frac{1}{3}}, \tag{5.2}$$

where the constants M_0, M_1 will be specified later. The difficulty is that for now we only have a guess as to the NT region. It is expected to be close to the region $[0,T]\times[\theta + \ell_{1,0}\lambda^{\frac{1}{3}} + \sqrt{\varepsilon}\ell_{0,1}, \theta + u_{1,0}\lambda^{\frac{1}{3}} + \sqrt{\varepsilon}u_{0,1}]\times\mathbb{R}$.

However, this is not entirely accurate, as it is needed that the boundary conditions (2.15) and (2.17) will be satisfied at the boundaries. We temporarily, do not specify exactly where the definitions (5.1) and (5.2) hold. To define the NT we utilize the following lemma.

Lemma 5.3. *Set*

$$\hat{v}^{\pm}(t,\xi) = \left(\xi - \left(\theta + \ell_{1,0}\lambda^{\frac{1}{3}} + \sqrt{\varepsilon}\ell_{0,1}\right)\right)^2 B_1^{\pm}(t) + \left(\xi - \left(\theta + u_{1,0}\lambda^{\frac{1}{3}} + \sqrt{\varepsilon}u_{0,1}\right)\right)^2 B_2^{\pm}(t) \tag{5.3}$$
$$+ \left(\xi - \left(\theta + \sqrt{\varepsilon}\ell_{0,1}\right)\right)^2 B_3^{\pm}(t) + \left(\xi - \left(\theta + \sqrt{\varepsilon}u_{0,1}\right)\right)^2 B_4^{\pm}(t),$$

where $B_i^{\pm}(t),\ t \in [0,T]\ i = 1,2,3,4$ are functions. Then under the Assumptions 5.1, there exist $B_i^{\pm}(t),\ t \in [0,T]\ i = 1,2,3,4$ smooth and bounded and $\tilde{\ell}$ and $\tilde{u}$ satisfying

$$\tilde{\ell}(t) = \theta + \ell_{1,0}\lambda^{\frac{1}{3}} + \ell_{0,1}\sqrt{\varepsilon} + \tilde{\ell}_{1,0}(t)\lambda^{\frac{2}{3}} + \tilde{\ell}_{0,1}(t)\sqrt{\varepsilon}\lambda^{\frac{1}{3}}, \tag{5.4}$$
$$\tilde{u}(t) = \theta + u_{1,0}\lambda^{\frac{1}{3}} + u_{0,1}\sqrt{\varepsilon} + \tilde{u}_{1,0}(t)\lambda^{\frac{2}{3}} + \tilde{u}_{0,1}(t)\sqrt{\varepsilon}\lambda^{\frac{1}{3}}, \tag{5.5}$$

where the non-zero functions $\tilde{\ell}_{1,0}, \tilde{u}_{1,0}$, and $\tilde{\ell}_{0,1}, \tilde{u}_{0,1}$ satisfy

$$\mathcal{B}\left(\tilde{v}^{\lambda,0,\pm} + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,\pm} + \varepsilon\hat{v}^{\pm}\right)(t,\tilde{\ell}) = 0, \tag{5.6}$$
$$\mathcal{S}\left(\tilde{v}^{\lambda,0,\pm} + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,\pm} + \varepsilon\hat{v}^{\pm}\right)(t,\tilde{u}) = 0.$$

Using the results of the Lemma 5.3, we define the NT region that we will use throughout the rest of the paper, which is a good approximation to the true NT region

$$\widetilde{\mathrm{NT}} := \left\{(t,\xi,z) | t \in [0,T], z \in \mathbb{R}, \xi \in (\tilde{\ell}(t), \tilde{u}(t))\right\}. \tag{5.7}$$

Additionally, we define the (approximate) buy and sell regions as

$$\widetilde{\mathrm{B}} := \left\{(t,\xi,z) | t \in [0,T], z \in \mathbb{R}, -\frac{1}{\lambda} < \xi < \tilde{\ell}\right\},$$
$$\widetilde{\mathrm{S}} := \left\{(t,\xi,z) | t \in [0,T], z \in \mathbb{R}, \frac{1}{\lambda} > \xi > \tilde{u}\right\},$$

where for convenience we omit the t dependency of $\tilde{\ell}, \tilde{u}$. Next, we define the approximations to the functions $v^{\lambda,i},\ i = 0,1$ as given in (5.1) and (5.2), but only inside the closure of the $\widetilde{\mathrm{NT}}$. Additionally, outside the closure of $\widetilde{\mathrm{NT}}$ region, we define for $i = 0,1$:

$$\tilde{v}^{\lambda,i,\pm}(t,\xi) := \begin{cases} \left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma} \tilde{v}^{\lambda,i,\pm}(t,\tilde{\ell}) & -\frac{1}{\lambda} \le \xi < \tilde{\ell} \\ \left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma} \tilde{v}^{\lambda,i,\pm}(t,\tilde{u}) & \frac{1}{\lambda} \ge \xi > \tilde{u} \end{cases}, \tag{5.8}$$

with similar definition for $\hat{v}^{\pm}$

$$\hat{v}^{\pm}(t,\xi) := \begin{cases} \left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma} \hat{v}^{\pm}(t,\tilde{\ell}) & -\frac{1}{\lambda} \le \xi < \tilde{\ell} \\ \left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma} \hat{v}^{\pm}(t,\tilde{u}) & \frac{1}{\lambda} \ge \xi > \tilde{u} \end{cases}. \tag{5.9}$$

5.2. Formulation of the Main Theorem. We are now able to formulate the main theorem of this paper. Its goal is to find the first terms in the expansion of the value function (3.1) and of the optimal strategies (3.2). Recall the definition of the value function $\widehat{V}$ in (2.5) and the change of variables, which defined $v^{\lambda,\varepsilon}$ in (2.9).

Theorem 5.4. *Let $\tilde{v}^{\lambda,i}$ be the approximations of the functions $v^{\lambda,i},\ i = 0,1$ as were computed in Sections 4.1 and 4.2, and defined in (5.1) , (5.2) and (5.8). Fix a compact $\mathcal{K}_0 \subset \mathcal{S}$, where $\mathcal{S}$ is the solvency region* *defined in* (2.2)*. Then for* $(x, y) \in \mathcal{K}_0$*,* $z \in \mathbb{R}$ *and* $t \in [0, T)$*, and setting* $\xi = \frac{y}{x+y}$*, we have that for* $\varepsilon, \lambda > 0$ *small enough, and under Assumption 5.1,*

$$(5.10) \qquad \left| v^{\lambda,\varepsilon}(t, \xi, z) - \tilde{v}^{\lambda,0,\pm}(t, \xi) + \sqrt{\varepsilon}\, \tilde{v}^{\lambda,1,\pm}(t, \xi) \right| = O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{1}{3}}\right) + O(\varepsilon).$$

Moreover, the strategy given by $\tilde{\ell}$ *and* $\tilde{u}$ *of the approximate NT region* $\widetilde{NT}$ *defined in* (5.7) *is* nearly optimal*, that is, if followed, the error would be of order* $O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{1}{3}}\right) + O(\varepsilon)$*.*

Before we continue, we need to formulate a couple of helpful remarks and auxiliary lemmas, that would be used throughout the proof.

5.3. Intermediate Calculations and Proof of Lemma 5.3. First, we define

$$(5.11) \qquad \tilde{v}^{\lambda,2,\pm}(t, \xi, z) := -\frac{\xi^2}{2}\, \phi(z) \left(D_2 - \Gamma I - 2\gamma D_1\right) \tilde{v}^{\lambda,0,\pm}(t, \xi),$$

where we recall that ϕ is given by (3.9). Note, that as opposed the definitions (5.1) – (5.8), the definition (5.11) is valid in the entire solvency region $(t, \xi, z) \in [0, T] \times [-\lambda^{-1}, \lambda^{-1}] \times \mathbb{R}$. Thus, inside the closure of $\widetilde{\mathrm{NT}}$ region, we define

$$(5.12) \qquad w^{1,\pm}(t, \xi) = \tilde{v}^{\lambda,0,\pm}(t, \xi) + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,\pm}(t, \xi) + \varepsilon\hat{v}^{\pm}(t, \xi),$$

$$(5.13) \qquad w^{2,\pm}(t, \xi, z) = \varepsilon\tilde{v}^{\lambda,2,\pm}(t, \xi, z) \pm M_3\varepsilon^{\frac{1+q}{2}},$$

$$(5.14) \qquad w^{\pm}(t, \xi, z) = w^{1,\pm}(t, \xi) + w^{2,\pm}(t, \xi, z).$$

for any fixed $0 < q < \frac{1}{2}$, and where the constant M_3 remains to be chosen.

Remark 5.5. It can be easily shown in the case $0 < \gamma < 1$ that $v^{\lambda,\varepsilon}\left(t, \frac{y}{x+y}, z\right) = 0$ if and only if $(x, y) \in \partial\mathcal{S}$ and strictly positive if $(x, y) \in \mathcal{S}$. Moreover, similar to [10] it can be shown that if $(x, y) \in \partial\mathcal{S}$, then the only admissible strategy is to liquidate the position, and stop all consumption and trading. Hence, for the rest of this paper, we may assume that $(x, y) \in \mathcal{S}$. Furthermore, since one strategy is to liquidate all the cash, and consume the accrued interest, then similar to [10], it follows that $v(t, x, y, z) > 0$. Moreover, we may also assume that all admissible strategies produce a strictly positive expected utility. We will also write $C_t = (X_t + Y_t)c_t$, where c_t is consumption written as proportion of wealth. Without loss of generality we may assume that c_t is bounded. It follows that there exists a constant $C > 0$ such that

$$\begin{aligned}
&(1-\gamma)\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\,\mathcal{U}(C_t)dt + \mathrm{e}^{-\nu T}\,\mathcal{U}\left(X_T + Y_T - \lambda\left|Y_T\right|\right)\right] \\
&\leq C\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t + Y_t)^{1-\gamma}dt + \mathrm{e}^{-\nu T}\left(X_T + Y_T - \lambda\left|Y_T\right|\right)^{1-\gamma}\right].
\end{aligned}$$

From this, together with the fact that $v^{\lambda,\varepsilon} > 0$ in $[0, T] \times \mathcal{S} \times \mathbb{R}$, it follows that

$$(5.15) \qquad \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t + Y_t)^{1-\gamma}dt + \mathrm{e}^{-\nu T}\left(X_T + Y_T - \lambda\left|Y_T\right|\right)^{1-\gamma}\right] > 0.$$

Remark 5.6. Note that the γ and $\tilde{\gamma}$ functions defined above are all bounded, as functions of $0 \leq t \leq T$.

Remark 5.7. With the definitions (5.12)–(5.14), for future reference we remark that inside the buy region $\widetilde{\mathrm{B}}$ we have that

$$w_t^{1,\pm}(t, \xi, z) = \left(\frac{1 + \lambda\xi}{1 + \lambda\tilde{\ell}}\right)^{1-\gamma}\left(w_t^{1,\pm}(t, \tilde{\ell}(t), z) + \frac{\mathcal{B}w_t^{1,\pm}(t, \tilde{\ell}(t), z)}{1 + \lambda\tilde{\ell}}\partial_t\tilde{\ell}(t)\right) = \left(\frac{1 + \lambda\xi}{1 + \lambda\tilde{\ell}}\right)^{1-\gamma} w_t^{1,\pm}(t, \tilde{\ell}(t), z).$$

A similar identity holds in the sell region $\widetilde{\mathrm{S}}$. Moreover, because of Lemma 5.3, $w^{1,\pm}$ is continuously differentiable across the boundaries of the $\widetilde{\mathrm{NT}}$ region, so it is $C^{1,1}\left((0,T)\times\mathcal{S}_\xi\right)$, where $\mathcal{S}_\xi$ was given in (2.10). It follows that

$$w^{1,\pm}_\xi(t,\xi)=\begin{cases}\frac{\lambda(1-\gamma)}{1+\lambda\xi}w^{1,\pm}(t,\tilde{\ell},z) & -\frac{1}{\lambda}\le\xi<\tilde{\ell}\\ \frac{\lambda(1-\gamma)}{1-\lambda\xi}w^{1,\pm}(t,\tilde{\ell},z) & \frac{1}{\lambda}\ge\xi>\tilde{u}\end{cases}. \tag{5.16}$$

However, $w^{1,\pm}$ is not twice continuously differentiable across the boundary. Using the fact that $\tilde{\ell}$ satisfies (5.4), and using the definitions (5.1), (5.2), (5.8), (5.12) and (5.16) to evaluate $w^{1,\pm}_{\xi\xi}$ in $\widetilde{\mathrm{NT}}$ and $\widetilde{\mathrm{B}}$ regions respectively, allows us to calculate the limits of the second derivative on the boundary, from both sides of the boundary. This technical computation reveals that

$$\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}} w^{1,\pm}_{\xi\xi}(t,\xi)=\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}} w^{1,\pm}_{\xi\xi}(t,\xi)+O(\lambda)+O\left(\sqrt{\varepsilon}\lambda^{\frac{2}{3}}\right)+O(\varepsilon), \tag{5.17}$$

and with a similar equality holding across the boundary of the sell region $\widetilde{\mathrm{S}}$.

Remark 5.8. Note that $\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}+\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)-\left(\left\langle\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)\right\rangle\right)=0$. Hence,

$$\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}+\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)=\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right). \tag{5.18}$$

Proof of Lemma 5.3. Consider the buy boundary $\tilde{\ell}$. We calculate next

$$\begin{aligned}&\mathcal{B}\left(\tilde{v}^{\lambda,0,+}+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}+\varepsilon\hat{v}^{+}\right)(t,\theta+\ell_{1,0}\lambda^{\frac{1}{3}}+\ell_{0,1}\sqrt{\varepsilon}+\tilde{\ell}_{1,0}\lambda^{\frac{2}{3}}+\tilde{\ell}_{0,1}\sqrt{\varepsilon}\lambda^{\frac{1}{3}})\\
&=(1-\gamma)\left(\frac{\gamma\ell_{1,0}\tilde{\ell}_{1,0}^2\gamma_0(t)}{(\theta-1)^2\theta^2}+\gamma_2(t)\right)\lambda^{\frac{5}{4}}+O(\lambda^2)-M_1(1-\gamma)\sqrt{\varepsilon}\lambda^{\frac{4}{3}}\\
&+\tilde{\ell}_{1,0}\left(\frac{(1-\gamma)\gamma\gamma_0(t)(\tilde{\ell}_{1,0}\ell_{0,1}+2\ell_{1,0}\tilde{\ell}_{0,1})}{(\theta-1)^2\theta^2}-6\ell_{1,0}(2\ell_{1,0}\tilde{\gamma}_{40}(t)+\tilde{\gamma}_{41}(t))+2\tilde{\gamma}_{42}(t)-3\tilde{\ell}_{1,0}\tilde{\gamma}_{30}(t)\right)\sqrt{\varepsilon}\lambda^{\frac{4}{3}}\\
&+O\left(\sqrt{\varepsilon}\lambda^{\frac{5}{3}}\right)+\left(2\ell_{1,0}(2B_2^++B_4^+-3\ell_{0,1}\tilde{\gamma}_{30}(t))-\frac{(\gamma-1)\gamma\ell_{0,1}^2\ell_{1,0}\gamma_0(t)}{(\theta-1)^2\theta^2}\right)\varepsilon\lambda^{\frac{1}{3}}+O\left(\varepsilon\lambda^{\frac{2}{3}}\right)\\
&+\left(4\ell_{0,1}(B_2^++B_4^+)-\frac{(\gamma-1)\gamma\ell_{0,1}^3\gamma_0(t)}{3(\theta-1)^2\theta^2}-3\ell_{0,1}^2\tilde{\gamma}_{30}(t)\right)\varepsilon^{\frac{3}{2}}+o\left(\varepsilon^{\frac{3}{2}}\right).\end{aligned}$$

From the fact that $\ell_{0,1}<0$, we conclude that there exists $\tilde{\ell}_{1,0}=\sqrt{-\frac{\gamma_2(t)(\theta-1)^2\theta^2}{\gamma\ell_{1,0}\gamma_0(t)}}+o(1)\neq 0$ such that the first line on the right hand side of the above is zero. Similarly, there exists $\tilde{\ell}_{0,1}$ such that the $O\left(\sqrt{\varepsilon}\right)$ term is zero. Finally, there also exist smooth and bounded functions $B_2^+(t), B_4^+(t)$, that make the other terms zero. It follows as desired that (5.6) holds. Similar calculations can be made for $\tilde{v}^{\lambda,0,-}+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,-}+\varepsilon\hat{v}^{-}$ and for the sell boundary $\tilde{u}$. ∎

5.4. Auxiliary Lemmas.

LEMMA 5.9. *Assume $J_1\colon\mathbb{R}\to\mathbb{R}$ and $J_2\colon\mathcal{S}\to\mathbb{R}$ be two real functions, such that $J_1(z)$ is at most of a polynomial growth, and $J_2(x,y)$ is bounded then there exists $C>0$ such that*

$$\sup_{t\le T}\sup_{\varepsilon<1}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}J_1(Z_t)J_2(X_t,Y_t)\right]\le C.$$

Proof. The proof follows from Hölder's inequality. Let $p>1$ be such that $p(1-\gamma)<1$, and set $q=\frac{p}{p-1}>1$, we get that $\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}J_1(Z_t)J_2(X_t,Y_t)\right]\le C\left(\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{p(1-\gamma)}\right]\right)^{\frac{1}{p}}\left(\mathbb{E}_0^{x,y,z}\left[|J_1(Z_t)|\right]\right)^{\frac{1}{q}}$. The expectation, $\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{p(1-\gamma)}\right]$ is uniformly bounded for all $t\le T$ and all $\varepsilon<1$, by the solution to the Merton problem with zero transaction costs. The second expectation is uniformly bounded, because of the Assumption 5.1 on the finiteness of the moments of Z.

LEMMA 5.10. *In addition to the assumptions of Lemma 5.9, assume also that $\langle J_1 \rangle = 0$. Then we have for any admissible strategy and for any $0 < t \leq T$ that for any $0 < q < 1$*

$$\mathbb{E}_0^{x,y,z}\left[J_1(Z_t)J_2(X_t,Y_t)\right] = O\left(\varepsilon^{\frac{q}{2}}\right), \tag{5.19}$$

$$\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}J_1(Z_t)J_2(X_t,Y_t)\right] = O\left(\varepsilon^{\frac{q}{2}}\right). \tag{5.20}$$

Proof. The first equation (5.19) follows from Lemma A.4 of [7], while the second equation (5.20) requires first an application of Hölder's inequality, before applying the same lemma. ∎

LEMMA 5.11. *For any $0 < q < \frac{1}{2}$, and under the assumptions of Lemma 5.10 and assuming that J_1, J_2 are smooth and $\langle J_1 \rangle = 0$, we have that for any admissible strategy (L, M, C), and for any $0 < t \leq T$ it holds that*

$$\varepsilon^{q/2}\mathbb{E}_0^{x,y,z}\left[\int_0^T (X_t+Y_t)^{1-\gamma}J_1(Z_t)J_2(X_t,Y_t)dL_t\right] \leq C,$$

$$\varepsilon^{q/2}\mathbb{E}_0^{x,y,z}\left[\int_0^T (X_t+Y_t)^{1-\gamma}J_1(Z_t)J_2(X_t,Y_t)dM_t\right] \leq C.$$

Proof. First, assume without loss of generality that $J_2 = 1$, then we have

$$\begin{aligned}
&\varepsilon^{q/2}\mathbb{E}_0^{x,y,z}\left[\int_0^T (X_t+Y_t)^{1-\gamma}J_1(Z_t)dL_t\right] \\
&= \varepsilon^{q/2}\mathbb{E}_0^{x,y,z}\left[\sum_{i=0}^{1/\varepsilon^q-1}\left((X_{iT\varepsilon^q}+Y_{iT\varepsilon^q})^{1-\gamma}+O\left(\varepsilon^{q/2}\right)\right)\mathbb{E}\left[\int_{iT\varepsilon^q}^{(i+1)T\varepsilon^q} J_1(Z_t)dL_t\middle|\mathcal{F}_{iT\varepsilon^q}\right]\right],
\end{aligned}$$

which follows form linear growth coefficients of X and Y, and the uniform finiteness of moments of all orders of Z, as follows from Assumption 5.1. Hence, it is sufficient to show that

$$\mathbb{E}\left[\int_{iT\varepsilon^q}^{(i+1)T\varepsilon^q} J_1(Z_t)dL_t\middle|\mathcal{F}_{iT\varepsilon^q}\right] = O\left(\varepsilon^{q/2}\right). \tag{5.21}$$

Integration by parts gives

$$\begin{aligned}
&\mathbb{E}\left[\int_{iT\varepsilon^q}^{(i+1)T\varepsilon^q} J_1(Z_t)dL_t\middle|\mathcal{F}_{iT\varepsilon^q}\right] \\
&= \mathbb{E}\left[J_1(Z_{(i+1)T\varepsilon^q})L_{(i+1)T\varepsilon^q}\middle|\mathcal{F}_{iT\varepsilon^q}\right] - J_1(Z_{iT\varepsilon^q})L_{iT\varepsilon^q} - \mathbb{E}\left[\int_{iT\varepsilon^q}^{(i+1)T\varepsilon^q} L_t d\left(J_1(Z_t)\right)\middle|\mathcal{F}_{iT\varepsilon^q}\right] \\
&= \left(L_{iT\varepsilon^q}+O\left(\varepsilon^{q/2}\right)\right)\mathbb{E}\left[J_1(Z_{(i+1)T\varepsilon^q})\middle|\mathcal{F}_{iT\varepsilon^q}\right] - J_1(Z_{iT\varepsilon^q})L_{iT\varepsilon^q} \\
&\quad - \left(L_{iT\varepsilon^q}+O\left(\varepsilon^{q/2}\right)\right)\mathbb{E}\left[\int_{iT\varepsilon^q}^{(i+1)T\varepsilon^q} d\left(J_1(Z_t)\right)\middle|\mathcal{F}_{iT\varepsilon^q}\right]
\end{aligned}$$

The last term is zero, since Z has an invariant distribution, whereas from Lemma A.5 of [7] it follows that $E\left[J_1(Z_{(i+1)T\varepsilon^q}) - J_1(Z_{iT\varepsilon^q})\middle|\mathcal{F}_{iT\varepsilon^q}\right] = O\left(\sqrt{\varepsilon}\right)$. Hence, (5.21) follows from existence of moments of Z. ∎

5.5. Proof of Theorem 5.4 .

5.5.1. Step 1: Terminal time condition. For sufficiently big constants M_i, and sufficiently small $\lambda, \varepsilon > 0$ we have the following inequalities at the terminal time:

$$\pm\mathbb{E}_0^{x,y,z}\left[(X_T+Y_T)^{1-\gamma}w^{\pm}\left(T, \frac{X_T}{X_T+Y_T}, Z_T\right)\right] \geq \pm\mathbb{E}_0^{x,y,z}\left[\mathcal{U}\left(X_T+Y_T-\lambda\left|Y_T\right|\right)\right]. \tag{5.22}$$

We show (5.22) going over different orders of ε. First, we assert that for M_0 and M_3 sufficiently big, we have that for any $0<q<1$

$$\tilde{v}^{\lambda,0,+}(T,\xi)+\tilde{v}^{\lambda,1,+}(T,\xi)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\geq\mathcal{U}(1-\lambda\,|\xi|). \tag{5.23}$$

Case I: $\tilde{\ell}\leq\xi\leq\tilde{u}$. Using the definition (5.1), and recalling that $\gamma_0(T)=\frac{1}{1-\gamma}$ and $\gamma_2(T)=0$, we calculate in this case that for M_0, M_3 big enough

$$\begin{aligned}
&\tilde{v}^{\lambda,0,+}(T,\xi)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\xi)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}=\frac{1}{1-\gamma}+M_0\lambda+O\left(\lambda^{\frac{4}{3}}\right)+O(\varepsilon)+O\left(\sqrt{\varepsilon}\lambda\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad\geq\frac{(1-\lambda\xi)^{1-\gamma}}{1-\gamma}=\mathcal{U}(1-\lambda\xi).
\end{aligned}$$

Case II: $\tilde{u}<\xi\leq\frac{1}{\lambda}$. It follows from the definition (5.14) and *Case I* that

$$\begin{aligned}
&\tilde{v}^{\lambda,0,+}(T,\xi)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\xi)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}=\left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(T,\tilde{u})+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\tilde{u})\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad\geq\left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(T,\tilde{u})+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\tilde{u})+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\right)\\
&\quad\geq\left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma}\mathcal{U}(1-\lambda\tilde{u})=\mathcal{U}(1-\lambda\xi),
\end{aligned}$$

where we have also used the fact that $0<\left(\frac{1-\lambda\xi}{1-\lambda\tilde{u}}\right)^{1-\gamma}<1$.

Case III: $0\leq\xi<\tilde{\ell}$.

$$\begin{aligned}
&\tilde{v}^{\lambda,0,+}(T,\xi)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\xi)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}=\left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(T,\tilde{\ell})+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\tilde{\ell})\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad=\left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma}\left(\frac{1}{1-\gamma}+M_0\lambda+O\left(\lambda^{\frac{4}{3}}\right)+O\left(\sqrt{\varepsilon}\lambda\right)+O(\varepsilon)\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad=\left(1-\lambda(1-\gamma)(\tilde{\ell}-\xi)+O(\lambda^2)\right)\left(\frac{1}{1-\gamma}+M_0\lambda+O\left(\lambda^{\frac{4}{3}}\right)\right)+O\left(\sqrt{\varepsilon}\lambda\right)+O(\varepsilon)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad\geq\frac{1}{1-\gamma}\geq\frac{(1-\lambda\xi)^{1-\gamma}}{1-\gamma}=\mathcal{U}(1-\lambda\xi).
\end{aligned}$$

where for the first inequality M_0 and M_3 need to be sufficiently large.

Case IV: $-\frac{1}{\lambda}<\xi<0$.

$$\begin{aligned}
&\tilde{v}^{\lambda,0,+}(T,\xi)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,\xi)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}=(1+\lambda\xi)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(T,0)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,0)\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\\
&\quad\geq(1+\lambda\xi)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(T,0)+\sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(T,0)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\right)\geq(1+\lambda\xi)^{1-\gamma}\mathcal{U}(1)=\mathcal{U}(1+\lambda\xi),
\end{aligned}$$

where the second inequality follows from application of case III.

To see that

$$\mathbb{E}_0^{x,y,z}\left[\varepsilon\tilde{v}^{\lambda,2,+}\left(T,\frac{Y_T}{X_T+Y_T},Z_T\right)\right]=O\left(\varepsilon^{1+\frac{q}{2}}\right), \tag{5.24}$$

recall definition (5.11), together with the facts that $\langle\phi\rangle=0$ and $\frac{\xi^2}{2}\left(D_2-\Gamma I-2\gamma D_1\right)\tilde{v}^{\lambda,0,\pm}(t,\xi)$ is bounded for any admissible strategy. Setting J_1 and J_2 to be the these functions respectively, equation (5.24) follows now by utilizing Lemma 5.10. Similarly, recalling that by definition of $\hat{v}$ in (5.3) and (5.9) and employing Lemma 5.9 with $J_1=1$ and $J_2=\hat{v}^+$, we conclude that

$$\mathbb{E}_0^{x,y,z}\left[(X_T+Y_T)^{1-\gamma}\varepsilon\hat{v}^+\left(T,\frac{Y_T}{X_T+Y_T}\right)\right]=O\left(\varepsilon\right).$$

It follows that for M_3 sufficiently large,

$$\mathbb{E}_0^{x,y,z}\left[\varepsilon\tilde{v}^{\lambda,2,+}\left(T,\frac{Y_T}{X_T+Y_T},Z_T\right)+\frac{M_3}{2}\varepsilon^{\frac{1+q}{2}}\right]\geq\frac{M_3}{4}\varepsilon^{\frac{1+q}{2}}>0. \tag{5.25}$$

Adding up (5.23) and (5.25) we can even strengthen the desired result (5.22), to be

$$\begin{aligned}\mathbb{E}_0^{x,y,z}&\left[(X_T+Y_T)^{1-\gamma}w^+\left(T,\frac{Y_T}{X_T+Y_T},Z_T\right)\right]\\ &\geq\mathbb{E}_0^{x,y,z}\left[\mathcal{U}\left(X_T+Y_T-\lambda\left|Y_T\right|\right)+\left(X_T+Y_T-\lambda\left|Y_T\right|\right)^{1-\gamma}\frac{M_3}{4}\varepsilon^{\frac{1+q}{2}}\right].\end{aligned} \tag{5.26}$$

The proof for w^- follows the same outline.

5.5.2. Step 2: Inside the $\widetilde{\mathrm{NT}}$ region. By construction of Section 4.1 of the $O(\varepsilon^{-1})$ term , it follows that

$$\pm\left\langle\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_\xi^{\lambda,0,\pm}\right)\right)\right\rangle=O(\lambda)+FM_0\lambda+O(\varepsilon)$$

where the left hand side is evaluated at $(t,\xi,z)\in\widetilde{\mathrm{NT}}$, and where the function F is defined by

$$F:=(1-\gamma)A-(1-\gamma)\Big((1-\gamma)\gamma_0(t)\Big)^{-\frac{1}{\gamma}}.$$

By Assumption 5.1.i, and using the fact that $\gamma_0(t)>0$, it follows that $F<0$. Then

$$\begin{aligned}&\pm\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_\xi^{\lambda,0,\pm}\right)+\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}\right)\\ &=\pm\left\langle\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_\xi^{\lambda,0,\pm}\right)\right)\right\rangle=FM_0\lambda+O(\lambda)+O(\varepsilon)<O(\varepsilon).\end{aligned} \tag{5.27}$$

For the last inequality Assumption 5.1.ii was used, and $M_0>0$ is taken sufficiently large, to satisfy the last inequality, as $F<0$. For the next term of $O\left(\sqrt{\varepsilon}\right)$ we have that for M_1 sufficiently large,

$$\pm\left\langle\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}\right)+\mathcal{L}_1\tilde{v}^{\lambda,2}\right\rangle=FM_1\lambda^{\frac{1}{3}}+O\left(\lambda^{\frac{2}{3}}\right)+O(\sqrt{\varepsilon})<O(\sqrt{\varepsilon}),$$

where again the fact that $F<0$ was used to conclude the last inequality. This, together with (5.27) we get

$$\begin{aligned}&\pm\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)+\sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\right)\right)\\ &=\pm\left(\mathcal{L}_{\mathrm{NT}}w^{1,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{1,\pm}\right)+\sqrt{\varepsilon}\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\right)+O(\varepsilon).\end{aligned}$$

It follows that for any $C>0$ and for $\varepsilon>0$ sufficiently small, for any $0<q<1$, we have that

$$\pm\left(\mathcal{L}_{\mathrm{NT}}w^{1,\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{1,\pm}\right)+\sqrt{\varepsilon}\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\right)\leq C\varepsilon^{\frac{1+q}{2}}.$$

The proof that $\mathcal{B}\tilde{v}^{\lambda,0,+}\leq O(\varepsilon)$ inside the $\widetilde{\mathrm{NT}}$ follows from a similar calculation in [3]. Moreover, inside the $\widetilde{\mathrm{NT}}$ region we have that

$$\mathcal{B}\tilde{v}^{\lambda,1,+}\leq O(\sqrt{\varepsilon}). \tag{5.28}$$

Indeed, consider the buy boundary $(t,\tilde{\ell})$ where (5.28) is satisfied as shown in Lemma 5.3. Furthermore,

$$\begin{aligned}\mathcal{B}\tilde{v}^{\lambda,1,+}(t,\xi)&=-(\xi-\theta)^2(3\tilde{\gamma}_{30}(t)+4\tilde{\gamma}_{40}(t)(\xi-\theta))-3\tilde{\gamma}_{41}(t)(\xi-\theta)^2\lambda^{\frac{1}{3}}\\ &\quad-\left(2\tilde{\gamma}_{42}(t)(\xi-\theta)+\tilde{\gamma}_{32}(t)\right)\lambda^{\frac{2}{3}}-(1-\gamma)\tilde{\gamma}_0(t)\lambda-\tilde{\gamma}_{43}(t)\lambda+O\left(\lambda^{\frac{4}{3}}\right)+O(\sqrt{\varepsilon})\\ &=-3\tilde{\gamma}_{30}(t)(\xi-\theta)^2\tilde{\gamma}_{32}(t)-\tilde{\gamma}_{32}(t)\lambda^{\frac{2}{3}}+O(\lambda)+O(\sqrt{\varepsilon}).\end{aligned}$$

Specifically, ignoring the $O(\sqrt{\varepsilon})$ terms, $\mathcal{B}\tilde{v}^{\lambda,1,+}(t,\xi)$ is a negative quadratic function for $|\xi-\theta| = o\left(\lambda^{\frac{1}{3}}\right) + O(\sqrt{\varepsilon})$, and its derivative $\mathcal{B}'$ at the buy boundary $\mathcal{B}'\tilde{v}^{\lambda,1,+}(t,\xi) = -6(\xi-\theta)\tilde{\gamma}_{30}(t) + O\left(\lambda^{\frac{2}{3}}\right)$, is negative for $\xi < \theta$ and $|\xi-\theta| = O\left(\lambda^{\frac{1}{3}}\right) + O(\sqrt{\varepsilon})$. Similar conclusions hold true for the sell boundary $(t,\tilde{u})$. Together, it follows that (5.28) holds true everywhere inside the $\widetilde{\mathrm{NT}}$ region.

Similarly it follows that $\mathcal{S}\left(\tilde{v}^{\lambda,0,+} + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}\right) \leq O(\sqrt{\varepsilon})$.

5.5.3. Step 3a: Inside the $\widetilde{\mathrm{B}}$ region. From the boundary conditions in Section 3.2, inside the $\widetilde{\mathrm{B}}$ region we have that $\mathcal{B}\left(\left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(t,\xi,z) + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(t,\xi,z)\right)\right) = O(\varepsilon)$. It easily follows that $\mathcal{S}\left(\left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{1-\gamma}\left(\tilde{v}^{\lambda,0,+}(t,\xi,z) + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}(t,\xi,z)\right)\right) \leq O(\varepsilon)$. Next, we will show that there exists C such that

$$\begin{aligned}(5.29)\qquad &\mathcal{L}_0\tilde{v}^{\lambda,2,+} + \mathcal{L}_2\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)\\ &+ \sqrt{\varepsilon}\left(\langle\mathcal{L}_2\rangle\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right) \leq C\varepsilon^{\frac{1+q}{2}}.\end{aligned}$$

First, note that by from (5.18) of Remark 5.8 this is equivalent to showing that inside the $\widetilde{\mathrm{B}}$ region

$$(5.30)\quad \mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right) + \sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right) \leq C\varepsilon^{\frac{1+q}{2}}.$$

The goal is to show that both terms $\sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)$ and $\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)$ in (5.30) above are dominated by $O(\varepsilon^{\frac{1+q}{2}})$. The proof that the term $\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)$ is dominated by $O(\varepsilon^{\frac{1+q}{2}})$ follows the logic of [3], more specifically,

$$\left(\frac{1+\lambda\xi}{1+\lambda\tilde{\ell}}\right)^{\gamma-1}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right) + FM_0\lambda + O(\lambda)\right)(t,\xi,z) \leq O(\varepsilon^{\frac{1+q}{2}}).$$

We now show that $\sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle + O(\lambda)\right) \leq O\left(\varepsilon^{\frac{1+q}{2}}\right)$. We do this in two steps. First, we show that it is dominated by $O(\varepsilon^{\frac{1+q}{2}})$ on the boundary of the buy region $\partial\widetilde{\mathrm{B}} \cap \partial\widetilde{\mathrm{NT}}$. Fix $(t,\tilde{\ell},z)$ there. Using the fact that $w^{1,+}$ is continuously differentiable across the boundary as shown in Remark 5.7 and using (5.17) it follows that

$$\begin{aligned}&\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}}\left(\mathcal{L}_{\mathrm{NT}}w^{1,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{1,+}\right)\right)(t,\xi,z)\\ &= \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}}\left(\mathcal{L}_{\mathrm{NT}}w^{1,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{1,+}\right)\right)(t,\xi,z) + O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{2}{3}}\right) + O(\varepsilon).\end{aligned}$$

Moreover, since $\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}}\sqrt{\varepsilon}\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle(t,\xi,z) = \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}}\sqrt{\varepsilon}\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle(t,\xi,z) + O\left(\sqrt{\varepsilon}\lambda^{\frac{1}{3}}\right)$, it follows that

$$\begin{aligned}&\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)\right)(t,\xi,z)\\ &\quad + \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}}\left(\sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)\right)(t,\xi,z)\\ &= \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{B}}}\left(\mathcal{L}_{\mathrm{NT}}w^{1,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{1,+}\right) + \sqrt{\varepsilon}\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z) + O(\varepsilon)\\ &= \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)\right)(t,\xi,z) + O(\varepsilon) + O(\sqrt{\varepsilon}\lambda^{\frac{1}{3}}) + O(\lambda)\\ &\quad + \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}}\sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z).\end{aligned}$$

From Subsection 5.5.2, inside the $\widetilde{\mathrm{NT}}$ region, it follows that by possibly increasing $M_i,\ i = 1, 2, 3$ we have for $\lambda > 0$ small enough that

$$\lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}} \sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z)$$
$$+ \lim_{\xi\to\tilde{\ell},\ \xi\in\widetilde{\mathrm{NT}}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,0,+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)\right)(t,\xi,z) \le -\nu\frac{M_1}{2}\sqrt{\varepsilon}\lambda^{\frac{1}{3}} + O(\sqrt{\varepsilon}\lambda^{\frac{1}{3}}) + O(\varepsilon) \le C\varepsilon^{\frac{1+q}{2}}. \tag{5.31}$$

Finally, in the rest of $\widetilde{\mathrm{B}}$ region we calculate:

$$\sqrt{\varepsilon}\frac{(1+\lambda\xi)^{\gamma}}{(1+\lambda\tilde{\ell})^{\gamma-1}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z)$$
$$= \sqrt{\varepsilon}(1-\gamma)r\tilde{v}^{\lambda,1,+}(1+\lambda\xi)^3$$
$$-\left((1-\gamma)\left(\xi(1+\lambda)(1+\lambda\xi)\left(\gamma\xi(1+\lambda)\frac{\sigma^2}{2} - \mu(1+\lambda\xi)\right)\right.\right.$$
$$\left.\left.+(1+\lambda\xi)^3\left((1-\gamma)(1+\lambda\theta)^{\gamma-1}\tilde{v}^{\lambda,0,+}\right)^{-1/\gamma}\right)\right)\tilde{v}^{\lambda,1,+}$$
$$-\nu(1+\lambda\xi)^3\tilde{v}^{\lambda,1,+} - (1-\gamma)\gamma\frac{\xi^2}{2}(1+\lambda)^2\bar{V}_3(\xi(1+\gamma-(1-\gamma)\lambda)-2)\tilde{v}^{\lambda,0,+} + (1+\lambda\xi)^3\partial_t\tilde{v}^{\lambda,0,+}.$$

Here on the right hand side, and for the rest of this section for convenience, unless the arguments of $\tilde{v}^{\lambda,0,+}, \tilde{v}^{\lambda,1,+}$ are explicitly specified, it will be assumed to be $(t, \tilde{\ell}, z)$. It follows that

$$\sqrt{\varepsilon}\frac{(1+\lambda\xi)^{\gamma}}{(1+\lambda\tilde{\ell})^{\gamma-1}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z)$$
$$= \sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)$$
$$+\sqrt{\varepsilon}(\xi-\tilde{\ell})\left[-(1-\gamma^2)\gamma\bar{V}_3\tilde{v}^{\lambda,0,+}\left(\xi^2+\xi\tilde{\ell}+\tilde{\ell}^2\right)\right.$$
$$\left.+\left(2(1-\gamma)\gamma\bar{V}_3\tilde{v}^{\lambda,0,+} - \gamma(1-\gamma)^2\sigma^2\tilde{v}^{\lambda,1,+}\right)\left(\xi+\tilde{\ell}\right) + 2(1-\gamma)\mu\tilde{v}^{\lambda,1,+}\right] + O(\varepsilon) + O\left(\sqrt{\varepsilon}\lambda\right).$$

In case $\bar{V}_3 = 0$, the whole polynomial is zero. Whereas if $\bar{V}_3 < 0$, it can then be shown that under the Assumption 5.1.i the quadratic polynomial in the ξ variable term on the right hand side in square brackets, has both of its roots greater than θ. Using the fact the the leading coefficient of the entire cubic polynomial is positive, it follows that the entire polynomial is negative for $\xi < \tilde{\ell}$, and we conclude using (5.31) that

$$\sqrt{\varepsilon}\frac{(1+\lambda\xi)^{\gamma}}{(1+\lambda\tilde{\ell})^{\gamma-1}}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right)(t,\xi,z)$$
$$\le \sqrt{\varepsilon}\left(\mathcal{L}_{\mathrm{NT}}\tilde{v}^{\lambda,1,+} - \left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,+}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,+} + \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,+}\right\rangle\right) + O(\varepsilon) + O\left(\sqrt{\varepsilon}\lambda\right) \le C\varepsilon^{\frac{1+q}{2}}.$$

5.5.4. Step 3b: Inside the $\widetilde{\mathrm{S}}$ region. Similarly, it can be shown that for appropriate choices of the various constants M_i, and for $\lambda, \varepsilon > 0$ small enough $\mathcal{B}\left(\tilde{v}^{\lambda,0,+} + \sqrt{\varepsilon}\tilde{v}^{\lambda,1,+}\right) \le O(\varepsilon)$, and (5.29) also holds there.

5.5.5. Summary of Steps 1-3. To summarize the above steps we see that

$$\pm\max\left\{\mathcal{L}_{\mathrm{NT}}w^{1,\pm} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right) - \sqrt{\varepsilon}\left(\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm} - \left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\right), \mathcal{B}w^{1,\pm}, \mathcal{S}w^{1,\pm}\right\} \le \pm C\varepsilon^{\frac{1+q}{2}},$$
$$\pm\mathbb{E}_0^{x,y,z}\left[(X_T+Y_T)^{1-\gamma}w^{\pm}\left(T, \frac{X_T}{X_T+Y_T}, Z_T\right)\right] \ge \pm\mathbb{E}_0^{x,y,z}\left[\mathcal{U}\left(X_T+Y_T-\lambda\left|Y_T\right|\right)\right].$$

5.5.6. Proof of Subsolution Property. We next evaluate the second order operator from the HJB equation (2.12). For any fixed $0 \le t < T$, and using the (5.12) - (5.14) to substitute for $w^{\pm}$, we have that:

$$
\begin{aligned}
(5.32)\qquad & \pm\,\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^{\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{\pm}\right)\right)\right)\Big] \\
&= \pm\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_{\xi}^{\lambda,0,\pm}\right)+\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}\right)\right] \\
&\quad \pm\sqrt{\varepsilon}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right)\right] \\
&\quad \pm\varepsilon\mathbb{E}_0^{x,y,z}\Big[(X_t+Y_t)^{1-\gamma}\mathcal{L}_2\left(\tilde{v}^{\lambda,2,\pm}+\hat{v}^{\pm}\right) \\
&\qquad +\frac{1}{2\gamma}\left(\xi(t,\xi_t,Z_t)\right)^{-\frac{1+\gamma}{\gamma}}\left(\mathcal{L}_{\mathcal{U}}\left(\tilde{v}^{\lambda,1,\pm}(t,\xi)+\sqrt{\varepsilon}\left(\tilde{v}^{\lambda,2,\pm}+\hat{v}^{\pm}\right)(t,\xi,z)\pm M_3\varepsilon^{\frac{q}{2}}\right)\right)^2\Big)\Big] \\
&\quad +\varepsilon^{\frac{1+q}{2}}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\right)M_3\right],
\end{aligned}
$$

where $\xi^{\pm}(t,\xi,z)$ is in between $w^{\pm}(t,\xi,z)$ and $\tilde{v}^{\lambda,0,\pm}(t,\xi)$ used to write the remainder in Lagrange form in the Taylor series of $\tilde{\mathcal{U}}$. Notice, that since both $w^{\pm}$ and $\tilde{v}^{\lambda,0,\pm}$ are bounded by the definition of an admissible strategy, then so is $\xi^{\pm}$.

We want to conclude that the expression in (5.32) is non positive. In order to do that first note that the $O\left(\varepsilon^{\frac{1+q}{2}}\right)$ term in (5.32) evaluates to

$$
\pm\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\right)M_3\right]=\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}FM_3\right]
$$

From the definition of admissible strategy and using Lemma 5.9 the two expectations in the $O(\varepsilon)$ term: $\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\xi(t,\xi_t,Z_t)\right)^{-\frac{1+\gamma}{\gamma}}\left(\mathcal{L}_{\mathcal{U}}\left(\tilde{v}^{\lambda,1,\pm}(t,\xi)+\sqrt{\varepsilon}\left(\tilde{v}^{\lambda,2}+\hat{v}^{\pm}\right)(t,\xi,z)\pm M_3\varepsilon^{\frac{q}{2}}\right)\right)^2\right]$ and $\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\mathcal{L}_2\left(\tilde{v}^{\lambda,2}+\hat{v}^{\pm}\right)\right]$ are both bounded.

Next, we rewrite the $O(1)$ and $O(\sqrt{\varepsilon})$ terms in (5.32) as

$$
\begin{aligned}
& \pm\,\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_{\xi}^{\lambda,0,\pm}\right)+\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}\right)\right] \\
& \pm\sqrt{\varepsilon}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right)\right] \\
&= \pm\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\mathcal{L}_2\tilde{v}^{\lambda,0,\pm}+\tilde{\mathcal{U}}\left((1-\gamma)\tilde{v}^{\lambda,0,\pm}-\xi\tilde{v}_{\xi}^{\lambda,0,\pm}\right)+\mathcal{L}_0\tilde{v}^{\lambda,2,\pm}\right)\right] \\
& \pm\sqrt{\varepsilon}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left\langle\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\right] \\
& \pm\sqrt{\varepsilon}\mathbb{E}_0^{x,y,z}\Big[(X_t+Y_t)^{1-\gamma}\Big(\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\mathcal{L}_1\tilde{v}^{\lambda,2,\pm} \\
&\qquad -\left\langle\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}+\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle\Big)\Big]
\end{aligned}
$$

From the Subsection 5.5.5, the sum of the first two term on the right hand side, i.e sum of the $O(1)$ and $O(\sqrt{\varepsilon})$ terms, is dominated by $O\left(\varepsilon^{\frac{1+q}{2}}\right)$. While the last term on the right hand side is of $O\left(\varepsilon^{\frac{1+q}{2}}\right)$ as follows from Lemma 5.10. Indeed, for the fixed t from above, we have that $\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}-\left\langle\mathcal{L}_2\tilde{v}^{\lambda,1,\pm}\right\rangle=(f^2(z)-\bar{\sigma}^2)\frac{\xi^2}{2}\left(D_2-2\gamma D_1-\Gamma I\right)\tilde{v}^{\lambda,1,\pm}$, thus Lemma 5.10 is first applied with $J_1(z)=f^2(z)-\bar{\sigma}^2$, $J_2(x,y)=\frac{\xi^2}{2}\left(D_2-2\gamma D_1-\Gamma I\right)\tilde{v}^{\lambda,1,\pm}\left(t,\frac{y}{x+y}\right)$, which fit the assumptions of the Lemma, as $\langle J_1\rangle=0$, and J_2 is bounded by definition of an admissible strategy. Similarly, the difference $\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}-\left\langle\mathcal{L}_1\tilde{v}^{\lambda,2,\pm}\right\rangle=J_1J_2$, where $J_1(z)=\bar{V}_3-\frac{\rho f(z)\beta(z)\phi'(z)}{2}$, $J_2(x,y)=\frac{y}{x+y}\left((1-\gamma)I+D_1\right)\left(\left(\frac{y}{x+y}\right)^2\left(D_2-\Gamma I-2\gamma D_1\right)\tilde{v}^{\lambda,0,\pm}\right)(t,\frac{y}{x+y})$. Lastly, note that $\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\times\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}=\left\langle\left(\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,0,\pm}\right)^{-\frac{1}{\gamma}}\mathcal{L}_{\mathcal{U}}\tilde{v}^{\lambda,1,\pm}\right\rangle$.

We conclude that (5.32) evaluates to

$$\pm\, \mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^{\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{\pm}\right)\right)\right]$$
$$\le \varepsilon^{\frac{1+q}{2}}\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}FM_3\right]+O\left(\varepsilon^{\frac{1+q}{2}}\right).$$

Using Tonelli's Theorem, we conclude that for appropriate choices of the constants M_i, and for $\lambda,\varepsilon>0$ small enough, we can insure that

$$\pm\, \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t+Y_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^{\pm}+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^{\pm}\right)\right)dt\right] \tag{5.33}$$
$$\le \varepsilon^{\frac{1+q}{2}}\int_0^T \mathrm{e}^{-\nu t}\,\mathbb{E}_0^{x,y,z}\left[(X_t+Y_t)^{1-\gamma}FM_3\right]dt+O\left(\varepsilon^{\frac{1+q}{2}}\right).$$

Next, we define the strategy associated with the buy, sell and NT region $\widetilde{\mathrm{B}},\widetilde{\mathrm{S}}$, and $\widetilde{\mathrm{NT}}$, $\tilde{L}$ and $\tilde{M}$ and the wealth diffusion processes $\tilde{X}_t$ and $\tilde{Y}_t$. Similar to (2.3), we set $\tilde{\xi}_t:=\frac{\tilde{Y}_t}{\tilde{X}_t+\tilde{Y}_t}$. Moreover, we define

$$\tilde{C}_t:=\left(\partial_x\left((\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\,\mathrm{e}^{-\nu t}\,w^-\left(t,\frac{\tilde{Y}_t}{\tilde{X}_t+\tilde{Y}_t},Z_t\right)\right)\right)^{-\frac{1}{\gamma}}$$
$$=\left((\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{L}_{\mathcal{U}}w^-\left(t,\frac{\tilde{Y}_t}{\tilde{X}_t+\tilde{Y}_t},Z_t\right)\right)^{-\frac{1}{\gamma}},$$

in which case we have that

$$(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^-\right)=\tilde{\mathcal{U}}\left((\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\mathcal{L}_{\mathcal{U}}w^-\right)=\mathcal{U}(\tilde{C}_t)-\tilde{C}_t(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\mathcal{L}_{\mathcal{U}}w^-. \tag{5.34}$$

Note, that $\frac{\tilde{C}_t}{(\tilde{X}_t+\tilde{Y}_t)}$ is bounded. Thus the strategy $(\tilde{L},\tilde{M},\tilde{C})$ is admissible.

Our goal is now to conclude that $(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\,\mathrm{e}^{-\nu t}\,w^-\left(t,\frac{\tilde{Y}_t}{\tilde{X}_t+\tilde{Y}_t},Z_t\right)$ is a submartingale. The last piece of the puzzle is what happens at the boundaries of $\widetilde{\mathrm{NT}}$ where w^- may not be twice differentiable. Using Itô-Tanaka formula and applying (5.34), we calculate that

$$\mathbb{E}_0^{x,y,z}\left[(X_T+Y_T)^{1-\gamma}\,\mathrm{e}^{-\nu T}\,w^-\left(T,\frac{\tilde{Y}_T}{\tilde{X}_T+\tilde{Y}_T},Z_T\right)\right]=(x+y)^{1-\gamma}w^-\left(0,\frac{y}{x+y},z\right) \tag{5.35}$$
$$+\,\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2-\frac{\tilde{C}_t}{\tilde{X}_t+\tilde{Y}_t}\mathcal{L}_{\mathcal{U}}\right)w^-dt\right]$$
$$+\,\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{B}w^-(t,\tilde{\ell},Z_t)d\tilde{L}_t+\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{S}w^-(t,\tilde{u},Z_t)d\tilde{M}_t\right]$$
$$=(x+y)^{1-\gamma}w^-\left(0,\frac{y}{x+y},z\right)-\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\,\mathcal{U}(\tilde{C}_t)dt\right]$$
$$+\,\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^-+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^-\right)\right)dt\right]$$
$$+\,\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{B}w^{2,-}(t,\tilde{\ell},Z_t)d\tilde{L}_t+\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{S}w^{2,-}(t,\tilde{u},Z_t)d\tilde{M}_t\right],$$

Remark 5.12. In the equalities above, we have omitted the Itô integral term, that has zero expectation. Indeed, this Itô integral involves the terms $(x+y)^{1-\gamma}w_y^{1,-}$ and $(x+y)^{1-\gamma}\frac{y}{x+y}w^{1,-}$. Recall, that the strategy

$(\tilde{L},\tilde{M},\tilde{C})$ is admissible, and thus by definition of an admissible strategy the process $\frac{\hat{Y}_t}{\hat{X}_t+\hat{Y}_t}$ stays in a compact (that may depend on the strategy), and thus is bounded, unless $\hat{X}_t=\hat{Y}_t=0$. In the latter case, both terms $(x+y)^{1-\gamma}w_y^{1,-}$ and $(x+y)^{1-\gamma}\frac{y}{x+y}w^{1,-}$ are zero , whereas in the former case, the functions $w^{1,-},w_y^{1,-}$ are both bounded on this compact, and so are $(x+y)^{1-\gamma}$ and $\frac{y}{x+y}$.

From (5.14) and Lemma 5.3 it follows that $\mathcal{B}w^-(t,\tilde{\ell},z)=\mathcal{B}w^{2,-}(t,\tilde{\ell},z)$, and similarly $\mathcal{S}w^-(t,\tilde{u},z)=\mathcal{S}w^{2,-}(t,\tilde{u},z)$. However, $w^{2,-}$ may not be necessarily smooth across the boundaries of the $\widetilde{\mathrm{NT}}$ region, hence $\mathcal{B}w^-(t,\tilde{\ell}),\mathcal{S}w^-(t,\tilde{u})$ may not be zero. Here the derivatives in these operators are evaluated from inside the $\widetilde{\mathrm{NT}}$ region.

It follows from (5.35) that

$$(x+y)^{1-\gamma}w^-\left(0,\frac{y}{x+y},z\right)=\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\,\mathcal{U}(\tilde{C}_t)dt+(X_T+Y_T)^{1-\gamma}\,\mathrm{e}^{-\nu T}\,w^-\left(T,\frac{\tilde{Y}_T}{\tilde{X}_T+\tilde{Y}_T},Z_T\right)\right]$$
$$-\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^-+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^-\right)\right)dt\right]$$
$$-\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{B}w^{2,-}(t,\tilde{\ell},Z_t)d\tilde{L}_t+\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{S}w^{2,-}(t,\tilde{u},Z_t)d\tilde{M}_t\right] \tag{5.36}$$

Since $\mathcal{B}w^{2,-}(t,\tilde{\ell},z)=\varepsilon\mathcal{B}\tilde{v}^{\lambda,2}(t,\tilde{\ell},z)+(1-\gamma)M_3\lambda\varepsilon^{1+\frac{q}{2}}$, using the fact that $\langle\phi\rangle=0$ from Lemma 5.11, we conclude that for $q<\frac{1}{2}$

$$\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\mathcal{B}w^{2,-}(t,\tilde{\ell},Z_t)d\tilde{L}_t\right] \tag{5.37}$$
$$=\varepsilon\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\mathcal{B}\tilde{v}^{\lambda,2}(t,\tilde{\ell},Z_t)d\tilde{L}_t\right]+O\left(\varepsilon^{1+\frac{q}{2}}\right)=O\left(\varepsilon^{1+\frac{q}{2}}\right),$$

$$\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\mathcal{S}w^-(t,\tilde{\ell},Z_t)d\tilde{M}_t\right]=O\left(\varepsilon^{1+\frac{q}{2}}\right). \tag{5.38}$$

Using the estimates in (5.33), (5.37) and (5.38) it follows that the constants M_i can be chosen so that for λ and ε small enough we have that

$$\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)w^-+\tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}}w^-\right)\right)dt\right]$$
$$+\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{B}w^{2,-}(t,\tilde{\ell},Z_t)d\tilde{L}_t+\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{-\gamma}\,\mathrm{e}^{-\nu t}\,\mathcal{S}w^{2,-}(t,\tilde{u},Z_t)d\tilde{M}_t\right]$$
$$\geq-\varepsilon^{\frac{1+q}{2}}\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\,\mathrm{e}^{-\nu t}\,FM_3dt\right]+O\left(\varepsilon^{\frac{1+q}{2}}\right).$$

It follows from (5.36) that

$$(x+y)^{1-\gamma}w^-\left(0,\frac{y}{x+y},z\right)$$
$$\leq\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\,\mathcal{U}(\tilde{C}_t)dt+(\tilde{X}_T+\tilde{Y}_T)^{1-\gamma}\,\mathrm{e}^{-\nu T}\,w^-\left(T,\frac{\tilde{Y}_T}{\tilde{X}_T+\tilde{Y}_T},Z_T\right)\right]$$
$$+\varepsilon^{\frac{1+q}{2}}\mathbb{E}_0^{x,y,z}\left[\int_0^T(\tilde{X}_t+\tilde{Y}_t)^{1-\gamma}\,\mathrm{e}^{-\nu t}\,FM_3dt\right]+O\left(\varepsilon^{\frac{1+q}{2}}\right).$$

Thus

$$(5.39)\quad (x+y)^{1-\gamma} w^{-}\left(0, \frac{y}{x+y}, z\right)$$
$$\leq \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\, \mathcal{U}(\tilde{C}_t)dt + \mathrm{e}^{-\nu T}\, \mathcal{U}\left(\tilde{X}_T + \tilde{Y}_T - \lambda\left|\tilde{Y}_T\right|\right)\right]$$
$$+\varepsilon^{\frac{1+q}{2}} \mathbb{E}_0^{x,y,z}\left[\int_0^T (\tilde{X}_t + \tilde{Y}_t)^{1-\gamma}\, \mathrm{e}^{-\nu t}\, F M_3 dt + \mathrm{e}^{-\nu T}\left(\tilde{X}_T + \tilde{Y}_T - \lambda\left|\tilde{Y}_T\right|\right)^{1-\gamma} \frac{M_3}{4}\right] + O\left(\varepsilon^{\frac{1+q}{2}}\right)$$
$$\leq \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\, \mathcal{U}(\tilde{C}_t)dt + \mathrm{e}^{-\nu T}\, \mathcal{U}\left(\tilde{X}_T + \tilde{Y}_T - \lambda\left|\tilde{Y}_T\right|\right)\right],$$

where the first inequality above follows from (5.26), and the last inequality follows from Remark 5.5, specifically (5.15). From the definition of $\widehat{V}$, we conclude that

$$(5.40)\qquad (x+y)^{1-\gamma} w^{-}\left(0, \frac{y}{x+y}, z\right) \leq \widehat{V}(0, x, y, z).$$

5.5.7. Proof of Supersolution Property. For the other direction, for any admissible trading strategy, we have from Itô-Tanaka formula that

$$(5.41)\quad \mathbb{E}_0^{x,y,z}\left[(X_T+Y_T)^{1-\gamma}\, \mathrm{e}^{-\nu T}\, w^{+}\left(T, \frac{Y_T}{X_T+Y_T}, Z_T\right)\right] - (x+y)^{1-\gamma} w^{+}\left(0, \frac{y}{x+y}, z\right)$$
$$= \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t+Y_t)^{1-\gamma}\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 - \frac{C_t}{X_t+Y_t}\mathcal{L}_{\mathcal{U}}\right) w^{+} dt\right]$$
$$+\mathbb{E}_0^{x,y,z}\left[\int_0^T (X_t+Y_t)^{-\gamma}\, \mathrm{e}^{-\nu t}\, \mathcal{B} w^{+} dL_t + \int_0^T (X_t+Y_t)^{-\gamma}\, \mathrm{e}^{-\nu t}\, \mathcal{S} w^{+} dM_t\right]$$
$$\leq -\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\, \mathcal{U}(C_t)dt\right]$$
$$+\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t+Y_t)^{1-\gamma}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) w^{+} + \tilde{\mathcal{U}}\left(\mathcal{L}_{\mathcal{U}} w^{+}\right)\right) dt\right]$$
$$+\mathbb{E}_0^{x,y,z}\left[\int_0^T (X_t+Y_t)^{-\gamma}\, \mathrm{e}^{-\nu t}\, \mathcal{B} w^{2,+} dL_t + \int_0^T (X_t+Y_t)^{-\gamma}\, \mathrm{e}^{-\nu t}\, \mathcal{S} w^{2,+} dM_t\right],$$

where the expectation of the Itô integral is zero, because of Remark 5.12. Next, note that similar to estimates (5.37) and (5.38), from Lemma 5.11 for $q < \frac{1}{2}$ we can conclude that

$$\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t+Y_t)^{-\gamma} \mathcal{B} w^{2,-}\left(t, \frac{Y_t}{X_t+Y_t}\right) dL_t\right] = O\left(\varepsilon^{1+\frac{q}{2}}\right),$$
$$\mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}(X_t+Y_t)^{-\gamma} \mathcal{S} w^{2,-}\left(t, \frac{Y_t}{X_t+Y_t}\right) dM_t\right] = O\left(\varepsilon^{1+\frac{q}{2}}\right).$$

We proceed similar to the above calculation, to conclude that

$$(x+y)^{1-\gamma} w^{+}\left(0, \frac{y}{x+y}, z\right) \geq \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\, \mathcal{U}(C_t)dt + (X_T+Y_T)^{1-\gamma}\, \mathrm{e}^{-\nu T}\, w^{+}\left(T, \frac{Y_T}{X_T+Y_T}, Z_T\right)\right].$$

Taking supremum over the set of all admissible strategies we conclude that

$$(5.42)\qquad (x+y)^{1-\gamma} w^{+}\left(0, \frac{y}{x+y}, z\right) \geq \widehat{V}(0, x, y, z).$$

5.5.8. Final Steps of Proof of Theorem 5.4. The proof now easily follows from the above results. Using the fact that inside $\mathcal{K}_0$, $w^{\pm} = \tilde{v}^{\lambda,0}(t,\xi) + \sqrt{\varepsilon}\,\tilde{v}^{\lambda,1}(t,\xi) + O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{2}{3}}\right) + O(\varepsilon)$, we obtain from (5.40) that $v^{\lambda,\varepsilon}(t,\xi,z) - w^{-}(t,\xi,z) \geq 0$, and thus inside $\mathcal{K}$, $v^{\lambda,\varepsilon}(t,\xi,z) - \left(\tilde{v}^{\lambda,0}(t,\xi) + \sqrt{\varepsilon}\,\tilde{v}^{\lambda,1}(t,\xi)\right) \geq O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{2}{3}}\right) + O(\varepsilon)$. Whereas, similarly from (5.42) $v^{\lambda,\varepsilon}(t,\xi,z) - w^{+}(t,\xi,z) \leq 0$, and thus inside $\mathcal{K}$, $v^{\lambda,\varepsilon}(t,\xi,z) - \left(\tilde{v}^{\lambda,0}(t,\xi) + \sqrt{\varepsilon}\,\tilde{v}^{\lambda,1}(t,\xi)\right) \leq O(\lambda) + O\left(\sqrt{\varepsilon}\lambda^{\frac{2}{3}}\right) + O(\varepsilon)$. Thus (5.10), the first assertion of the theorem, follows. Additionally, from (5.39) it also follows that the strategy $\widetilde{\mathrm{NT}}$, $\widetilde{\mathrm{B}}$ and $\widetilde{\mathrm{S}}$, whose expansion is $\tilde{\ell}$ and $\tilde{u}$ is nearly optimal, since

$$(x+y)^{1-\gamma} w^{-}\left(0, \frac{y}{x+y}, z\right) \leq \mathbb{E}_0^{x,y,z}\left[\int_0^T \mathrm{e}^{-\nu t}\,\mathcal{U}(\tilde{C}_t)dt + \mathrm{e}^{-\nu T}\,\mathcal{U}\left(\tilde{X}_T + \tilde{Y}_T - \lambda\left|\tilde{Y}_T\right|\right)\right] \leq \widehat{V}(0,x,y,z),$$

which is the second assertion of the theorem.

Remark 5.13. Note that the case $\gamma > 1$ is another case that can be handled by this approach. However, this case requires a whole new set of proofs and constraints, since current proofs are built on the assumption that $0 < \gamma < 1$, as well as some other constraints in Assumption 5.1. While the details of the proof will change, the methodology is the same. For example, the most crucial step in the proof is that the Itô integral terms that appear in (5.35) and (5.41) are still evaluate to zero, as discussed in Remark 5.12. This is handled using a localization technique, and the convergence is then obtained through the dominated convergence theorem and Fatou's lemma as in [3]. Note that the assumption on the admissible strategy being in a compact also plays a crucial part here.

It also important to note that the constraints on A and θ in Assumption 5.1.i,ii, specifically the constraints that $-\frac{\gamma}{A(1-\gamma)} \leq 1$ and $\theta < \frac{4}{3(1+\gamma)}$ are sufficient, but not necessary constraints, and could be relaxed a little, while still having this proof hold. These relaxed constrains are more complicated, and hence we chose to state the simpler constraints instead. It's very likely that the result is true even without them, but unfortunately we are not aware of the way to get rid of them completely. The constraint $0 < \theta < 1$ is not necessary, as the other cases when $\theta < 0$ or $\theta > 1$ can be considered, but are omitted here, as they require a whole new set of inequalities to be worked out, as was mentioned in Section 2.3. Finally the cases $\theta = 0, 1$ are explicitly excluded, as in those case, the optimal strategies become trivial by holding all the wealth in stock or cash, and not trading except for possibly at the initial and terminal times.

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