OPTIMAL INVESTMENT WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY PART II: FINITE HORIZON

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1. Appendix. We now show the verification theorem that the value function $\widehat{V}$ is a viscosity solution of [1] eq. (2.8). The proof is divided into the following three lemmas:

LEMMA 1.1. *The value function $\widehat{V}$ is continuous function on $[0,T]\times\bar{\mathcal{S}}\times\mathbb{R}$.*

Proof. The continuity in the (x,y) variables follows from concavity in the same way as in [4] in Corollaries 3.2, 5.5 and 5.8. For the continuity in time t, note that it is the same as continuity in terminal time T. Clearly, the integral term in [1] eq. (2.5) is continuous. Regarding the second term $\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)$, note that the same admissible strategy (C,L,M) that liquidates the position at time T and consumes nothing after that, can be used for longer maturity $T+\Delta T$, for $\Delta T>0$. Thus $\widehat{V}$ is upper semi-continuous in T. The lower semi-continuity of $\widehat{V}$ follows from the the lower semi-continuity of the process $\mathrm{Liq}(X,Y)$.

The continuity in the z variable, is the result of stability of strong solutions. Fix (t_0,x_0,y_0,z_0) such that $(x_0,y_0)\in\mathcal{S}$ and $\varepsilon_0>0$. For $|z-z_0|<1$, let $(C^{(t_0,x_0,y_0,z)},L^{(t_0,x_0,y_0,z)},M^{(t_0,x_0,y_0,z)})$ be a nearly-optimal admissible strategy, such that

$$\widehat{V}(t_0,x_0,y_0,z)-\mathbb{E}^{x_0,y_0,z}_{t_0}\left[\int_{t_0}^T \mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}\left(C_s^{(t_0,x_0,y_0,z)}\right)ds+\mathrm{e}^{-\nu(T-t_0)}\,\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)\right]<\varepsilon_0.$$

Then, $\left\{\int_{t_0}^T \mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}\left(C_s^{(t_0,x_0,y_0,z)}\right)ds+\mathrm{e}^{-\nu(T-t_0)}\,\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)\right\}_{(t_0,x_0,y_0,z)}$ is uniformly integrable, since it is dominated by zero transaction costs value function, which is shown by [3] to be continuous. Hence, let $\delta_0>0$ be such that $(x_0+\frac{\delta}{1-\lambda},y_0)\in\mathcal{S}$, and that any A satisfying $\mathbb{P}(A)>1-\delta_0$ we have that

$$\widehat{V}(t_0,x_0,y_0,z)-\mathbb{E}^{x_0,y_0,z}_{t_0}\left[\mathbb{I}_A\int_{t_0}^T \mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}\left(C_s^{(t_0,x_0,y_0,z)}\right)ds+\mathrm{e}^{-\nu(T-t_0)}\,\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)\right]<2\varepsilon_0.$$

Next, from a calculation similar to the classical result of strong solution (see [2] Theorem 9.2.1) it follows that $\mathbb{E}_{t_0}\left[\sup_{t_0\le s\le T}||(Y_s^{y_0},Z_s^{z_0})-(Y_s^{y_0},Z_s^{z})||^2\right]<(1-\delta_0)\delta_0$ for (possibly even smaller) $\delta_0>0$. It follows that there exists a set $A_0=A_0^{(y_0,z)}$, such that $\mathbb{P}(A_0)>1-\delta_0$, and $\sup_{t_0\le s\le T}||(Y_s^{y_0},Z_s^{z_0})-(Y_s^{y_0},Z_s^{z})||^2<\delta_0$ on A_0. Using the fact that

$$Y_t=Y_{t_0}\,\mathrm{e}^{(\mu+r)(t-t_0)-\frac12\int_{t_0}^t f^2(Z_s)ds+\int_{t_0}^t f(Z_s)dB_s^1}+\int_{t_0}^t \mathrm{e}^{(\mu+r)(t-u)-\frac12\int_u^t f^2(Z_s)ds+\int_u^t f(Z_s)dB_s^1}(dL_u-dM_u),$$

It follows that there exists some $\delta_1>0$, such that using the control $(C^{(t_0,x_0,y_0,z)},L^{(t_0,x_0,y_0,z)},M^{(t_0,x_0,y_0,z)})$ we have that $\mathrm{Liq}(X_t^{x_0}+\delta_1,Y_t^{y_0,z_0})\ge\mathrm{Liq}\left(X_t^{x_0},Y_t^{y_0,z}\right)$, for all $t\in[t_0,T]$. Moreover, δ_0 can be chosen small enough, so that $(x_0+\delta_1,y_0)\in\mathcal{S}$, and thus the control $(C^{(t_0,x_0,y_0,z)},L^{(t_0,x_0,y_0,z)},M^{(t_0,x_0,y_0,z)})$ is also admissible for the starting point $(t_0,x_0+\delta_1,y_0,z_0)$. Thus,

$$\begin{aligned}\widehat{V}(t_0,x_0,y_0,z)-2\varepsilon_0&<\mathbb{E}^{x_0,y_0,z}_{t_0}\left[\mathbb{I}_{A_0}\int_{t_0}^T \mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}\left(C_s^{(t_0,x_0,y_0,z)}\right)ds+\mathrm{e}^{-\nu(T-t_0)}\,\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)\right]\\&\le\mathbb{E}^{x_0+\delta_1,y_0,z_0}_{t_0}\left[\mathbb{I}_{A_0}\int_{t_0}^T \mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}\left(C_s^{(t_0,x_0,y_0,z)}\right)ds+\mathrm{e}^{-\nu(T-t_0)}\,\mathcal{U}\left(\mathrm{Liq}(X_T,Y_T)\right)\right]\\&\le\widehat{V}(t_0,x_0+\delta_1,y_0,z_0),\end{aligned}$$

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Similar calculation implies

$$\widehat{V}(t_0,x_0-\delta_1,y_0,z_0)-2\varepsilon_0 \le \widehat{V}(t_0,x_0,y_0,z) \le \widehat{V}(t_0,x_0+\delta_1,y_0,z_0)+2\varepsilon_0.$$

The continuity of $\widehat{V}$ in the z variable, now follows from its continuity in the x variable. ☐

We note that similar to Proposition 3.5 of [4] we have that

Remark 1.2. Let $(x_0,y_0),(x,y)\in\overline{\mathcal{S}}$ be two points, such that (x,y) can be reached from (x_0,y_0) by a trade. Then $\widehat{V}(t_0,x_0,y_0,z_0)\ge\widehat{V}(t_0,x,y,z_0)$, for any $t_0\in[0,T]$ and $z_0\in\mathbb{R}$.

We now show that the value function $\hat{V}$ is a viscosity solution of the HJB equation. This proof follows along the lines of the proof of Lemmas 7.8 and 7.9 of [4]. We also divide our proof into two parts:

LEMMA 1.3. *The value function $\widehat{V}$ is a viscosity supersolution of [1] eq.* (2.8) *on* $[0,T)\times\mathcal{S}\times\mathbb{R}$.

Proof. Let $(t_0,x_0,y_0,z_0)\in[0,T)\times\mathcal{S}\times\mathbb{R}$ and smooth ϕ satisfying $\phi\le\widehat{V}$ and $\phi(t_0,x_0,y_0,z_0)=\widehat{V}(t_0,x_0,y_0,z_0)$. Then for $\delta_0>0$ sufficiently small, such that $(t_0,x_0-(1+\lambda)\delta_0,y_0+\delta_0,z_0)\in[0,T)\times\mathcal{S}\times\mathbb{R}$, then from Remark 1.2 we have that

$$\begin{aligned}&\phi(t_0,x_0,y_0,z_0)-\phi(t_0,x_0-(1+\lambda)\delta_0,y_0+\delta_0,z_0)\\&\ge\widehat{V}(t_0,x_0,y_0,z_0)-\widehat{V}(t_0,x_0-(1+\lambda)\delta_0,y_0+\delta_0,z_0)\ge 0\end{aligned}$$

Thus, dividing by δ_0 and letting $\delta_0\to0$ and it follows that $(1+\lambda)\phi_x-\phi_y\ge0$. Similarly, it can be concluded that $-(1-\lambda)\phi_x+\phi_y\ge0$.

To show that $-(\partial_t+\mathcal{D}^\varepsilon)\phi-\tilde{\mathcal{U}}(\phi_x)\ge0$, let $\epsilon_0>0$ be such that $\{(x,y)|\,|x-x_0|\le\epsilon_0,|y_y-0|\le\epsilon_0\}\subset\mathcal{S}$. For $c>0$ let (an admissible) strategy be given by $(0,0,c)$, i.e. the strategy $L=M=0,C=c$, for all times $t_0\le t\le\tau$, where $\tau=\varepsilon_0\wedge\inf\{t_0\le t\le T|\,|X_t-x_0|>\varepsilon_0,\text{ or }|Y_t-y_0|>\varepsilon_0\}$. Then

$$\phi(t_0,x_0,y_0,z_0)=\mathbb{E}_{t_0}\left[\mathrm{e}^{-\nu(\tau-t_0)}\,\phi\left(\tau,X_\tau,Y_\tau,Z_\tau\right)-\int_{t_0}^{\tau}\mathrm{e}^{-\nu(s-t_0)}((\partial_t+\mathcal{D}^\varepsilon-c\partial_x)\phi)ds\right]$$

From the dynamic programming principle it follows that

$$\begin{aligned}\widehat{V}(t_0,x_0,y_0,z_0)&\ge\mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}(c)ds+\mathrm{e}^{-\nu(\tau-t_0)}\,\widehat{V}\left(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau},Z_{t\wedge\tau}\right)\right]\\&\ge\mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}(c)ds+\mathrm{e}^{-\nu(\tau-t_0)}\,\phi\left(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau},Z_{t\wedge\tau}\right)\right]\\&=\widehat{V}(t_0,x_0,y_0,z_0)-\mathbb{E}_{t_0}\left[\int_{t_0}^{\tau}\mathrm{e}^{-\nu(s-t_0)}\left((\partial_t+\mathcal{D}^\varepsilon-c\partial_x)\phi+\mathcal{U}(c)\right)ds\right].\end{aligned}$$ ☐

Thus $\mathbb{E}_{t_0}\left[\int_{t_0}^{\tau}\mathrm{e}^{-\nu(s-t_0)}\left((\partial_t+\mathcal{D}^\varepsilon-c\partial_x)\phi+\mathcal{U}(c)\right)ds\right]\ge0$. So we must have $(\partial_t+\mathcal{D}^\varepsilon-c\partial_x)\,\phi+\mathcal{U}(c)\ge0$ in some neighborhood of (t_0,x_0,y_0,z_0). Letting $\varepsilon_0\to0$ and minimizing over $c>0$ we obtain $-(\partial_t+\mathcal{D}^\varepsilon)\phi(t_0,x_0,y_0,z_0)-\tilde{\mathcal{U}}(\phi_x(t_0,x_0,y_0,z_0))\ge0$.

LEMMA 1.4. *The value function $\widehat{V}$ is a viscosity subsolution of [1] eq.* (2.8) *on* $[0,T)\times\mathcal{S}\times\mathbb{R}$.

Proof. Let $(t_0,x_0,y_0,z_0)\in[0,T)\times\mathcal{S}\times\mathbb{R}$ and smooth ϕ satisfying $\phi\ge\widehat{V}$ and $\phi(t_0,x_0,y_0,z_0)=\widehat{V}(t_0,x_0,y_0,z_0)$. We prove the subsolution property by contradiction. Assume that

$$\min\left\{-(\partial_t+\mathcal{D}^\varepsilon)\phi-\tilde{\mathcal{U}}(\phi_x),((1+\lambda)\,\partial_x-\partial_y)\phi,(\partial_y-(1-\lambda)\,\partial_x)\,\phi\right\}\le0$$

does not hold. By continuity of ϕ it means that for some $\varepsilon_0>0$ small enough, we have that on the set $H=\{(t,x,y,z)|\,|x-x_0|<\epsilon_0,|y-y_0|<\epsilon_0,|z-z_0|<\epsilon_0,t_0\le t<(t_0+\epsilon_0)\wedge T\}$ we have that $\min\left\{-(\partial_t+\mathcal{D}^\varepsilon)\phi-\tilde{\mathcal{U}}(\phi_x),((1+\lambda)\,\partial_x-\partial_y)\phi,(\partial_y-(1-\lambda)\,\partial_x)\,\phi\right\}\ge\delta_0$ for some $\delta_0>0$. Set $\tau=\inf\{t|t\ge t_0,|X_t-x_0|\ge\epsilon_0,|y-y_0|\ge\epsilon_0,|z-z_0|\ge\epsilon_0\}$. By [1] Remark 1.2 we may also assume that $(\tau,X_\tau,Y_\tau,Z_\tau)$ is on the boundary of H, as any discontinuity can happen only due to trading. Then by dynamic programming principle for $t_0 \le t \le T$:

$$
\begin{aligned}
(1.1)\qquad & \phi(t_0,x_0,y_0,z_0) \\
&= \mathbb{E}_{t_0}\left[\mathrm{e}^{-\nu(t\wedge\tau-t_0)}\,\phi\left(t\wedge\tau, X_{t\wedge\tau}, Y_{t\wedge\tau}, Z_{t\wedge\tau}\right) - \int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(t\wedge\tau-t_0)}((\partial_t+\mathcal{D}^\varepsilon - C_s\partial_x)\phi)ds\right] \\
&+ \mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}((1+\lambda)\,\partial_x-\partial_y)\phi dL_s + \int_{t_0}^{t\wedge\tau}\left(\partial_y-(1-\lambda)\,\partial_x\right)\phi dM_s\right] \\
&\ge \mathbb{E}_{t_0}\left[\mathrm{e}^{-\nu(t\wedge\tau-t_0)}\,\widehat{V}\left(t\wedge\tau, X_{t\wedge\tau}, Y_{t\wedge\tau}, Z_{t\wedge\tau}\right) - \int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}((\partial_t+\mathcal{D}^\varepsilon)\phi - \tilde{\mathcal{U}}(\phi_x))ds\right] \\
&- \mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\left(\tilde{\mathcal{U}}(\phi_x)) - C_s\phi_x\right)ds\right] \\
&+ \mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}((1+\lambda)\,\partial_x-\partial_y)\phi dL_s + \int_{t_0}^{t\wedge\tau}\left(\partial_y-(1-\lambda)\,\partial_x\right)\phi dM_s\right] \\
&\ge \mathbb{E}_{t_0}\left[\mathrm{e}^{-\nu(t\wedge\tau-t_0)}\,\widehat{V}\left(t\wedge\tau, X_{t\wedge\tau}, Y_{t\wedge\tau}, Z_{t\wedge\tau}\right) + \int_{t_0}^{t\wedge T}\mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}(C_s)ds\right] \\
&+ \mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\left(\tilde{\mathcal{U}}(\phi_x) + C_s\phi_x - \mathcal{U}(C_s)\right)ds\right] + \delta_0\,\mathrm{e}^{-\nu(T-t_0)}\,\mathbb{E}_{t_0}\left[t\wedge\tau + L_{t\wedge\tau} + M_{t\wedge\tau}\right] \\
&\ge \kappa(t) + \mathbb{E}_{t_0}\left[\mathrm{e}^{-\nu(t\wedge\tau-t_0)}\,\widehat{V}\left(t\wedge\tau, X_{t\wedge\tau}, Y_{t\wedge\tau}, Z_{t\wedge\tau}\right) + \int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\,\mathcal{U}(C_s)ds\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa(t) = \inf_{(C,L,M)\in\mathcal{A}(t_0,x_0,y_0,z_0)} \Bigg\{ & \mathbb{E}_{t_0}\Bigg[\delta_0\,\mathrm{e}^{-\nu(T-t_0)}\left(t\wedge\tau + L_{t\wedge\tau} + M_{t\wedge\tau}\right) \\
& + \int_{t_0}^{t\wedge\tau}\mathrm{e}^{-\nu(s-t_0)}\left(\tilde{\mathcal{U}}(\phi_x) + C_s\phi_x - \mathcal{U}(C_s)\right)ds\Bigg]\Bigg\}
\end{aligned}
$$

where the infimum above is taken over all admissible strategies, for which $(\tau, X_\tau, Y_\tau, Z_\tau)$ is on the boundary of H. Maximizing the right-hand side of (1.1) over admissible strategies, we obtain that

$$
\phi(t_0,x_0,y_0,z_0) \ge \kappa(t) + \phi(t_0,x_0,y_0,z_0).
$$

We will now obtain contradiction by showing that $\kappa(t) > 0$ for $t > 0$ sufficiently small.

Using Doob's maximal inequality we obtain that for $\delta_1 > 0$

$$
\mathbb{P}\left(\max_{t_0\le s\le u\le t}\int_{s\wedge\tau}^{u\wedge\tau} f(Z_s)dB^1_s \ge \delta_1\right) \le \frac{4}{\delta_1^2}\mathbb{E}_{t_0}\left[\int_{t_0}^{t\wedge\tau} f^2(Z_s)ds\right] \le \frac{4\left(f^2\right)^*}{\delta_1^2}(t-t_0),
$$

where $\left(f^2\right)^* = \max_{z_0-\varepsilon_0\le z\le z_0+\varepsilon_0} f^2(z)$, and similarly $\mathbb{P}\left(\min_{t_0\le s\le u\le t}\int_{s\wedge\tau}^{u\wedge\tau} f(Z_s)dB^1_s \le -\delta_1\right) \le \frac{4\left(f^2\right)^*}{\delta_1^2}(t-t_0)$. Similar inequalities for the diffusion of the Z process give

$$
\mathbb{P}\left(\max_{t_0\le s\le u\le t}\frac{1}{\sqrt{\varepsilon}}\int_{s\wedge\tau}^{u\wedge\tau}\beta(Z_s)dB^2_s \ge \delta_1\right),\ \mathbb{P}\left(\min_{t_0\le s\le u\le t}\frac{1}{\sqrt{\varepsilon}}\int_{s\wedge\tau}^{u\wedge\tau}\beta(Z_s)dB^2_s \le -\delta_1\right) \le \frac{4(\beta^2)^*}{\varepsilon}(t-t_0),
$$

where $\left(\beta^2\right)^* = \max_{z_0-\varepsilon_0\le z\le z_0+\varepsilon_0}\beta^2(z)$.

Let also, $\left(f^2\right)_* = \min_{z_0-\varepsilon_0\le z\le z_0+\varepsilon_0} f^2(z)$, and $\alpha^* = \max_{z_0-\varepsilon_0\le z\le z_0+\varepsilon_0}|\alpha(z)|$. Thus define the set

$F(t) \subset \Omega$ as

$$
\begin{aligned}
F(t) = \Bigg\{ \max \Bigg\{ & \mathrm{e}^{(\mu + r - \frac{(f^2)_*}{2})(t-t_0) + \max_{t_0 \le s \le u \le t} \int_{s\wedge\tau}^{u\wedge\tau} f(Z_s) dB^1_s} - 1, \\
& 1 - \mathrm{e}^{(\mu + r - \frac{(f^2)^*}{2})(t-t_0) + \min_{t_0 \le s \le u \le t} \int_{s\wedge\tau}^{u\wedge\tau} f(Z_s) dB^1_s}, . \\
& \frac{1}{\varepsilon}\alpha^*(t-t_0) + \max_{t_0 \le s \le u \le t} \frac{1}{\sqrt{\varepsilon}} \int_{s\wedge\tau}^{u\wedge\tau} \beta(Z_s) dB_s, \\
& -\frac{1}{\varepsilon}\alpha^*(t-t_0) + \min_{t_0 \le s \le u \le t} \frac{1}{\sqrt{\varepsilon}} \int_{s\wedge\tau}^{u\wedge\tau} \beta(Z_s) dB_s \Bigg\} \le \min\{ \frac{\varepsilon_0}{2((|x_0| + |y_0|) \vee 1)}, 1\} \Bigg\}.
\end{aligned}
$$

The previous argument shows that $\mathbb{P}(F(t)) \nearrow 1$ as $t \searrow t_0$. Using the fact that

$$
\begin{aligned}
X_t &= x_0 \,\mathrm{e}^{r(t-t_0)} - \int_{t_0}^t \mathrm{e}^{r(t-s)} C_s ds - (1+\lambda) \int_{t_0}^t \mathrm{e}^{r(t-s)} dL_s + (1-\lambda) \int_{t_0}^t \mathrm{e}^{r(t-s)} dM_s, \\
Y_t &= y_0 \,\mathrm{e}^{(\mu+r)(t-t_0) - \frac{1}{2}\int_{t_0}^t f^2(Z_s) ds + \int_{t_0}^t f(Z_s) dB^1_s} \\
&\quad + \int_{t_0}^t \mathrm{e}^{(\mu+r)(t-u) - \frac{1}{2}\int_u^t f^2(Z_s) ds + \int_u^t f(Z_s) dB^1_s} (dL_u - dM_u),
\end{aligned}
$$

it follows that on $\{t \le \tau\}$

$$
\begin{aligned}
|X_t - x_0| &\le |x_0| \left(\mathrm{e}^{r(t-t_0)} - 1 \right) + \int_{t_0}^t \mathrm{e}^{r(t-s)} C_s ds + (1+\lambda) \int_{t_0}^t \mathrm{e}^{r(t-s)} dL_s + (1-\lambda) \int_{t_0}^t \mathrm{e}^{r(t-s)} dM_s, \\
|Y_t - y_0| &\le |y_0| \max \Bigg\{ \mathrm{e}^{(\mu + r - \frac{(f^2)_*}{2})(t-t_0) + \max_{t_0 \le s \le u \le t} \int_{s\wedge\tau}^{u\wedge\tau} f(Z_s) dB^1_s} - 1, \\
&\qquad 1 - \mathrm{e}^{(\mu + r - \frac{(f^2)^*}{2})(t-t_0) + \min_{t_0 \le s \le u \le t} \int_{s\wedge\tau}^{u\wedge\tau} f(Z_s) dB^1_s} \Bigg\} \\
&\quad + \int_{t_0}^t \mathrm{e}^{\left(\mu + r - \frac{(f^2)_*}{2}\right)(t-u) + \max_{t_0 \le s \le u \le t} \int_{s\wedge\tau}^{u\wedge\tau} f(Z_s) dB^1_s} (L_t + M_t).
\end{aligned} \tag{1.2}
$$

Next, since $Z_t - z_0 = \frac{1}{\varepsilon}\int_{t_0}^t \alpha(Z_s)\, ds + \frac{1}{\sqrt{\varepsilon}}\int_{t_0}^t \beta(Z_t)\, dB^2_s$, it follows $|Z_t - z_0| < \varepsilon_0$ on $F(t)$.

Assume also that $\varepsilon_0 > 0$ is small enough, so that $[0,T) \times \partial S \times \mathbb{R} \cap \partial H = \emptyset$. Then, similarly to Corollary 3.7 of [4], we also have that ϕ_x is bounded on H. Then let $\psi > \max_H (\mathcal{U}')^{-1}(\phi_x)$, and define $\varphi = \min_H \phi_x - \mathcal{U}'(\psi) > 0$, from the definition of ψ.

Then, by the convexity of $\mathcal{U}$, it follows that for any $C \ge 0$

$$
\tilde{\mathcal{U}}(\phi_x) + C\phi_x - \mathcal{U}(C) \ge \varphi(C - \psi)^+.
$$

Define $\Xi = \delta_0 \,\mathrm{e}^{-\nu(T-t_0)} (t \wedge \tau + L_{t\wedge\tau} + M_{t\wedge\tau}) + \int_{t_0}^{t\wedge\tau} \mathrm{e}^{-\nu(s-t_0)} \left(\tilde{\mathcal{U}}(\phi_x) + C_s\phi_x - \mathcal{U}(C_s) \right) ds$ to be a random variable.

Let $0 < t < \frac{\varepsilon_0}{16\psi}$ and small enough such that $\mathrm{e}^{rt} < 2$ and $\mathrm{e}^{rt} - 1 < \frac{\varepsilon_0}{2((|x_0|+|y_0|))}$. Then the set $F(t)$ consists of the following cases:

Case I: $\tau(\omega) \ge t$. In this case, $\Xi(\omega) \ge \delta_0 \,\mathrm{e}^{-\nu(T-t_0)} t$.

Case II: $\tau(\omega) < t$. In this case, we either have $|X_\tau - x_0| \ge \varepsilon_0$, or $|Y_\tau - y_0| \ge \varepsilon_0$.

Case II.a: $\tau(\omega) < t$, and $|X_\tau - x_0| \ge \varepsilon_0$. In this case, we must have that either $(1+\lambda)\int_{t_0}^\tau \mathrm{e}^{r(t-s)} dL_s + (1-\lambda)\int_{t_0}^\tau \mathrm{e}^{r(t-s)} dM_s \ge \frac{\varepsilon_0}{4}$ or $2\int_{t_0}^\tau C_s ds \ge \frac{\varepsilon_0}{4}$.

In the former case,

$$
\Xi \ge \delta_0 \,\mathrm{e}^{-\nu(T-t_0)} (L_{t\wedge\tau} + M_{t\wedge\tau}) \ge \delta_0 \,\mathrm{e}^{-\nu(T-t_0)} \frac{\varepsilon_0}{4(1+\lambda) \wedge (1-\lambda)}.
$$

In the latter case, by the choice of t

$$\frac{\varepsilon_0}{8} \leq \int_{t_0}^{\tau} C_s ds \leq \psi t + \int_{t_0}^{\tau} (C_s - \psi)^+ ds \leq \frac{\varepsilon_0}{16} + \int_{t_0}^{\tau} (C_s - \psi)^+ ds.$$

Thus, $\Xi \geq \mathrm{e}^{-\nu(T-t_0)} \frac{\varphi \varepsilon_0}{16}$.

Case II.b: $\tau(\omega) < t$, and $|Y_\tau - y_0| \geq \varepsilon_0$. In this case, on the set $F(t)$, from (1.2) we have that

$$|Y_t - y_0| \leq \frac{\varepsilon_0}{2} + 2(L_t + M_t).$$

Thus, we must have that $L_\tau + M_\tau \geq \frac{\varepsilon_0}{4}$, and thus $\Xi \geq \delta_0 \, \mathrm{e}^{-\nu(T-t_0)} \frac{\varepsilon_0}{4}$.

Together, on $F(t)$ we must have that $\Xi \geq \delta_0 \, \mathrm{e}^{-\nu(T-t_0)} \left(t \wedge \frac{\varepsilon_0}{4(1+\lambda)\wedge(1-\lambda)} \wedge \frac{\varphi\varepsilon_0}{16} \wedge \frac{\varepsilon_0}{4} \right)$. We reach a contradiction since,

$$\kappa(t) \geq \mathbb{P}(F(t))\delta_0 \, \mathrm{e}^{-\nu(T-t_0)} \left(t \wedge \frac{\varepsilon_0}{4(1+\lambda)\wedge(1-\lambda)} \wedge \frac{\varphi\varepsilon_0}{16} \wedge \frac{\varepsilon_0}{4} \right) > 0. \qquad \square$$

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