Portfolio Benchmarking under Drawdown Constraint and Stochastic Sharpe Ratio

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Abstract

We consider an investor who seeks to maximize her expected utility derived from her terminal wealth relative to the maximum wealth achieved over a fixed time horizon, and under a portfolio *drawdown* constraint, in a market with local stochastic volatility (LSV). The newly proposed investment objective paradigm also allows the investor to set portfolio benchmark targets. In the absence of closed-form formulas for the value function and optimal portfolio strategy, we obtain approximations for these quantities through the use of a coefficient expansion technique and nonlinear transformations. We utilize regularity properties of the *risk tolerance* function to numerically compute the estimates for our approximations. In order to achieve similar value functions, we illustrate that, compared to a constant volatility model, the investor must deploy a quite different portfolio strategy which depends on the current level of volatility in the stochastic volatility model.

Keywords and phrases. portfolio optimization, drawdown, stochastic volatility, local volatility

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1 Introduction

1.1 Background and motivation

In the vast and long-dated literature on dynamic portfolio optimization, different types of terminal utility paradigms under various portfolio constraints have been considered to understand the investor behaviour (see, for instance, Rogers [23] for a detailed exposition). The solutions to these problems provide optimal investment strategies which aid institutional investors, and at times help to reveal deep insights about the market observed phenomenons. The classical problem of continuous-time portfolio optimization dates back to Samuelson [24] and Merton [20, 19]. In his seminal paper, Merton [20] considered a market where the prices of risky assets are given by geometric Brownian motions (with *constant* volatilities), and the objective is to maximize the expected utility of terminal wealth by investing capital between the risky assets and a risk-free bank account. For constant relative risk aversion (CRRA) utility functions, the author showed that the optimal strategy is a "fixed mix" investment in the risky assets and the bank account.

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Merton's landmark result provided structural market insight but the restrictive problem setting – investor objective and market dynamics – prevented application of the results to practical situations. As a result, subsequent research has focused upon relaxing the assumptions made in [20], incorporating various market constraints and considering more realistic model settings.

Portfolio managers typically use a *stop-loss* level on the portfolio value to prevent a complete wipe-out of wealth in the face of falling prices. A very low value of the portfolio is a real concern which can be avoided by using the *drawdown* constraint. Under this constraint, the wealth in the portfolio must always remain above a certain fraction of the current maximum wealth value achieved. Furthermore, in several instances, portfolio managers commit to return a certain percentage of the starting wealth to the pooling investors. This situation can also be covered by imposing a drawdown constraint on the portfolio wealth.

In this article, we propose a new framework to study the dynamic portfolio optimization under a drawdown portfolio constraint in a stochastic volatility market model. In many empirical studies it has been well established that stochastic volatility is a reasonable asset price modelling tool to capture the market observed volatility smiles and volatility clustering. Our principal innovation is to introduce a new terminal investor objective paradigm which allows for a reduction in the dimensionality of the problem. As our central objective in this work is to numerically study the impact of stochastic volatility on the value function and optimal portfolio strategy, the dimensionality reduction serves as a crucial feature to allow for an efficient implementation of the numerical procedures used to solve the problem and study the effects of stochastic volatility.

1.2 Literature review

Several authors have considered the optimal portfolio problem under drawdown constraint. Grossman and Zhou [12] were the first to comprehensively study this problem over infinite time horizon in a market setting with single risky asset modelled as a geometric Brownian motion with constant volatility (*lognormal model*). They investigated to maximize the long term growth rate of the expected utility of the wealth and used dynamic programming principle to solve the problem. Cvitanic and Karatzas [8] streamlined the analysis of Grossman and Zhou [12] and extended the results to the case when there are multiple risky assets whose dynamics are governed by a lognormal model. By defining an auxiliary process, they were able to show that the solution of optimization problem with drawdown constraint can be linked to an unconstrained optimization problem whose solution follows from the work of Karatzas et al. [14]. They further showed that in the case of logarithmic utility function, the results hold even if the coefficients in the geometric Brownian motion model are random and satisfy some ergodicity conditions. More recently, Cherny and Obłój [6] studied the optimal portfolio problem in an abstract semimartingale model with a generalized drawdown constraint. They utilized the properties of Azéma-Yor processes to show that the value function of the constrained problem, where the investor objective is to maximize the long term growth rate of the expected utility, has the same value function as an unconstrained problem with a suitably modified utility function. Moreover, they showed that the optimal wealth process can also be obtained as an explicit path-wise transformation of the optimal wealth process in the unconstrained problem.

The portfolio optimization problem with drawdown constraint has also been studied in a continuous-time framework with consumption. Roche [22] studied the problem of maximizing the expected utility of consumption over an infinite time horizon for a power utility function under a linear drawdown constraint. This analysis was performed in the setting of a lognormal model with single asset. Elie and Touzi [10] subsequently generalized the result to a general class of utility functions in the setting of zero risk-free interest rate and obtained an explicit representation of the solution. Elie [9] also studied a finite time version of the same problem and in the absence of an analytical representation, provided a numerical solution to the problem.

In the financial literature, different problem settings with a drawdown constraint have received considerable attention due to their significance. Magdon-Ismail and Atiya [18] considered the problem of optimal portfolio choice when the drawdown is minimized in the single asset lognormal model. Chekhlov et al. [4] analyzed the portfolio optimization problem in discrete time where the investor objective is to maximize the expected return from the portfolio subject to risk constraints given in terms of drawdowns. They considered a multi-asset lognormal model and reduced the problem to a linear programming problem which can be solved numerically. In the insurance literature, drawdown constraint has been incorporated to study problems of lifetime investments. In [5], Chen et al. considered the optimization problem of minimizing the probability of a significant drawdown occurring over a lifetime investment, i.e. the probability that portfolio wealth hits the drawdown barrier before a random time which represents the death time of a client. A relevant benchmarking problem was studied by Boyle and Tian [2] in which the investor is concerned with selecting the optimal portfolio investment strategy such that over a finite time horizon, she obtains a return which beats a certain benchmark with a specified confidence level in a multi asset market model. This work extends the analysis of Basak and Shapiro [1] in which the investor is specificallty concerned about a value-at-risk constraint in a single risky asset model.

1.3 Our contributions

In this article, we consider an investor who at any time is worried about her wealth falling below a fixed fraction of the running maximum wealth. Furthermore, the investor also wishes to attain the portfolio value benchmark which she sets at the beginning of her investment period. Therefore, it is reasonable to consider a bounded terminal utility where the maximum value is achieved when the portfolio benchmark is attained. For this reason, we propose that the investor is interested to maximize utility of the ratio of the two quantities at the end of a fixed investment horizon. At the beginning of an investment period, the investor starts with a certain value of the initial wealth and fixes an initial value for the maximum wealth such that it satisfies the drawdown constraint. Note that this value of the maximum wealth also serves as the portfolio benchmark or target. An investor will liquidate the position in the risky asset if the maximum wealth target is reached.

We consider the basic setting of a frictionless financial market with a single underlying asset and a risk-free money market account. We study this problem in a stochastic volatility environment to demonstrate how uncertainty in the volatility impacts the optimal portfolio strategy. This problem has no explicit solution and thus, we look for accurate approximations to the value function and optimal strategy. We use the technique of coefficient expansion to formulate separate problems for different terms in the expansion of value function. The solutions to these problems allow us to derive an expansion for the optimal portfolio strategy. Due to the presence of portfolio constraints, the terms in the value function expansion are not available in closed-form. We numerically solve for the leading term in the value function expansion and use the regularity properties of the so-called *risk tolerance* function to compute the remaining higher order terms. The numerical estimates for the optimal portfolio strategy are derived similarly.

We show that the leading terms in the expansion of value function and optimal strategy are related to the solution of our problem in a lognormal model. The optimal strategy in this case suggests to liquidate the risky position when portfolio wealth approaches its maximum value. Also, close to the drawdown constraint, the optimal strategy instructs to steadily build up a position in the risky asset to drive away the portfolio value from the lower barrier. In the stochastic volatility model chosen for our numerical example, we observe that the stochastic volatility correction term for the value function approximation suggests very small loss or gain due to the uncertainty in volatility. However, depending on the current level of stochastic volatility, we observe that the optimal strategy approximation with volatility correction is remarkably different than the case with constant volatility. Close to the maximum wealth value, the corrected optimal strategy approximation suggests to hold onto the risky assets longer than in the constant volatility case. This clearly illustrates the impact of stochastic volatility on the optimal investment strategy. However, near the drawdown barrier, the behaviour of corrected optimal strategy approximation depends on the level of current stochastic volatility in the model when compared to the optimal strategy in the constant volatility case.

1.4 Organization

In Section 2 we introduce the continuous-time model setting and formulate the problem. We derive the Hamilton-Jacobi-Bellman equation for the optimal portfolio problem and under certain assumptions, give the analytical formula for the optimal portfolio strategy in terms of the value function. We provide the approximation formulas for the value function and optimal portfolio strategy in Section 3 and summarize our main results. In Section 4, we discuss the numerical implementation of our results and provide practical insights with the help of popular numerical examples considered in the literature. Section 5 concludes the article and suggests directions for future research. The proofs are included in Appendix A.

2 Problem Formulation

We consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ endowed with a two dimensional Brownian motion $W = \big((W_t^{(1)}, W_t^{(2)}), 0 \leq t \leq T\big)$. The filtration generated by W is denoted as $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$. Here $T < \infty$ is a finite time horizon. We suppose that there is a risky asset whose dynamics under $\mathbb{P}$ is given by the following local stochastic volatility (LSV) model:

$$\begin{aligned}
\frac{\mathrm{d}S_t}{S_t} &= \tilde{\mu}(S_t, Y_t)\mathrm{d}t + \tilde{\sigma}(S_t, Y_t)\mathrm{d}B_t^{(1)}, \\
\mathrm{d}Y_t &= c(Y_t)\mathrm{d}t + \beta(Y_t)\mathrm{d}B_t^{(2)},
\end{aligned}$$

where $B_t^{(1)} := W_t^{(1)}$ and $B_t^{(2)} := \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(2)}$ are standard Brownian motions under measure $\mathbb{P}$ with correlation $\rho \in [-1, 1] : \mathrm{d}\langle B_t^{(1)} B_t^{(2)}\rangle = \rho\, \mathrm{d}t$. From Itô's formula, the log price process $X = \log S$ is described as following:

$$\mathrm{d}X_t = b(X_t, Y_t)\mathrm{d}t + \sigma(X_t, Y_t)\mathrm{d}B_t^{(1)},$$

where $\mu(X_t, Y_t) := \tilde{\mu}(\mathrm{e}^{X_t}, Y_t), \sigma(X_t, Y_t) := \tilde{\sigma}(\mathrm{e}^{X_t}, Y_t)$ and

$$b(X_t, Y_t) := \mu(X_t, Y_t) - \frac{1}{2}\sigma^2(X_t, Y_t).$$

We assume that the model coefficient functions μ, σ, c and β are Borel-measurable and possess sufficient regularity to ensure that a unique strong solution exists for (X, Y) which is adapted to the augmentation of $\mathbb{F}$.

Further, we suppose the existence of a frictionless financial market with the price of a single risky asset given by S and the risk-free rate of interest given by a scalar constant $r > 0$. In this market, we denote the wealth process of an investor by $\bar{L}$ who invests $\bar{\pi}_t$ units of currency in risky asset S at time t and the remaining $(\bar{L}_t - \bar{\pi}_t)$ units of currency in the risk-free bank account. Then, the self-financing portfolio, $\bar{L}$ satisfies the following stochastic differential equation (SDE)

$$\begin{aligned}
\mathrm{d}\bar{L}_t &= r(\bar{L}_t - \bar{\pi}_t)\mathrm{d}t + \bar{\pi}_t\frac{\mathrm{d}S_t}{S_t} \\
&= \big(r\bar{L}_t + \bar{\pi}_t(\mu(X_t, Y_t) - r)\big)\, \mathrm{d}t + \bar{\pi}_t\sigma(X_t, Y_t)\, \mathrm{d}B_t^{(1)}.
\end{aligned}$$

The running maximum wealth in time t dollars is given by $\bar{M}_t := \max\{\bar{L}_s e^{r(t-s)}; s \le t\}$. In this work, we propose an investment framework that encourages exiting the market in the face of a sizeable drawdown, while also targeting a benchmark that is related to the running maximum, or high watermark of the investment performance. The investor's risk preferences are given by a utility function U satisfying:

Assumption 1. The terminal utility function $U : (0,1) \to \mathbb{R}$ is smooth. It is also strictly increasing and strictly concave.

We solve the utility maximization problem with the following *drawdown constraint*:

$$\bar{L}_t \ge \alpha \bar{M}_t \text{ a.s.}, \quad 0 \le t \le T, \quad \text{where } \alpha \in (0,1) \text{ is a fixed drawdown parameter.}$$

2.1 The discounted formulation

We look to formulate the problem in the setting where the wealth process is discounted with respect to the risk-free rate of interest. This allows us to clearly study the impact of stochastic volatility on the optimal strategy and value function. For this purpose, we define, $L_t := \bar{L}_t e^{-rt}$ and $M_t := \bar{M}_t e^{-rt} = \max\{L_s; s \le t\}$. The discounted wealth process satisfies the following SDE

$$dL_t = \pi_t\big((\mu(X_t, Y_t) - r)dt + \sigma(X_t, Y_t)dB_t^{(1)}\big),$$

where $\pi_t := e^{-rt}\bar{\pi}_t$ is the risky-asset trading strategy.

The investor's utility maximization problem under drawdown constraint is expressed by defining the value function as follows:

$$V(t,l,m,x,y) = \sup_{\pi \in \Pi_{\alpha,t,l,m,x,y}} \mathbb{E}\left[U\left(\frac{L_T}{M_T}\right) \Big| L_t = l, M_t = m, X_t = x, Y_t = y\right], \tag{1}$$

where the admissible strategies are given by

$$\begin{aligned}\Pi_{\alpha,t,l,m,x,y} :=& \Big\{\pi : \text{measurable }, \mathbb{F} - \text{adapted}, \mathbb{E}_{t,l,m,x,y} \int_t^T \pi_s^2 \sigma^2(X_s, Y_s) ds < \infty, \\ & \text{s.t. } L_s \ge \alpha M_s > 0 \text{ a.s.}, t \le s \le T\Big\}.\end{aligned}$$

For an integrable random variable Z on $(\Omega, \mathbb{F}, \mathbb{P})$, we have employed the short-hand notation $\mathbb{E}_{t,l,m,x,y}[Z]$ to denote the conditional expectation $\mathbb{E}[Z|L_t = l, M_t = m, X_t = x, Y_t = y]$ where (l,m,x,y) stands for the initial condition of the state processes (L, M, X, Y) with $l \le m$. Further, we define the domain in $\mathbb{R}_+ \times \mathbb{R}^4$ as $[0,T] \times \overline{\tilde{\mathcal{O}}}_\alpha$ where

$$\tilde{\mathcal{O}}_\alpha := \{(l,m,x,y) : 0 < \alpha m < l < m\}.$$

Here, $\overline{\mathcal{A}}$ denotes the closure of set $\mathcal{A}$. The definition (1) of the value function V is for any $5-$tuple $(t,l,m,x,y) \in [0,T] \times \overline{\tilde{\mathcal{O}}}_\alpha$. Next, we suppose the following:

Assumption 2. The value function given in (1), $V \in C^{1,2,1,2,2}([0,T] \times \overline{\tilde{\mathcal{O}}}_\alpha)$.

Then, under Assumption 2, by following the usual dynamic programming principle (see, for example, Pham [21, Chapter 3]), V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$(\partial_t + \mathcal{A})V + \sup_{\pi \in \mathbb{R}} \mathcal{A}^\pi V = 0, \tag{2}$$

where $(\mathcal{A}+\mathcal{A}^\pi)$ is the generator of the process (X,Y,L) with

$$\mathcal{A} = b(x,y)\frac{\partial}{\partial x} + c(y)\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2(x,y)\frac{\partial^2}{\partial x^2} + \frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial y^2} + \sigma(x,y)\beta(y)\rho\frac{\partial^2}{\partial x\partial y},$$
$$\mathcal{A}^\pi = \pi\Big[(\mu(x,y)-r)\frac{\partial}{\partial l} + \sigma^2(x,y)\frac{\partial^2}{\partial x\partial l} + \rho\sigma(x,y)\beta(y)\frac{\partial^2}{\partial y\partial l}\Big] + \frac{1}{2}\pi^2\sigma^2(x,y)\frac{\partial^2}{\partial l^2}.$$

In the above, for any $\mathcal{O}\subset\mathbb{R}$, $C^{1,n}([0,T]\times\mathcal{O})$ denotes the space of real-valued function f on $[0,T]\times\mathcal{O}$ whose partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial^i f}{\partial x^i}, 1\le i\le n$, exist and are continuous on $[0,T]\times\mathcal{O}$.

By inspecting the quadratic expression above in π, it is clear that the unique optimal strategy exists and is given by $\pi^* := \underset{\pi\in\mathbb{R}}{\arg\max}\,\mathcal{A}^\pi V$, i.e.,

$$\pi^* = -\frac{(\mu(x,y)-r)V_l + \rho\beta(y)\sigma(x,y)V_{yl} + \sigma^2(x,y)V_{xl}}{\sigma^2(x,y)V_{ll}}, \tag{3}$$

where the subscripts indicate partial derivatives with respect to the corresponding variables. The HJB equation becomes

$$(\partial_t+\mathcal{A})V + \tilde{\mathcal{N}}(V) = 0, \tag{4}$$

with the nonlinear term given as

$$\tilde{\mathcal{N}}(V) = -\frac{1}{2V_{ll}}\Big(\lambda(x,y)V_l + \sigma(x,y)V_{xl} + \rho\beta(y)V_{yl}\Big)^2,$$

where

$$\lambda(x,y) := \frac{\mu(x,y)-r}{\sigma(x,y)}$$

is the Sharpe ratio function. The boundary conditions are

$$\text{(Terminal condition):}\qquad V(T,l,m,x,y) = U\left(\frac{l}{m}\right), \tag{5}$$
$$\text{(Neumann condition):}\qquad V_m(t,m,m,x,y) = 0, \tag{6}$$
$$\text{(Dirichlet condition):}\qquad V(t,\alpha m,m,x,y) = U(\alpha). \tag{7}$$

The above Dirichlet condition signifies that when the drawdown constraint is hit, the investor stops trading in the risky asset ($\pi_t = 0$). In the discounted formulation when the investor stops trading, it signifies that the wealth process stops varying and the investor accepts the utility which is given at the drawdown barrier.

Remark 1. In the constant volatility case, the value function $V = V(t,l,m)$ does not depend on x and y, and the solution of the HJB equation (2) with the boundary conditions (5)–(7) can be obtained in the viscosity sense as introduced by Crandall et al. [7] (also, see Pham [21, Chapter 4] for a concise treatment). However, a similar viscosity solution analysis in the presence of stochastic volatility is not available in the literature. We do not pursue this direction as our aim is to provide numerical estimates for the value function and optimal strategy under stochastic volatility. In addition, the numerical analysis via finite difference schemes of the viscosity solution in stochastic volatility case will not be possible due to the high dimensionality of the problem. Thus, we suppose that Assumption 2 is valid, that is, the existence of a classical solution with sufficient regularity which allows us to apply the coefficient expansion method. A similar assumption has been made in the recent literature on portfolio optimization problems under stochastic parameters, for example, in Fouque et. al. [11], Liu and Muhle-Karbe [15] and Lorig and Sircar [16].

2.2 Dimensionality reduction

The nonlinear PDE in (4) with boundary conditions (5), (6) and (7) is difficult to solve numerically because the domain $\tilde{\mathcal{O}}_\alpha$ is a wedge in (L, M) space requiring a non-rectangular finite-difference grid. However, we notice that given the structure of our problem, we could perform a change of variable which reduces the dimensionality of the problem. We introduce

$$\xi = \frac{l}{m}, \qquad \text{and define} \qquad Q(t, \xi, x, y) := V(t, l, m, x, y),$$

which results in a new nonlinear PDE for $Q \in C^{1,2,2,2}\big([0, T] \times [\alpha, 1] \times \mathbb{R}^2\big)$:

$$(\partial_t + \mathcal{A})Q + \mathcal{N}(Q) = 0, \text{ on } [0, T) \times (\alpha, 1) \times \mathbb{R}^2, \tag{8}$$

where

$$\mathcal{N}(Q) = -\frac{1}{2Q_{\xi\xi}}\Big(\lambda(x, y)Q_\xi + \sigma(x, y)Q_{x\xi} + \rho\beta(y)Q_{y\xi}\Big)^2,$$

and the terminal and boundary conditions are

$$Q(T, \xi, x, y) = U(\xi), \quad Q_\xi(t, 1, x, y) = 0, \quad Q(t, \alpha, x, y) = U(\alpha). \tag{9}$$

Apart from providing a reduction in dimensionality, the above change of variable also transforms the space domain of the problem from a high-dimensional wedge to a semi-rectangular domain which typically helps to get more accurate numerical estimates for the solution.

3 Value Function and Optimal Strategy Approximation

Even under the lognormal model for the asset price, no closed form solution is available for the nonlinear PDE (8) and one needs to rely on accurate numerical approximations. In this paper, we propose to find an approximation for the value function as

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + \dots, \tag{10}$$

as well as an approximation for the optimal investment strategy

$$\pi^* = \pi_0 + \pi_1 + \pi_2 + \dots, \tag{11}$$

by using the coefficient expansion technique. This approach has been developed for the linear European option pricing problem in a general LSV model setting by Lorig et al. [17], and for the classical (unconstrained) Merton problem by Lorig and Sircar [16].

3.1 Coefficient polynomial expansions

The main idea of the coefficient expansion technique is to first fix a point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ and then for any function $\chi(x, y)$, which is locally analytic around $(\bar{x}, \bar{y})$, define the following family of functions indexed by $a \in [0, 1]$:

$$\chi^a(x, y) := \sum_{n=0}^{\infty} a^n \chi_n(x, y)$$

where

$$\chi_n(x, y) := \sum_{k=0}^{n} \chi_{n-k,k}(x - \bar{x})^{n-k}(y - \bar{y})^k, \quad \chi_{n-k,k} := \frac{1}{(n-k)!k!}\frac{\partial^{n-k}}{\partial x^{n-k}}\frac{\partial^k}{\partial y^k}\chi(x, y)\Big|_{x=\bar{x}, y=\bar{y}}.$$

Note that for $n = 0$, $\chi_0 := \chi_{0,0} = \chi(\bar{x}, \bar{y})$ is a constant. We can observe that $\chi^a\Big|_{a=1}$ is the Taylor series expansion of χ about the point $(\bar{x}, \bar{y})$. Here, a is seen as a *perturbation parameter* which is used to identify the successive terms in the approximation.

To apply this technique in PDE (8), we first replace each of the coefficient functions

$$\chi \in \{b, c, \sigma^2, \beta^2, \sigma\beta, \lambda, \sigma, \beta\}$$

with their respective series expansion for some $a \in (0, 1)$ and $(\bar{x}, \bar{y}) \in \mathbb{R}^2$. Next, to obtain approximations as in (10) and (11), we define a series expansion of value function as $Q = Q^a = \sum_{n=0}^{\infty} a^n Q^{(n)}$, linear operator $\mathcal{A} = \mathcal{A}^a = \sum_{n=0}^{\infty} a^n \mathcal{A}_n$ and replace the nonlinear operator $\mathcal{N}(Q)$ by $\mathcal{N}^a(Q^a)$ which involves series expansions for the coefficient functions and the value function. Then from (8), we consider the PDE problem

$$(\partial_t + \mathcal{A}^a)Q^a + \mathcal{N}^a(Q^a) = 0, \text{ on } [0, T) \times (\alpha, 1) \times \mathbb{R}^2, \tag{12}$$

with the boundary conditions

$$Q^a(T, \xi, x, y) = U(\xi), \quad Q^a_\xi(t, 1, x, y) = 0, \quad Q^a(t, \alpha, x, y) = U(\alpha). \tag{13}$$

Now, to obtain the successive terms of approximation in expansions (10) and (11), we compare the corresponding degree terms in the polynomial of perturbation parameter a in (12) and the boundary conditions (13). The approximations are then obtained by setting $a = 1$ and choosing a particular value of $(\bar{x}, \bar{y})$ as different choices provide different approximations.

3.2 Zeroth and first order approximation

The first term in approximation (10) is obtained by collecting the zeroth order terms w.r.t. a in the expansion of (12). We get

$$(\partial_t + \mathcal{A}_0)Q^{(0)} - \frac{1}{2Q^{(0)}_{\xi\xi}} \left(\lambda_0 Q^{(0)}_\xi + \rho\beta_0 Q^{(0)}_{y\xi} \right)^2 = 0,$$

with

$$\mathcal{A}_0 := b_0 \frac{\partial}{\partial x} + c_0 \frac{\partial}{\partial y} + \frac{1}{2}\sigma_0^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\beta_0^2 \frac{\partial^2}{\partial y^2} + \rho\sigma_0\beta_0 \frac{\partial^2}{\partial x \partial y}, \tag{14}$$

and the corresponding order boundary conditions are

$$Q^{(0)}(T, \xi, x, y) = U(\xi), \quad Q^{(0)}_\xi(t, 1, x, y) = 0, \quad Q^{(0)}(t, \alpha, x, y) = U(\alpha).$$

As the linear operator $\mathcal{A}_0$ has only constant coefficients and the boundary conditions do not depend on (x, y), the solution $Q^{(0)}(t, \xi, x, y)$ is independent of (x, y). Therefore, in this case we get:

Definition 1. *The leading order term $Q^{(0)} = Q^{(0)}(t, \xi)$ satisfies the following nonlinear PDE*

$$Q^{(0)}_t - \frac{1}{2}\lambda_0^2 \frac{\left(Q^{(0)}_\xi\right)^2}{Q^{(0)}_{\xi\xi}} = 0, \text{ on } [0, T) \times (\alpha, 1), \tag{15}$$

with the boundary conditions

$$Q^{(0)}(T, \xi) = U(\xi), \quad Q^{(0)}(t, \alpha) = U(\alpha), \quad Q^{(0)}_\xi(t, 1) = 0. \tag{16}$$

Remark 2. Due to the presence of boundary conditions, an explicit formula for $Q^{(0)}$ is inaccessible, even for a power utility function. In Elie [9], the author performed a classical viscosity solution analysis by suitably adapting the ideas proposed in Zariphopoulou [25] to obtain a viscosity solution for an HJB equation which closely resembles the PDE in (15) with boundary conditions (16). By formulating our utility maximization problem in the setting of a lognormal model with constant Sharpe ratio λ_0 (as shown later in the proof of Lemma 1), we can repeat the arguments of Elie [9] to obtain a viscosity solution for the nonlinear PDE which is similar to (15), i.e. the PDE in (t, l, m) space without the dimensionality reduction. Unlike in [9], where *asymptotic elasticity* of U has to be smaller than $1-\alpha$ for the existence of the viscosity solution, we do not require such an assumption as the utility always remains bounded in our setting. To finally obtain the viscosity solution for PDE in (15), we use the dimensionality reduction as defined in Section 2.2 and verify that the conditions required for its existence and uniqueness remain satisfied.

However, to use our approach for numerical approximations, we need classical regular solutions to PDE (15) and for this reason we make the following assumption throughout:

Assumption 3. The PDE problem (15)-(16) has a unique classical solution $Q^{(0)} \in C_b^{1,5}([0,T) \times [\alpha, 1])$, that is $Q^{(0)}$ has at least five derivatives in ξ which are continuous and bounded up to the boundaries at $\xi = \alpha, 1$.

In the unconstrained case, with no drawdown restrictions, the PDE (15) is simply the constant Sharpe ratio Merton value function PDE on the half-space $\xi > 0$, where ξ would denote the wealth level. As is well-known, given a smooth and strictly concave utility function satisfying the usual conditions ($U'(0^+) = \infty$ and $U'(\infty) = 0$), smoothness of the value function follows from Legendre transform to a linear parabolic PDE. In our restricted drawdown problem we assume regularity of the solution when restricted to a finite domain. Our value function approximation, summarized in Section 3.2.2, and our optimal portfolio approximation in Section 3.3, are given in terms of (up to 5th order) partial derivatives of $Q^{(0)}$.

In order to find the first order correction term, we introduce the following *risk tolerance* function

$$\mathcal{R}(t, \xi) := \left(-\frac{Q_{\xi}^{(0)}}{Q_{\xi\xi}^{(0)}} \right) (t, \xi). \tag{17}$$

This function has been well studied in the unconstrained case by Källblad and Zariphopoulou [13] and has been recently used to study the classical Merton problem in a stochastic volatility environment by Fouque et al. [11]. It satisfies an autonomous PDE of fast-diffusion type:

Proposition 1. *The risk tolerance function $\mathcal{R}(t, \xi)$ satisfies the nonlinear PDE*

$$\mathcal{R}_t + \frac{1}{2}\lambda_0^2 \mathcal{R}^2 \mathcal{R}_{\xi\xi} = 0, \ \text{on } [0,T) \times (\alpha, 1), \tag{18}$$

with the boundary conditions

$$\mathcal{R}(T, \xi) = -\frac{U'(\xi)}{U''(\xi)}, \quad \mathcal{R}(t, \alpha) = 0, \quad \mathcal{R}(t, 1) = 0. \tag{19}$$

The proof is given in Appendix A.1.

As we show later in Section 3.3, Proposition 1 is also crucial to compute the leading order terms in the approximation of optimal strategy π^*. Next, we define the differential operators

$$\mathcal{D}_k := \mathcal{R}^k \frac{\partial^k}{\partial \xi^k}, \quad k = 1, 2, \ldots, \tag{20}$$

which allows us to write equation (15) as

$$\Big(\partial_t + \frac{\lambda_0^2}{2}\mathcal{D}_2 + \lambda_0^2\mathcal{D}_1\Big)Q^{(0)} = 0. \tag{21}$$

To obtain first order correction term to the value function, we collect the first order terms w.r.t. a in expansion (12). As $Q^{(0)}$ does not depend on y, the linear term contributes

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0\right) Q^{(1)},$$

and the nonlinear term contributes

$$\lambda_0^2\mathcal{D}_1 Q^{(1)} + \frac{1}{2}\lambda_0^2\mathcal{D}_2 Q^{(1)} + \lambda_0\lambda_1\mathcal{D}_1 Q^{(0)} + \beta_0\lambda_0\rho\mathcal{D}_1\frac{\partial}{\partial y}Q^{(1)} + \sigma_0\lambda_0\mathcal{D}_1\frac{\partial}{\partial x}Q^{(1)}.$$

Definition 2. *The first order correction term $Q^{(1)}$ satisfies the following PDE*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\right) Q^{(1)} + S_1 = 0, \text{ on } [0,T) \times (\alpha, 1) \times \mathbb{R}^2, \tag{22}$$

with linear operator $\mathcal{B}_0$ given as

$$\mathcal{B}_0 := \lambda_0^2\mathcal{D}_1 + \frac{1}{2}\lambda_0^2\mathcal{D}_2 + \beta_0\lambda_0\rho\mathcal{D}_1\frac{\partial}{\partial y} + \sigma_0\lambda_0\mathcal{D}_1\frac{\partial}{\partial x},$$

and the source term

$$S_1 = \big(\frac{1}{2}\lambda^2\big)_1(x,y)\mathcal{D}_1 Q^{(0)}(t,\xi).$$

The terminal and boundary conditions (13) for Q^a are already satisfied by $Q^{(0)}$, and so we have

$$Q^{(1)}(T,\xi,x,y) = 0, \quad Q_\xi^{(1)}(t,1,x,y) = 0, \quad Q^{(1)}(t,\alpha,x,y) = 0. \tag{23}$$

3.2.1 Explicit expression for the first order correction term

We now employ a transformation that enables us to find an explicit expression for $Q^{(1)}$ in terms of the partial derivatives of $Q^{(0)}$. For this purpose, we first note that $Q_\xi^{(0)}$ is a monotone function from the following result on the zeroth order term.

Lemma 1. *$Q^{(0)}(t,\xi)$ is a non-decreasing and concave function in ξ variable.*

The proof is given in Appendix A.2. This result allows us to define a change of variable.

Definition 3. *On $[0,T] \times [\alpha, 1]$, define,*

$$z(t,\xi) := -\log Q_\xi^{(0)}(t,\xi) + \frac{1}{2}\lambda_0^2(T-t),$$
$$\psi(t) := -\log Q_\xi^{(0)}(t,\alpha) + \frac{1}{2}\lambda_0^2(T-t), \quad \varphi(t) := -\log Q_\xi^{(0)}(t,1) + \frac{1}{2}\lambda_0^2(T-t),$$

and let

$$q^{(0)}(t, z(t,\xi)) := Q^{(0)}(t,\xi).$$

It is clear from the boundary condition (16) that we have $\varphi(t) = \infty$ for all $0 \le t < T$. Then, we obtain the following PDE problem for $q^{(0)}(t,z)$:

Proposition 2. *$q^{(0)}(t,z)$ satisfies the following linear PDE*

$$\Big(\frac{\partial}{\partial t}+\frac{1}{2}\lambda_0^2\frac{\partial}{\partial z^2}\Big)q^{(0)}=0,\ \ \textit{on}\ [0,T)\times(\psi(t),\infty),$$

with the terminal and boundary conditions

$$q^{(0)}(T,z)=U\Big(\big(U'\big)^{-1}\big(\mathrm{e}^{-z}\big)\Big),\quad \lim_{z\to\infty}q_z^{(0)}(t,z)=0,\quad q^{(0)}(t,\psi(t))=U\Big(\big(U'\big)^{-1}\big(\mathrm{e}^{-\psi(t)+\frac{\lambda_0^2}{2}(T-t)}\big)\Big).$$

The proof is given in Appendix A.3.

Lemma 2. *Denote $q\big(t,z(t,\xi),x,y\big):=\hat{Q}(t,\xi,x,y)$. Then, on $[0,T)\times(\psi(t),\infty)\times\mathbb{R}^2$, we have*

$$\left(\frac{\partial}{\partial t}+\mathcal{A}_0+\mathcal{B}_0\right)\hat{Q}=\left(\frac{\partial}{\partial t}+\mathcal{A}_0+\mathcal{C}_0\right)q,$$

where

$$\mathcal{C}_0=\frac{1}{2}\lambda_0^2\frac{\partial^2}{\partial z^2}+\rho\beta_0\lambda_0\frac{\partial^2}{\partial y\partial z}+\sigma_0\lambda_0\frac{\partial^2}{\partial x\partial z}. \tag{24}$$

The above result follows from the calculations performed in the proof of Proposition 2 (also see [16, Lemma 3.3]).

Next, we set $\hat{Q}=Q^{(0)}$ and $q=q^{(0)}$ in Lemma 2. Further we know that $q^{(0)}$ does not depend on (x,y) and $\mathcal{A}_0$ and the last two terms in $\mathcal{C}_0$ have derivatives w.r.t. (x,y). Then, we get the constant coefficient heat equation as in Proposition 2 by applying the operator $\mathcal{C}_0$. On $[0,T)\times(\psi(t),\infty)$, we have

$$\left(\frac{\partial}{\partial t}+\mathcal{A}_0+\mathcal{C}_0\right)q^{(0)}=0.$$

Finally, we define $q^{(1)}$ from $Q^{(1)}$ as

$$q^{(1)}(t,z(t,\xi),x,y):=Q^{(1)}(t,\xi,x,y). \tag{25}$$

Proposition 3. *The alternative representation $q^{(1)}(t,z,x,y)$ of the first order correction term satisfies*

$$\left(\frac{\partial}{\partial t}+\mathcal{A}_0+\mathcal{C}_0\right)q^{(1)}+\mathcal{S}_1=0,\ \ \textit{on}\ [0,T)\times(\psi(t),\infty)\times\mathbb{R}^2, \tag{26}$$

where

$$\mathcal{S}_1(t,z,x,y)=\big(\frac{1}{2}\lambda^2\big)_1(x,y)q_z^{(0)}(t,z,x,y). \tag{27}$$

The boundary conditions are

$$q^{(1)}(T,z,x,y)=0,\quad \lim_{z\to\infty}q_z^{(1)}(t,z,x,y)=0,\quad q^{(1)}(t,\psi(t),x,y)=0. \tag{28}$$

The above result follows from Definition 2. The solution to (26) with boundary conditions (28) is given in terms of derivatives of $q^{(0)}$ in the following proposition.

Proposition 4. *The solution of the PDE in* (26) *with boundary conditions* (28) *is given by*

$$q^{(1)}(t,z,x,y) = (T-t)\lambda_0 A(t,x,y) q_z^{(0)}(t,z) + \frac{1}{2}(T-t)^2 \lambda_0 B q_{zz}^{(0)}(t,z), \tag{29}$$

where

$$\begin{aligned} A(t,x,y) &= \lambda_{1,0}\left[(x-\bar{x}) + \frac{1}{2}(T-t)b_0\right] + \lambda_{0,1}\left[(y-\bar{y}) + \frac{1}{2}(T-t)c_0\right], \\ B &= \lambda_{1,0}\sigma_0\lambda_0 + \lambda_{0,1}\rho\beta_0\lambda_0. \end{aligned}$$

In the original variables, $Q^{(1)}$*, the solution of* (22) *with terminal and boundary conditions* (23)*, is given by*

$$Q^{(1)}(t,\xi,x,y) = (T-t)\lambda_0 A(t,x,y)\mathcal{D}_1 Q^{(0)} + \frac{1}{2}(T-t)^2\lambda_0 B\left(\mathcal{D}_3 - 2\mathcal{D}_1\right) Q^{(0)}. \tag{30}$$

The proof is given in Appendix A.4.

3.2.2 Summary of the first order value function approximation results

The coefficient polynomial approximation to the value function Q, solution to the PDE problem (8)-(9), is then defined by setting $a=1$: $Q \approx Q^{(0)} + Q^{(1)}$, where

- Zeroth order term: $Q^{(0)}(t,\xi)$ is estimated by numerically solving (15) with the boundary conditions (16).
- First order term: $Q^{(1)}(t,\xi,x,y)$ is obtained from Proposition 4 and is given by (30).

3.3 Optimal strategy approximation

Once we have the estimates for $Q^{(0)}$ and $Q^{(1)}$ in expansion (10) of the value function Q^a, we can find the first order approximation of the optimal strategy π^* from the formula in (3). In terms of $Q^a(t,\xi,x,y)$, the optimal strategy is given as

$$\pi^{*,a}(t,l,m,x,y) = -m\left[\frac{(\mu^a(x)-r)Q^a_\xi}{(\sigma^a(x,y))^2 Q^a_{\xi\xi}} + \frac{\rho\beta(y)Q^a_{y\xi}}{\sigma^a(x,y)Q^a_{\xi\xi}} + \frac{Q^a_{x\xi}}{Q^a_{\xi\xi}}\right], \text{ with } \xi = \frac{l}{m}.$$

To express the approximation for π^* in terms of $\mathcal{R}, Q^{(0)}$ and their spatial derivatives, we first replace Q^a by $Q^{(0)} + aQ^{(1)}$ in the above formula, use the results in (30) and following Lemma 3 and then set $a=1$.

Lemma 3. *From the definition* (17) *of* $\mathcal{R}$*, we have the following identities:*

(i) $(\mathcal{D}_1 + \mathcal{D}_2)\mathcal{D}_1 Q^{(0)} = \mathcal{R}\mathcal{R}_{\xi\xi}\mathcal{D}_1 Q^{(0)}$,

(ii) $(-2\mathcal{D}_1 + \mathcal{D}_3)Q^{(0)} = \mathcal{D}_1\mathcal{D}_1 Q^{(0)}$,

(iii) $\big(\mathcal{D}_1 + \mathcal{D}_2\big)\mathcal{D}_1\mathcal{D}_1 Q^{(0)} = \mathcal{R}\big(\mathcal{R}_{\xi\xi}(3\mathcal{R}_\xi - 2) + \mathcal{R}\mathcal{R}_{\xi\xi\xi}\big)\mathcal{D}_1 Q^{(0)}$.

Proof. We show the following using elementary manipulations. From (17) and (20), recall that

$$\mathcal{R} = -\frac{Q^{(0)}_\xi}{Q^{(0)}_{\xi\xi}}, \quad \mathcal{D}_k = \mathcal{R}^k \frac{\partial^k}{\partial \xi^k}, \quad k = 1, 2, \ldots.$$

(i) We have,

$$\mathcal{D}_1\mathcal{D}_1Q^{(0)} = \mathcal{D}_1(\mathcal{R}Q^{(0)}_\xi) = \mathcal{R}\mathcal{R}_\xi Q^{(0)}_\xi + \mathcal{R}^2 Q^{(0)}_{\xi\xi} = (\mathcal{R}_\xi - 1)\mathcal{D}_1 Q^{(0)}, \text{ and}$$
$$\mathcal{D}_2\mathcal{D}_1Q^{(0)} = \mathcal{R}\mathcal{R}_{\xi\xi}\mathcal{D}_1Q^{(0)} - (\mathcal{R}_\xi - 1)\mathcal{D}_1Q^{(0)}.$$

The above result and the distributive property of $\mathcal{D}_k$ operator completes the proof.
(ii) We have,

$$\mathcal{D}_3Q^{(0)} = \mathcal{R}^3\partial_\xi\Big(-\frac{Q^{(0)}_\xi}{\mathcal{R}}\Big) = \mathcal{R}^3\Big(-\frac{Q^{(0)}_{\xi\xi}}{\mathcal{R}} + \frac{Q^{(0)}_\xi\mathcal{R}_\xi}{\mathcal{R}^2}\Big) = (\mathcal{R}_\xi + 1)\mathcal{D}_1Q^{(0)}.$$

This gives,

$$-2\mathcal{D}_1Q^{(0)} + \mathcal{D}_3Q^{(0)} = -2\mathcal{D}_1Q^{(0)} + (\mathcal{R}_\xi + 1)\mathcal{D}_1Q^{(0)} = (\mathcal{R}_\xi - 1)\mathcal{D}_1Q^{(0)}.$$

The final conclusion follows from (i).
(iii) Using the previous calculations, we get

$$\begin{aligned}
\mathcal{D}_1\big((\mathcal{R}_\xi - 1)\mathcal{D}_1Q^{(0)}\big) &= \mathcal{R}^2\mathcal{R}_{\xi\xi}Q^{(0)}_\xi + (\mathcal{R}_\xi - 1)\mathcal{D}_1\mathcal{D}_1Q^{(0)} = \mathcal{R}\mathcal{R}_{\xi\xi}\mathcal{D}_1Q^{(0)} + (\mathcal{R}_\xi - 1)^2\mathcal{D}_1Q^{(0)},\\
\mathcal{D}_2\big((\mathcal{R}_\xi - 1)\mathcal{D}_1Q^{(0)}\big) &= \mathcal{R}\mathcal{D}_1\big(\mathcal{R}\mathcal{R}_{\xi\xi}Q^{(0)}_\xi + (\mathcal{R}_\xi - 1)^2Q^{(0)}_\xi\big)\\
&= \mathcal{R}\big(\mathcal{R}\mathcal{R}_{\xi\xi\xi} + \mathcal{R}_{\xi\xi}(\mathcal{R}_\xi - 1)\big)\mathcal{D}_1Q^{(0)} + \mathcal{R}\mathcal{D}_1\big((\mathcal{R}_\xi - 1)^2Q^{(0)}_\xi\big)\\
&= \mathcal{R}\big(\mathcal{R}\mathcal{R}_{\xi\xi\xi} + 3\mathcal{R}_{\xi\xi}(\mathcal{R}_\xi - 1)\big)\mathcal{D}_1Q^{(0)} - (\mathcal{R}_\xi - 1)^2\mathcal{D}_1Q^{(0)}.
\end{aligned}$$

The sum of above two results concludes the proof. ☐

Thus, we obtain the optimal strategy approximation as

$$\begin{aligned}
\pi^* \approx m\Bigg[&\frac{(\mu(x,y) - r)}{(\sigma(x,y))^2}\mathcal{R} + (T-t)\lambda_0 A(t,x,y)\frac{(\mu(x,y) - r)}{(\sigma(x,y))^2}\mathcal{R}^2\mathcal{R}_{\xi\xi}\\
&+ \frac{1}{2}(T-t)^2\lambda_0 B\frac{(\mu(x,y) - r)}{(\sigma(x,y))^2}\mathcal{R}^2\big(\mathcal{R}_{\xi\xi}(3\mathcal{R}_\xi - 2) + \mathcal{R}\mathcal{R}_{\xi\xi\xi}\big)\\
&+ (T-t)\lambda_0\Big(\frac{\lambda_{0,1}\rho\beta(y)}{\sigma(x,y)} + \lambda_{1,0}\Big)\mathcal{R}(\mathcal{R}_\xi - 1)\Bigg].
\end{aligned} \tag{31}$$

3.4 Higher order terms and accuracy of the approximation

To obtain higher order terms in the value function approximation (10), we first write the PDEs associated with $Q^{(n)}(t,\xi,x,y)$ as follows:

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\Big)Q^{(n)} + S_n = 0, \text{ on } [0,T)\times(\alpha,1)\times\mathbb{R}^2,$$

with the terminal and boundary conditions

$$Q^{(n)}(T,\xi,x,y) = 0, \quad Q^{(n)}_\xi(t,1,x,y) = 0, \quad Q^{(n)}(t,\alpha,x,y) = 0.$$

The source term S_n depends only on $Q^{(k)}$ $(k \le n-1)$ and its derivatives. This follows from the analysis of Section 4 in [16]. Furthermore, following the calculations in [16], if we define $q^{(n)}$ from $Q^{(n)}$ as

$$q^{(n)}(t,z,x,y) = Q^{(n)}(t,\xi,x,y),$$

by using Definition 3, we get constant coefficient equation for $q^{(n)}$:

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0\right) q^{(n)} + \mathcal{S}_n = 0, \text{ on } [0,T) \times (\psi(t), \infty) \times \mathbb{R}^2, \tag{32}$$

with the terminal and boundary conditions

$$q^{(n)}(T,z,x,y) = 0, \quad \lim_{z\to\infty} q_z^{(n)}(t,z,x,y) = 0, \quad q^{(n)}(t,\psi(t),x,y) = 0. \tag{33}$$

In Proposition 4, we obtained an explicit expression for the transformed first-order function $q^{(1)}$ in terms of a differential operator acting on $q^{(0)}$. However, for the higher order terms $q^{(n)}$ $(n \geq 2)$, this may not be possible as the source term $\mathcal{S}_n(t,z,x,y)$ calculated from $S_n(t,\xi,x,y)$ is composed of products and quotients of derivatives of $q^{(k)}(t,z,x,y)$ $(k \leq n-1)$. As shown in Lemma 4.1 [16], we need $q^{(0)}$ to have a specific form which allows to obtain higher order terms $q^{(n)}$ as a differential operator $\mathcal{L}_n$ acting on $q^{(0)}$, where $\mathcal{L}_n$ has coefficients that are polynomials in (x,y) and independent of z. Since, in our setting, we do not have a closed form formula for $q^{(0)}$, it is not possible to derive such expressions for higher order terms $q^{(n)}$ $(n \geq 2)$. Instead, the contribution of the higher order terms can be evaluated by first imposing further smoothness condition on the zeroth order term $Q^{(0)}$ and then numerically solving the PDEs of the type (32) with boundary conditions (33).

Therefore, in the absence of formulas for higher order terms $Q^{(n)}, n \geq 2$, it is difficult to compare the accuracy of the first order approximation with respect to higher order approximations. But intuitively it is clear that it performs better than the zeroth order approximation which is also made clear through a numerical comparison in Section 4.

4 Examples and Numerical Implementation

In this section, we consider the stochastic volatility model as in Chacko and Viceira [3] with their calibrated set of parameters and provide a detailed discussion of the application of our results obtained in Section 3. Even in the case of constant parameters, as we do not have explicit expressions to test the numerical accuracy, we demonstrate the superior performance of the first order optimal strategy approximation with respect to the zeroth order term in the approximation. We discuss the effect of stochastic volatility on the value function and optimal strategy for the case of *power* utility function and a *mixture* of two power utility functions, as introduced in [11]. The latter allows for relative aversion that declines with wealth, while for the former it is constant across wealth levels. Under the considered stochastic volatility model [3, Section 1], the coefficients (μ, σ, c, β) of Section 2 are independent of x and are given as

$$\mu(y) = \mu, \quad \sigma(y) = \frac{1}{\sqrt{y}}, \quad c(y) = \kappa(\theta - y), \quad \beta(y) = \delta\sqrt{y}.$$

The market calibrated values of the constants involved are:

$\mu - r$	κ	θ	δ	ρ
0.0811	0.3374	27.9345	0.6503	0.5241

Also, we set $\alpha = 0.4$ and $T = 1.0$. We first need to compute the estimates for zeroth order term $Q^{(0)}$ whose partial differential equation is degenerate. Hence, we choose explicit finite difference scheme to obtain its numerical estimates. We approximate the domain $[0,T] \times [\alpha, 1]$ with a uniform mesh of time step Δt and space step $\Delta\xi$. By setting $N\Delta t = T$ and $J\Delta\xi = (1-\alpha)$, the discretization grid is given as

$$\mathcal{M} = \left\{(t^n, \xi_j) : n = 0,1,\ldots,N, j = 0,1,\ldots,J\right\}, \quad t^n = T - n\Delta t, \quad \xi_j = \alpha + j\Delta\xi,$$

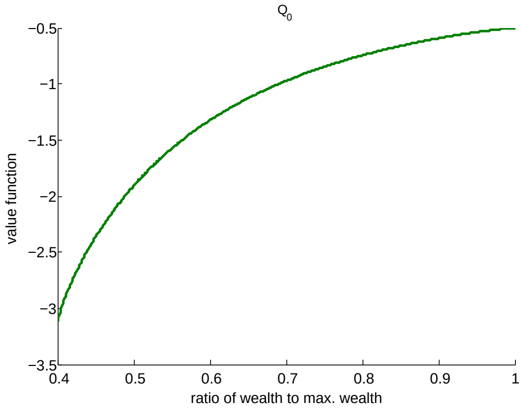
where Δt is of the order $(\Delta\xi)^2$ (monotonicity condition) to ensure convergence of the scheme. Let $\tilde{Q}_j^n$ denote the numerical approximation of $Q^{(0)}(t^n, \xi_j)$. Then the discretized equation for $Q^{(0)}$ in the interior is written as

$$\tilde{Q}_j^{n+1} = \tilde{Q}_j^n - \frac{1}{8}\lambda_0^2 \Delta t \frac{(\tilde{Q}_{j+1}^n - \tilde{Q}_{j-1}^n)^2}{(\tilde{Q}_{j+1}^n - 2\tilde{Q}_j^n + \tilde{Q}_{j-1}^n)}. \tag{34}$$

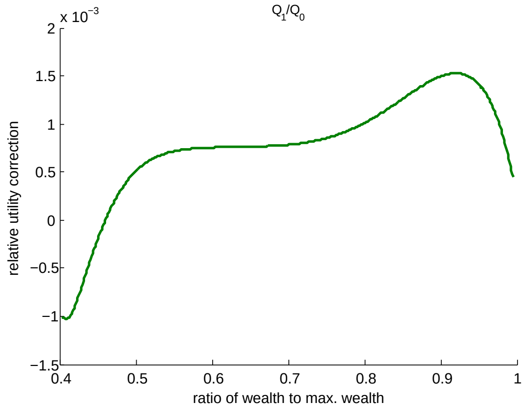
We start with the guess $\tilde{Q}_j^0 = U(\xi_j)$, for all $j = 0, 1, \ldots, J$, and the boundary conditions are

$$\tilde{Q}_J^{n+1} = \tilde{Q}_{J-1}^{n+1}, \text{ and, } \tilde{Q}_0^{n+1} = U(\xi_0).$$

In Figure 1(a) and 2(a), we plot the numerical solution for the leading order expansion term



(a)



(b)

Figure 1: *Numerical solutions to (a) zeroth order value function* $Q^{(0)}$ *(b) relative utility correction* $Q^{(1)}/Q^{(0)}$. *Utility function used* $U(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}, \gamma = 3.0$

$Q^{(0)}$ obtained from (34). We can see that the zeroth order term is concave and non-decreasing as expected from Lemma 1. To find the first order correction term (30), we use the risk-tolerance function $\mathcal{R}$ from Proposition 1 and Lemma 3 in the formula instead of using the derivatives of $Q^{(0)}$ to avoid high order numerical differentiation. We note that to obtain $Q^{(1)}$, we need to set the value for reference level $\bar{y}$. We choose to set $\bar{y} = y$, which gives us a particular correction term. We get

$$\begin{aligned} Q^{(1)} &= \left(\frac{1}{2}\lambda^2\right)_{0,1}(T-t)^2 c_0 \mathcal{R} Q_\xi^{(0)} + \left(\frac{1}{2}\lambda^2\right)_{0,1}(T-t)^2 \lambda_0 \rho \beta_0 \left(-2\mathcal{R} Q_\xi^{(0)} + \mathcal{R}^3 \partial_\xi^3 Q^{(0)}\right) \\ &= \frac{1}{2}(\mu - r)^2 (T-t)^2 \Big[\kappa(\theta - y)\mathcal{R} Q_\xi^{(0)} + \rho\delta(\mu - r) y \left(-2\mathcal{R} Q_\xi^{(0)} + \mathcal{R}^3 \partial_\xi^3 Q^{(0)}\right)\Big]. \end{aligned}$$

We use the regularity properties of $\mathcal{R}$ and $Q^{(0)}$ to compute the above expression. We obtain estimates of $\mathcal{R}$ by numerically solving (18) with boundary conditions (19) via explicit finite difference scheme. We define the discretization grid as in (34) and let $\tilde{R}_j^n$ denote the numerical approximation of $\mathcal{R}(t^n, \xi_j)$. The discretized equation in the interior is written as

$$\tilde{R}_j^{n+1} = \tilde{R}_j^n + \frac{1}{2}\lambda_0^2 \Delta t (\tilde{R}_j^n)^2 \frac{(\tilde{R}_{j+1}^n - 2\tilde{R}_j^n + \tilde{R}_{j-1}^n)}{(\Delta\xi)^2},$$

and the boundary conditions as $\tilde{R}_J^{n+1} = 0$, and $\tilde{R}_0^{n+1} = 0$. As we solve the scheme backward in time, we start with the guess $\tilde{R}_j^0 = -\frac{U'(\xi_j)}{U''(\xi_j)}$, for all $j = 0, 1, \ldots, J$. To ensure convergence, we

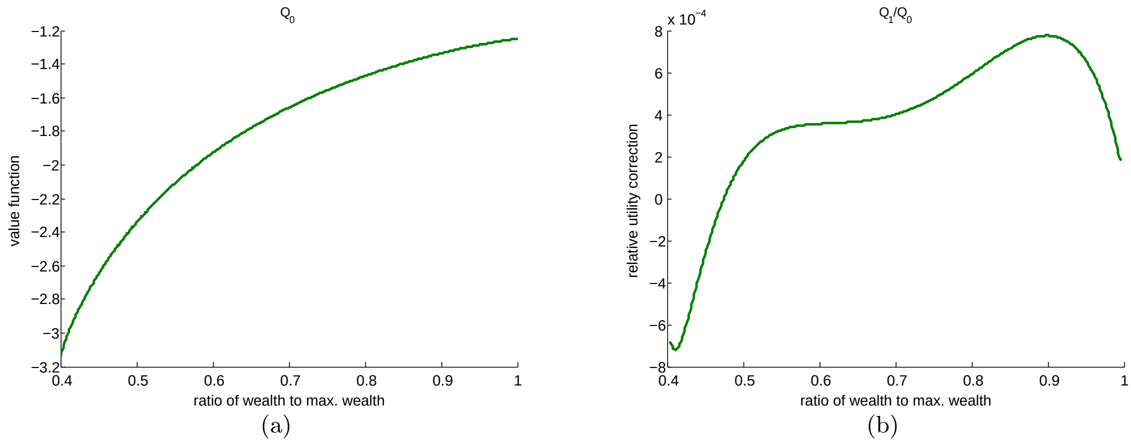


Figure 2: *Numerical solutions to (a) zeroth order value function $Q^{(0)}$ (b) relative utility correction $Q^{(1)}/Q^{(0)}$. Utility function used $U(\xi) = \frac{\xi^{1-\gamma_1}}{1-\gamma_1} + \frac{\xi^{1-\gamma_2}}{1-\gamma_2}, \gamma_1 = 3.0, \gamma_2 = 1.5$.*

choose $(\Delta\xi, \Delta t)$ such that the monotonicity condition holds

$$\frac{\Delta t}{(\Delta\xi)^2}\|\mathcal{R}\|_\infty^2 \leq \frac{1}{2}.$$

It can be seen that the above relationship between Δt and $\Delta\xi$ will also allow the numerical scheme for $Q^{(0)}$ to converge. In our market calibrated stochastic volatility model, we first set $y = \theta$ and plot the relative utility correction in Figure 1(B) and Figure 2(B). We observe that the change in the value function due to the introduction of stochastic volatility is negligible.

Next, we calculate the approximation to optimal strategy whose different terms are given from (31) as

$$\begin{aligned}
\frac{\pi_0}{m} &= (\mu - r)y\mathcal{R}, \qquad (35)\\
\frac{\pi_1}{m} &= \frac{(\mu-r)^3y^2}{2}(T-t)^2\Big[\kappa(\theta-y)\Big(\mathcal{R}^2Q^{(0)}_{\xi\xi}\Big)\\
&\quad + \rho\delta\Big(\mathcal{R}^2\mathcal{R}_{\xi\xi}(3\mathcal{R}_\xi - 2) + \mathcal{R}^3\mathcal{R}_{\xi\xi\xi}\Big)\Big] + (\mu-r)^2(T-t)\rho\delta y\mathcal{R}\Big(\mathcal{R}_\xi - 1\Big).
\end{aligned}$$

We suppose that the initial value of maximum wealth is unity, i.e. we set $m = 1.0$ and plot the numerical solution to leading order term π_0 and to the first order approximation $\pi_0 + \pi_1$ in Figure 3(A) and 3(B). It is interesting to note that to achieve similar value functions without and with the stochastic volatility correction, i.e. $Q^{(0)}$ and $Q^{(0)} + Q^{(1)}$, we clearly need to employ two very different investment policies, namely π_0 and $\pi_0 + \pi_1$.

In Figure 3(A) and 3(B), we note that as the current wealth approaches to the maximum wealth value, the optimal strategy is to gradually liquidate the position in the risky asset. In the presence of stochastic volatility, the optimal strategy approximation $\pi_0+\pi_1$ suggests to hold the risky position longer than without the stochastic volatility correction as in π_0. The corrected strategy also suggests to sharply liquidate the position in the risky asset to safeguard from the downside risk of stochastic volatility. On the other hand, when the current wealth moves away from the drawdown barrier, the optimal strategy approximation $\pi_0 + \pi_1$ suggests to build up a position in the risky asset at about the same trading rate to that in the case of constant volatility approximation π_0.

From the above results, we deduce that even in the presence of stochastic volatility, the investor does not lose much value in her portfolio. However, to achieve similar value functions, the investor has to deploy a remarkably different strategy corrected for stochastic volatility

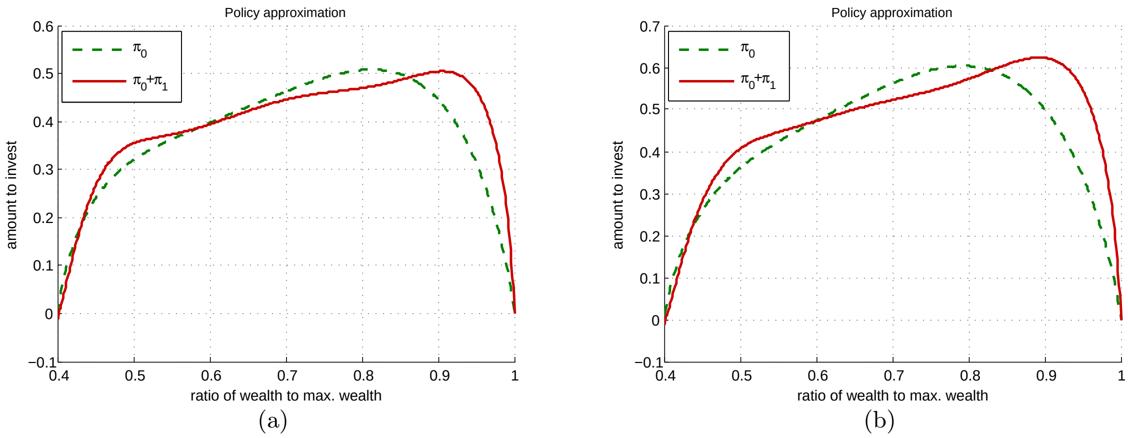


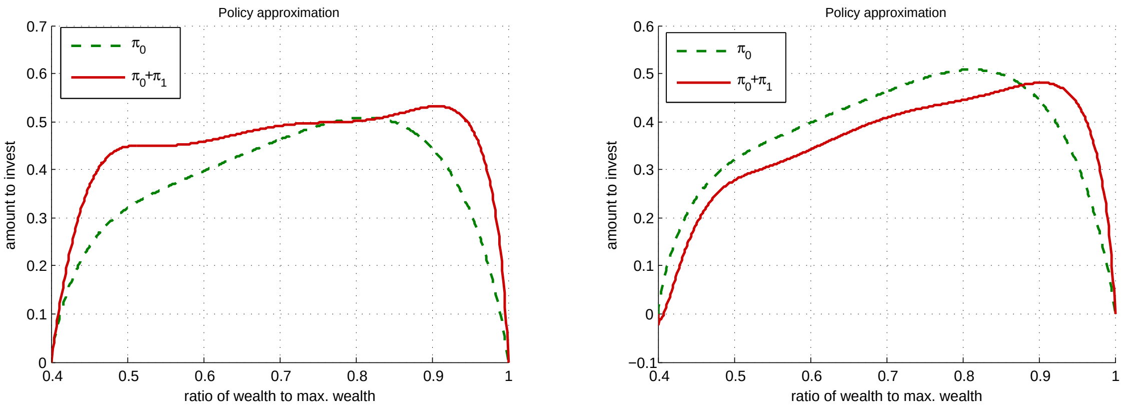
Figure 3: *Numerical solutions to the optimal strategy approximation in the case $y = \theta$ for utility function (a) $U(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}, \gamma = 3.0$ (b) $U(\xi) = \frac{\xi^{1-\gamma_1}}{1-\gamma_1} + \frac{\xi^{1-\gamma_2}}{1-\gamma_2}, \gamma_1 = 3.0, \gamma_2 = 1.5$.*

$\pi_0 + \pi_1$ when compared to the constant volatility strategy π_0. The larger position in the risky asset when moving away from the drawdown barrier suggests leveraging the possible upside due to stochastic volatility while holding on to the risky asset longer than in the constant volatility case when close to the optimal level suggests caution towards a possible downside risk.

In the above results, we have set the level of stochastic volatility factor y to be the same as the long term value θ. As it is clear that the level of stochastic volatility plays a crucial role in the correction terms, we studied the effects when y moves in either direction away from its long term value θ. We observed that even in the other cases, the relative utility correction remains small. However, the optimal strategy in these cases exhibit remarkably different behaviours due to the particular form of the correction term in (35). When the current level of volatility is higher than the long-term average $y = 1.05 \times \theta$, in Figure 4(A) the optimal strategy approximation suggests to invest more in the risky asset compared to the strategy without stochastic volatility correction. Also, as the portfolio wealth moves away the drawdown barrier, the corrected optimal strategy suggests to build up the position in risky asset at a much higher rate than suggested by π_0. Whereas, in the case when the current level of volatility is lower than the long-term average $y = 0.95 \times \theta$, in Figure 4(B) the optimal strategy approximation suggests to invest less in the risky asset compared to the strategy without stochastic volatility correction. Still close to the maximum wealth value, the corrected strategy suggests to hold more risky asset than the constant volatility strategy suggests.

Once we have derived the zeroth and first order approximations for the optimal strategy, we also demonstrate how these results can be used to guide an investment strategy in practice. Recall that we work in the model setting as discussed at the beginning of this section. We suppose that the portfolio rebalancing happens at N_{int} intermediate times over the investment horizon T. We utilize the zeroth and first order approximation of the optimal trading strategy to guide an investment strategy in the following way:

1. Choose initial value of starting wealth l and maximum wealth m such that for drawdown parameter $0 < \alpha < 1$, it satisfies $0 < \alpha m < l < m$.



(a) (b)

Figure 4: *Numerical solutions to the optimal strategy approximation for (a) $y = 1.05\times\theta$ (b) $y = 0.95\times\theta$. The utility function used is $U(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}, \gamma = 3.0$*

2. Create discretized sample paths $(\hat{Y}, \hat{L}^{(0)}, \hat{L}^{(1)})$ using the Euler scheme as

$$\begin{aligned}
\hat{Y}_{i\hat{\Delta}t} &:= \hat{Y}_{(i-1)\hat{\Delta}t} + \kappa\big(\theta - \hat{Y}_{(i-1)\hat{\Delta}t}\big)\hat{\Delta}t + \delta\sqrt{\hat{\Delta}t}\big(\max(0, \hat{Y}_{(i-1)\hat{\Delta}t})\big)^{\frac{1}{2}}\mathrm{Z}_i^{(2)}, \quad \hat{Y}_0 = \theta,\\
\hat{L}^{(0)}_{i\hat{\Delta}t} &:= \hat{L}^{(0)}_{(i-1)\hat{\Delta}t} + \pi_0\big((i-1)\hat{\Delta}t\big)\Big((\mu - r)\hat{\Delta}t + (\hat{Y}_{(i-1)\hat{\Delta}t})^{-\frac{1}{2}}\sqrt{\hat{\Delta}t}\mathrm{Z}_i^{(1)}\Big), \quad \hat{L}^{(0)}_0 := l,\\
\hat{L}^{(1)}_{i\hat{\Delta}t} &:= \hat{L}^{(1)}_{(i-1)\hat{\Delta}t} + \big(\pi_0 + \pi_1\big)\big((i-1)\hat{\Delta}t\big)\Big((\mu - r)\hat{\Delta}t + (\hat{Y}_{(i-1)\hat{\Delta}t})^{-\frac{1}{2}}\sqrt{\hat{\Delta}t}\mathrm{Z}_i^{(1)}\Big) \quad \hat{L}^{(1)}_0 := l.
\end{aligned}$$

   where $N_{\text{int}}\hat{\Delta}t := T$ and $(\mathrm{Z}_i^{(1)}, \mathrm{Z}_i^{(2)})_{i=1,2,\ldots,N}$ is a sequence of normal random numbers with correlation ρ. The values for $\pi_0(i\hat{\Delta}t)$ and $\pi_1(i\hat{\Delta}t)$ are calculated by plugging the value of $\hat{Y}_{i\hat{\Delta}t}$ and the ratio $\xi(i\hat{\Delta}t) := \hat{L}^{(j)}_{i\hat{\Delta}t}/m$, $j = 0, 1$, in the formulas (35).

Based on 10^5 sample paths, for $l = 0.5$, $m = 1.0$ and for different values of N_{int}, we plot the normalized histograms for terminal wealth values $\hat{L}^{(0)}_T$ and $\hat{L}^{(1)}_T$ which are approximations of wealth process $\{L_t, 0 \le t \le T\}$ using the optimal strategy approximations π_0 and $\pi_0 + \pi_1$, respectively. As higher terminal wealth leads to a higher utility value for the investor, from the results in Figure 5(A)-5(D), we can easily deduce the superior performance of the first order optimal strategy approximation, $\pi_0 + \pi_1$, over the zeroth order approximation π_0.

5 Conclusion

We studied the impact of stochastic Sharpe ratio in a dynamic portfolio optimization problem under a drawdown constraint. We proposed a new investor objective framework which allows for portfolio benchmarking and a dimensionality-reducing transformation of the problem. This new setting allowed us to employ coefficient expansion technique to solve for different terms in the approximation of the value function and optimal strategy. With the help of a nonlinear transformation we derived the value function expansion terms which can be numerically calculated and used those expansion terms to approximate the optimal portfolio strategy. In a popular stochastic volatility model with market calibrated parameters, we illustrated the remarkable differences between the optimal strategies with and without stochastic volatility correction.

The current problem requires further investigation in the direction of a multi-asset market model. We studied the portfolio optimization problem under drawdown constraint in a stochastic volatility model which provides a sensible guide towards informed investment decisions. However,

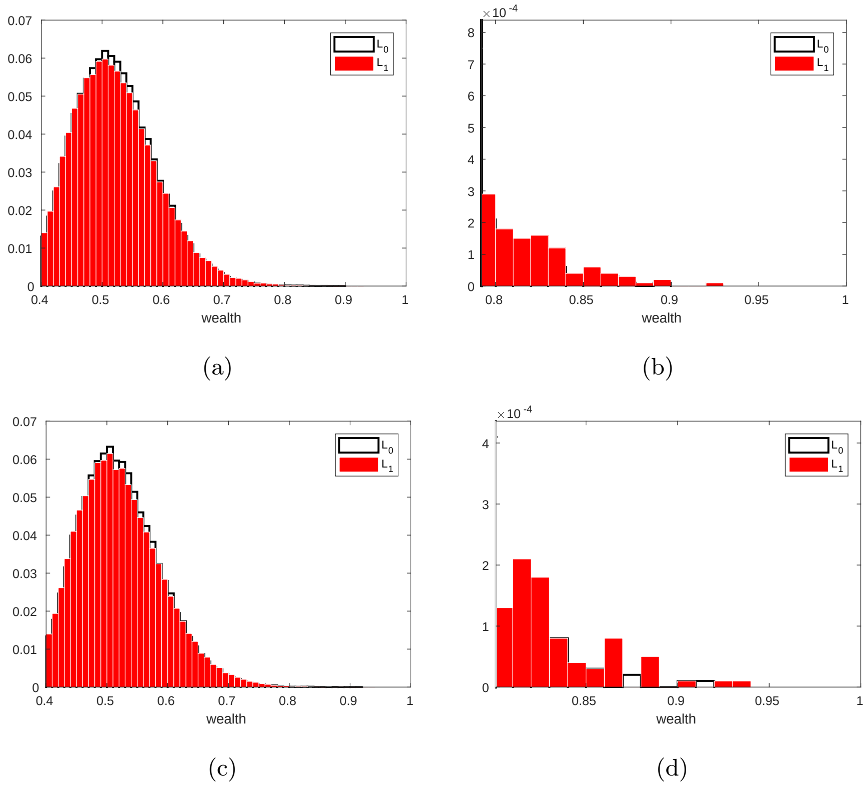


Figure 5: *Normalized histograms and plots of the tail for the values of* $\hat{L}_T^{(0)}$ *and* $\hat{L}_T^{(1)}$ *with (a)-(b)* $N_{int} = 64$ *(c)-(d)* $N_{int} = 128$.

in order to completely capture the market conditions, we plan to tackle the same problem in a multi-asset model setting and study the effect of stochastic volatility on investment strategies.

A Proofs

A.1 Proof of Proposition 1

Proof. We observe that PDE (21) can also be written as $Q_t^{(0)} = \frac{1}{2}\lambda_0^2 \mathcal{D}_2 Q^{(0)}$. Differentiating this w.r.t. ξ, we get

$$\partial_{t\xi} Q^{(0)} = \lambda_0^2 \left(\frac{1}{2} \mathcal{R}^2 Q_{\xi\xi\xi}^{(0)} + \mathcal{R}\mathcal{R}_\xi Q_{\xi\xi}^{(0)} \right).$$

Further, from the definition of $\mathcal{R}$, we get $\mathcal{R} Q_{\xi\xi}^{(0)} = -Q_\xi^{(0)}$ which after differentiating w.r.t. ξ gives

$$\mathcal{R}^2 Q_{\xi\xi\xi}^{(0)} = -\mathcal{R} Q_{\xi\xi}^{(0)} \Big(1 + \mathcal{R}_\xi \Big).$$

This provides us

$$\partial_{t\xi} Q^{(0)} = -\frac{\lambda_0^2}{2} Q_\xi^{(0)} \Big(-1 + \mathcal{R}_\xi \Big). \tag{36}$$

Differentiating (17) w.r.t. t gives

$$\mathcal{R}_t = -\frac{Q_{t\xi}^{(0)}}{Q_{\xi\xi}^{(0)}} + \frac{Q_\xi^{(0)}}{(Q_{\xi\xi}^{(0)})^2} Q_{t\xi\xi}^{(0)}. \tag{37}$$

Differentiating (36) w.r.t. ξ, we get

$$Q^{(0)}_{t\xi\xi} = -\frac{\lambda_0^2}{2} Q^{(0)}_{\xi\xi}\Big(-1 + \mathcal{R}_\xi\Big) - \frac{\lambda_0^2}{2} Q^{(0)}_{\xi}\mathcal{R}_{\xi\xi}.$$

Plugging back the above result and (36) into (37) gives the PDE for $\mathcal{R}$. The terminal condition at $t = T$ is straightforward from the terminal condition for $Q^{(0)}$. At the boundary, $\xi = \alpha, Q^{(0)} = U(\alpha)$ and due to the continuity of $Q^{(0)}$ across the boundary, it gives that $Q^{(0)}_t = 0$. Then, due to the continuity of derivatives w.r.t. space variables across the boundary, from (15) we get at $\xi = \alpha$,

$$\frac{(Q^{(0)}_\xi)^2}{Q^{(0)}_{\xi\xi}} = \mathcal{R} Q^{(0)}_\xi = 0.$$

As $Q^{(0)}_\xi\big|_{\xi=\alpha} \neq 0$, it gives that $\mathcal{R}\big|_{\xi=\alpha} = 0$.

It can be shown (as done in the proof of Lemma 1 in Section A.2) that the optimal strategy corresponding to the value function in constant parameter lognormal model with Sharpe ratio λ_0, after the dimensionality reduction, is given by $\pi_0 :=$ constant $\times\, \mathcal{R}$. It is clear that as the portfolio wealth approaches to its maximum value, i.e. at $\xi = 1$, the optimal strategy suggests to unwind the risky position, i.e. $\pi_0|_{\xi=1} = 0$. This gives us the right boundary condition for $\mathcal{R}$ as $\mathcal{R}\big|_{\xi=1} = 0$. □

A.2 Proof of Lemma 1

Proof. Let us consider a market with a risky asset whose dynamics is given by the following lognormal model:

$$\frac{\mathrm{d}S_t}{S_t} = \mu_0 \mathrm{d}t + \sigma_0 \mathrm{d}B^{(1)}_t.$$

With this risky asset in the market, we once again formulate our portfolio optimization problem (see Section 2.1) by defining the following value function

$$\mathcal{V}(t,l,m) = \sup_{\pi\in\Pi_{\alpha,t,l,m}} \mathbb{E}\Big[U\Big(\frac{L_T}{M_T}\Big)\Big| L_t = l, M_t = m\Big], \quad t \geq 0, m > l > \alpha m > 0,$$

where the admissible strategies are given by

$$\Pi_{\alpha,t,l,m} := \big\{\pi : \text{measurable}, \mathbb{F}^{(1)} - \text{adapted}, \mathbb{E}_{t,l,m}\int_t^T \pi_s^2 \mathrm{d}s < \infty \text{ s.t. } L_s \geq \alpha M_s > 0 \text{ a.s.}, t \leq s \leq T\big\},$$

and $\mathbb{F}^{(1)} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ is the augmentation of the filtration generated by $B^{(1)}$. We define the constant Sharpe ratio as $\lambda_0 := \frac{(\mu_0 - r)}{\sigma_0}$ and the space domain as $\mathscr{O}_\alpha := \{(l,m) : m > l > \alpha m > 0\} \subset \mathbb{R}^2$. Then, by proceeding as in Section 2.1, we assume that $\mathcal{V} \in C^{1,2,1}([0,T] \times \overline{\mathscr{O}}_\alpha)$, to obtain the following nonlinear PDE

$$\partial_t \mathcal{V} - \frac{1}{2}\lambda_0^2 \frac{(\mathcal{V}_l)^2}{\mathcal{V}_{ll}} = 0, \text{ on } [0,T) \times \mathscr{O}_\alpha,$$

with terminal and boundary conditions as

$$\mathcal{V}(T,l,m) = U(l/m), \quad \mathcal{V}_m(t,m,m) = 0, \quad \mathcal{V}(t,\alpha m,m) = U(\alpha).$$

Similar to Section 2.2, we perform a change of variable $\xi := l/m$. Then, it is clear that the leading order term in expansion (10) is $Q^{(0)}(t,\xi) = \mathcal{V}(t,l,m)$.

To first show that $Q^{(0)}(t,\cdot)$ is a non-decreasing function, we recall that in the lognormal model, for a portfolio strategy π, the discounted wealth process is given as

$$L_t^{l,\pi} = l + \int_0^t \pi_s\sigma_0\big(\lambda_0 \mathrm{d}s + \mathrm{d}B_s^{(1)}\big),$$

where l is the starting wealth value. Let $(L^{l,\pi})^*$ denote the maximum of wealth process $L^{l,\pi}$ over the time period $[0,T]$. Now, we consider l, l' for a fixed value of m such that $(t,l,m),(t,l',m) \in [0,T)\times\mathscr{O}_\alpha$. Then, for $l \le l'$, we choose $\pi \in \Pi_{\alpha,t,l,m}$ such that we have

$$\begin{aligned}
L^{l,\pi} &\ge \alpha(m \vee (L^{l,\pi})^*)\\
&= \alpha\Bigg(m \vee \Big(l + \big(\int_0^t \pi_s\sigma_0(\lambda_0\mathrm{d}s + \mathrm{d}B_s^{(1)})\big)^*\Big)\Bigg).
\end{aligned}$$

Here, we note that $\Pi_{\alpha,t,l,m}$ is a non-empty set from the result of Cvitanic and Karatzas [8, Appendix A]. Add $(l'-l)$ to both sides of the inequality above to write

$$\begin{aligned}
L^{l',\pi} &\ge \Bigg(\big(\alpha m + (l'-l)\big) \vee \Big(\alpha l' + \alpha\big(\int_0^t \pi_s\sigma_0(\lambda_0\mathrm{d}s + \mathrm{d}B_s^{(1)})\big)^* + (1-\alpha)(l'-l)\Big)\Bigg)\\
&\ge \alpha\Bigg(m \vee \Big(l' + \big(\int_0^t \pi_s\sigma_0(\lambda_0\mathrm{d}s + \mathrm{d}B_s^{(1)})\big)^*\Big)\Bigg)\\
&= \alpha(m \vee (L^{l',\pi})^*),
\end{aligned}$$

which gives that $\Pi_{\alpha,t,l,m} \subset \Pi_{\alpha,t,l',m}$. Thus, we get $\mathcal{V}(t,l,m) \le \mathcal{V}(t,l',m)$. For $\xi := \frac{l}{m}$ and $\xi' := \frac{l'}{m}$, this gives us

$$Q^{(0)}(t,\xi) \le Q^{(0)}(t,\xi').$$

Next, it follows from the arguments presented in Lemma 3.2 Elie [9] that $\mathcal{V}(t,l,m)$ is non-increasing in variable m. Thus, for fixed l and $m \le m'$ such that $(t,l,m),(t,l,m') \in [0,T)\times\mathscr{O}_\alpha$, we have $\mathcal{V}(t,l,m') \le \mathcal{V}(t,l,m)$. Once again by defining $\xi' := \frac{l}{m'}$ and $\xi := \frac{l}{m}$, we get

$$\mathcal{V}(t,l,m') \le \mathcal{V}(t,l,m) \implies Q^{(0)}(t,\xi') \le Q^{(0)}(t,\xi).$$

Therefore, we have shown that $Q^{(0)}(t,\cdot)$ is non-decreasing.

In order to show the concavity of value function $Q^{(0)}(t,\cdot)$, we take motivation from the arguments presented in Lemma 3.2 Elie [9]. First, we fix $\eta \in [0,1]$ and choose $\alpha \le \xi_1,\xi_2 \le 1$. Our aim is to show that $\mathcal{V}(t,l,m)$ is concave in its second argument, i.e.

$$\eta\mathcal{V}(t,l_1,m) + (1-\eta)\mathcal{V}(t,l_2,m) \le \mathcal{V}(t,\eta l_1 + (1-\eta)l_2,m), \tag{38}$$

where for a fixed value of m, we set $l_1 = m\xi_1$ and $l_2 = m\xi_2$. Now, suppose (38) is true. Then by reversing the change of variables, we get in (38)

$$\eta Q^{(0)}(t,\xi_1) + (1-\eta)Q^{(0)}(t,\xi_2) \le Q^{(0)}(t,\eta\xi_1 + (1-\eta)\xi_2)$$

which gives us the concavity of $Q^{(0)}(t,\cdot)$. It remains to show that (38) is indeed true.

We define process $L^{(1)}$ as the wealth process with starting wealth l_1 and portfolio strategy $\pi_1 \in \Pi_{\alpha,t,l_1,m}$. Similarly, we define the process $L^{(2)}$ with starting wealth l_2 and portfolio strategy $\pi_2 \in \Pi_{\alpha,t,l_2,m}$. Then, we have by definition

$$\begin{aligned}
\eta L^{(1)} + (1-\eta)L^{(2)} &\ge \eta\alpha(m \vee (L^{(1)})^*) + (1-\eta)\alpha(m \vee (L^{(2)})^*)\\
&\ge \alpha\Big(m \vee \big(\eta L^{(1)} + (1-\beta)L^{(2)}\big)^*\Big).
\end{aligned}$$

This gives us that $\eta\pi_1 + (1-\eta)\pi_2 \in \Pi_{\alpha,t,\eta l_1+(1-\eta)l_2,m}$. From the concavity property of utility function U, it follows

$$\begin{aligned}
&\eta\mathbb{E}_t\left[U\left(\frac{L_T^{(1)}}{(m \vee (L^{(1)})_T^*)}\right)\right] + (1-\eta)\mathbb{E}_t\left[U\left(\frac{L_T^{(1)}}{(m \vee (L^{(1)})_T^*)}\right)\right] \\
&\leq \mathbb{E}_t\left[U\left(\frac{\eta L_T^{(1)}}{(m \vee (L^{(1)})_T^*)} + \frac{(1-\eta)L_T^{(2)}}{(m \vee (L^{(2)})_T^*)}\right)\right].
\end{aligned}$$

Next, we intend to show that

$$\frac{\eta L_T^{(1)}}{(m \vee (L^{(1)})_T^*)} + \frac{(1-\eta)L_T^{(2)}}{(m \vee (L^{(2)})_T^*)} \leq \frac{\eta L_T^{(1)} + (1-\eta)L_T^{(2)}}{(m \vee (\eta L^{(1)} + (1-\eta)L^{(2)})_T^*)}. \tag{39}$$

Consider the following possible scenarios where we compare the respective terms with m and find the maximum

	$(L^{(1)})_T^*$	$(L^{(2)})_T^*$	$(\eta L^{(1)} + (1-\eta)L^{(2)})_T^*$
Case 1	m	m	m
Case 2	m	$(L^{(2)})_T^*$	m
Case 3	$(L^{(1)})_T^*$	m	m
Case 4	$(L^{(1)})_T^*$	$(L^{(2)})_T^*$	–

It is clear that the inequality in (39) holds for Case 1–3 and we only need to consider Case 4. We know from the optimality condition that for strategies π_1 and π_2 which attain the maximum, the position in the risky asset becomes zero thereafter as the maximum possible utility is achieved. It follows that for such strategies, we have

$$L_T^{(1)} = (L^{(1)})_T^*, \qquad L_T^{(2)} = (L^{(2)})_T^*.$$

Then, we get

$$\frac{\eta L_T^{(1)} + (1-\eta)L_T^{(2)}}{(m \vee (\eta L^{(1)} + (1-\eta)L^{(2)})_T^*)} = \frac{\eta (L^{(1)})_T^* + (1-\eta)(L^{(2)})_T^*}{(m \vee (\eta L^{(1)} + (1-\eta)L^{(2)})_T^*)} \geq 1,$$

due to

$$\eta (L^{(1)})_T^* + (1-\eta)(L^{(2)})_T^* \geq m, \qquad \eta (L^{(1)})_T^* + (1-\eta)(L^{(2)})_T^* \geq (\eta L^{(1)} + (1-\eta)L^{(2)})_T^*).$$

Thus, we have shown that (39) is indeed true. This gives us

$$\begin{aligned}
&\eta\mathbb{E}_t\left[U\left(\frac{L_T^{(1)}}{(m \vee (L^{(1)})_T^*)}\right)\right] + (1-\eta)\mathbb{E}_t\left[U\left(\frac{L_T^{(1)}}{(m \vee (L^{(1)})_T^*)}\right)\right] \\
&\leq \mathbb{E}_t\left[U\left(\frac{\eta L_T^{(1)} + (1-\eta)L_T^{(2)}}{(m \vee (\eta L^{(1)} + (1-\eta)L^{(2)})_T^*)}\right)\right] \\
&\leq \mathcal{V}(t, \eta l_1 + (1-\eta)l_2, m).
\end{aligned}$$

As, π_1, π_2 are arbitrary, we have have shown (38). This concludes the proof for concavity of $Q^{(0)}(t, \cdot)$. □

A.3 Proof of Proposition 2

Proof. In the definition, $q^{(0)}(t, z(t,\xi)) = Q^{(0)}(t,\xi)$, we differentiate w.r.t. t on both sides to write

$$\begin{aligned}\partial_t Q^{(0)} &= \partial_t q^{(0)} + q_z^{(0)} \frac{\partial z}{\partial t} \\ &= \partial_t q^{(0)} - \Big(\frac{Q_{t\xi}^{(0)}}{Q_\xi^{(0)}} + \frac{\lambda_0^2}{2} \Big) q_z^{(0)}.\end{aligned}$$

It is also straightforward to check from definition (20) of differential operators $\big(\mathcal{D}_k\big)_{k=1,2,\ldots}$ that

$$\mathcal{D}_1 Q^{(0)} = q_z^{(0)}, \quad \mathcal{D}_2 Q^{(0)} = q_{zz}^{(0)} - \mathcal{R}_\xi q_z^{(0)}. \tag{40}$$

From the calculations performed in Proposition 1, we have

$$\begin{aligned}\partial_{t\xi} Q^{(0)} &= \frac{\lambda_0^2}{2} \mathcal{R} Q_{\xi\xi}^{(0)} \Big(-1 + \mathcal{R}_\xi \Big) \\ &= -\frac{\lambda_0^2}{2} Q_\xi^{(0)} \Big(-1 + \mathcal{R}_\xi \Big).\end{aligned}$$

Finally, we collect all the expressions for $\partial_t Q^{(0)}, \mathcal{D}_1 Q^{(0)}$ and $\mathcal{D}_2 Q^{(0)}$ in terms of $q^{(0)}$ to write

$$\begin{aligned}&\Big(\partial_t + \lambda_0^2 \mathcal{D}_1 + \frac{\lambda_0^2}{2} \mathcal{D}_2 \Big) Q^{(0)} \\ &= \partial_t q^{(0)} - \Big(-\frac{\lambda_0^2}{2} \Big(-1 + \mathcal{R}_\xi \Big) + \frac{\lambda_0^2}{2} \Big) q_z^{(0)} + \lambda_0^2 q_z^{(0)} + \frac{\lambda_0^2}{2} \Big(q_{zz}^{(0)} - \mathcal{R}_\xi q_z^{(0)} \Big) \\ &= \Big(\frac{\partial}{\partial t} + \frac{1}{2} \lambda_0^2 \frac{\partial}{\partial z^2} \Big) q^{(0)}\end{aligned}$$

which gives us the desired PDE.

For the terminal boundary condition for $q^{(0)}$, it follows from the definition of $z(t,\xi)$ and terminal condition (16) that

$$q^{(0)}(T, z) = U\Big(\big(U' \big)^{-1} \big(\mathrm{e}^{-z} \big) \Big), \quad \psi(T) < z < \infty.$$

The left boundary condition in (16) can also be easily transformed. Next, for the right boundary condition in (16), we first note that

$$q_z^{(0)} \times \partial_\xi z = Q_\xi^{(0)}.$$

Now, as $Q_\xi^{(0)} = 0$, for $\xi = 1$, it holds only if in the above relation we have

$$\lim_{z \to \infty} q_z^{(0)}(t, z) = 0.$$

This completes the proof. ☐

A.4 Proof of Proposition 4

We first consider the PDE problem with a terminal condition

$$\mathcal{H} q + \mathcal{S} = 0, \quad q(T, z, x, y) = 0, \tag{41}$$

where $\mathcal{H}$ is a constant coefficient linear operator

$$\mathcal{H} := \frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0.$$

We suppose that the source term $\mathcal{S}$ is of the following special form

$$\mathcal{S}(t,z,x,y) = \sum_{k,l,n} (T-t)^n (x-\bar{x})^k (y-\bar{y})^l v(t,z,x,y) \tag{42}$$

where the sum has a finite number of terms, and v is a solution of the homogeneous equation $\mathcal{H}v = 0$.

Further, define the commutator of operators $\mathcal{H}$ and $(x-\bar{x})I$ (I is the identity operator), $\mathcal{L}_X = [\mathcal{H}, (x-\bar{x})I]$ as

$$\mathcal{L}_X v := \mathcal{H}((x-\bar{x})v) - (x-\bar{x})\mathcal{H}v,$$

which from the definition of $\mathcal{A}_0$ (14) and $\mathcal{C}_0$ (24) gives

$$\mathcal{L}_X = b_0 I + \sigma_0^2 \frac{\partial}{\partial x} + \rho\sigma_0\beta_0 \frac{\partial}{\partial y} + \sigma_0\lambda_0 \frac{\partial}{\partial z}. \tag{43}$$

Similarly, define $\mathcal{L}_Y = [\mathcal{H}, (y-\bar{y})I]$, which gives

$$\mathcal{L}_Y = c_0 I + \beta_0^2 \frac{\partial}{\partial y} + \rho\sigma_0\beta_0 \frac{\partial}{\partial x} + \rho\beta_0\lambda_0 \frac{\partial}{\partial z}. \tag{44}$$

Using $\mathcal{L}_X$ and $\mathcal{L}_Y$, we also define

$$\mathcal{M}_X(s) := (x-\bar{x})I + (s-t)\mathcal{L}_X, \quad \mathcal{M}_Y(s) := (y-\bar{y})I + (s-t)\mathcal{L}_Y.$$

Using these definitions, we first give the following result related to the homogeneous solution v, from [16, Lemma 3.4]. Here, we provide the proof for the sake of completeness.

Lemma 4. *For integers k, l, we have,*

$$\mathcal{H}\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v = 0.$$

Proof. We proceed by induction. We first calculate

$$\begin{aligned}
\mathcal{H}\mathcal{M}_X(s)v &= \mathcal{H}(x-\bar{x})v + \mathcal{H}(s-t)\mathcal{L}_X v \\
&= \mathcal{L}_X v + (x-\bar{x})\mathcal{H}v - \mathcal{L}_X v + (s-t)\mathcal{H}\mathcal{L}_X v \\
&= \mathcal{L}_X \mathcal{H} v = 0,
\end{aligned}$$

where we have used the definition of the commutator $\mathcal{L}_X$, the fact that $\mathcal{L}_X$ and $\mathcal{H}$ commute as they are constant coefficient operators and that $\mathcal{H}v = 0$. Thus, we can then iterate over integer k to show $\mathcal{H}\mathcal{M}_X(\mathcal{M}_X^{(k-1)}v) = 0$ (as $\mathcal{H}\mathcal{M}_X(s)v = 0$). Similarly, we can show that $\mathcal{H}\mathcal{M}_Y^l v = 0$ for integer l. Finally, we have $\mathcal{H}(\mathcal{M}_X^k(s)\ \mathcal{M}_Y^l(s)v) = 0$. ☐

Lemma 5. *The solution q of equation* (41) *with zero terminal condition is*

$$q(t,z,x,y) = \sum_{k,l,n} \int_t^T (T-s)^n \mathcal{M}_X^k(s)\mathcal{M}_Y^l(s) v(t,z,x,y)\, \mathrm{d}s. \tag{45}$$

Proof. This can be shown by using the form of source term (42) and Lemma 4. Let us suppose that the source term consists of a monomial and is given as $\mathcal{S}(t,z,x,y) = (T-t)^n(x-\bar{x})^k(y-\bar{y})^l v(t,z,x,y)$. In this case, from our claim, the solution should be given as

$$q(t,z,x,y) = \int_t^T (T-s)^n \mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v(t,z,x,y)\mathrm{d}s.$$

We verify by computing

$$\begin{aligned}\mathcal{H}q &= -(T-t)^n\mathcal{M}_X^k(t)\mathcal{M}_Y^l(t)v(t,z,x,y) + \int_t^T (T-s)^n\mathcal{H}\Big(\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v(t,z,x,y)\Big)\mathrm{d}s \\ &= -(T-t)^n(x-\bar{x})^k(y-\bar{y})^l v(t,z,x,y) \\ &= -\mathcal{S}.\end{aligned}$$

It is also easy to see that for the form of solution proposed in (45), the terminal condition at T is satisfied. The result follows from linearity of the PDE problem. □

Finally, we give the proof of Proposition 4.

Proof. We first observe that, since $q^{(0)}$ solves $\mathcal{H}q^{(0)} = 0$, then $q_z^{(0)}$ also solves the homogeneous equation, as the operator $\mathcal{H}$ has constant coefficients. We set $v = q_z^{(0)}$. From (27), the source term is

$$\mathcal{S}(t,z,x,y) = \Big(\big(\frac{1}{2}\lambda^2\big)_{1,0}(x-\bar{x}) + \big(\frac{1}{2}\lambda^2\big)_{0,1}(y-\bar{y})\Big)v,$$

and so from Lemma 5, we obtain the solution

$$\begin{aligned}q^{(1)}(t,z,x,y) =& \Big[\big(\frac{1}{2}\lambda^2\big)_{1,0}\big((T-t)(x-\bar{x}) + \frac{1}{2}(T-t)^2\mathcal{L}_X\big) \\ &+ \big(\frac{1}{2}\lambda^2\big)_{0,1}\big((T-t)(y-\bar{y}) + \frac{1}{2}(T-t)^2\mathcal{L}_Y\big)\Big]q_z^{(0)}(t,z).\end{aligned} \tag{46}$$

From the expansion for $\lambda(y)$, we get

$$\big(\frac{1}{2}\lambda^2\big)_{1,0} = \lambda_0\lambda_{1,0}, \quad \big(\frac{1}{2}\lambda^2\big)_{0,1} = \lambda_0\lambda_{0,1}.$$

Putting back the expression of $\mathcal{L}_X$ and $\mathcal{L}_Y$ from (43) and (44) into (46), we get the expression in (29). The terminal condition at $t = T$ is clearly satisfied.

It remains to check the boundary conditions for $q^{(1)}$. We show that the boundary conditions for $Q^{(1)}$, corresponding to the original variables (t,ξ), are satisfied. Using (25) and (40), we obtain (30). Now, due to the zero boundary condition at $\xi = \alpha$ for the risk-tolerance function $\mathcal{R}$, we get from (30) that $Q^{(1)}(t,\alpha,x,y) = 0$, which means that the left boundary condition in (23) is satisfied. Consequently, the left boundary condition in (28) is satisfied for $q^{(1)}$.

Next, we calculate

$$\begin{aligned}Q_\xi^{(1)}(t,\xi,x,y) =& (T-t)\lambda_0 A(t,x,y)\Big(\mathcal{R}_\xi Q_\xi^{(0)} + \mathcal{R}Q_{\xi\xi}^{(0)}\Big) + \frac{1}{2}(T-t)^2\lambda_0 B\Big(-2\big[\mathcal{R}_\xi Q_\xi^{(0)} + \mathcal{R}Q_{\xi\xi}^{(0)}\big] \\ &+ 3\mathcal{R}^2\partial_\xi^3 Q^{(0)} + \mathcal{R}^3\partial_\xi^4 Q^{(0)}\Big).\end{aligned} \tag{47}$$

From Assumption 3 on the boundedness of $\partial_\xi^k Q^{(0)}(t,1)$ for $k \le 5$, we have

$$\lim_{\xi\to 1}\mathcal{R}^k\partial_\xi^{(k+1)}Q^{(0)} = 0, \qquad k = 1,2,3.$$

Then, we can use the boundary condition of $Q_{\xi}^{(0)}$ and $\mathcal{R}$ at $\xi = 1$ to conclude from (47) that

$$Q_{\xi}^{(1)}(t, \xi, x, y)\Big|_{\xi=1} = 0,$$

which means that the right boundary condition in (23) is satisfied. This implies that the right boundary condition in (28) is satisfied for $q^{(1)}$. $\square$

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