MULTISCALE INTENSITY MODELS FOR SINGLE NAME CREDIT DERIVATIVES

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Abstract. We study the pricing of defaultable derivatives, such as bonds, bond options, and credit default swaps in the reduced form framework of intensity-based models. We use regular and singular perturbation expansions on the intensity of default from which we derive approximations for the pricing functions of these derivatives. In particular, we assume an Ornstein-Uhlenbeck process for the interest rate, and a two-factor diffusion model for the intensity of default. The approximation allows for computational efficiency in calibrating the model. Finally, empirical evidence on the existence of multiple scales is presented by the calibration of the model on corporate yield curves.

Contents

Date: June 26, 2006; revised February 7, 2007.

JEL Classification. G12, G13.

Key words and phrases. Defaultable bond, credit default swap, defaultable bond option, asymptotic approximation, time scales.

Work partially supported by NSF grants DMS-0306357 and DMS-0456195.

1. Introduction

In this article we study the effect of stochastic intensity of default in the pricing of defaultable derivatives in an intensity-based framework (reduced form models). We construct an asymptotic expansion to obtain approximations of the pricing functions of defaultable bonds, options on such defaultable bonds, and spreads of credit default swaps. These approximations are based on the modeling of the default intensity via two processes that vary on different time scales. The first process evolves on a fast scale, and the second on a slow one. This naturally allows flexibility in the intensity of default to capture empirically the short and long end of the yield curve (for defaultable bonds) or spread curve (for credit default swaps).

The approximations allow us to leave unspecified the precise form of the intensity of default in the model and instead we calibrate the group parameters that arise directly from data. Our expansions allow us to approximate the price of the securities outside of the usual affine-model specification.

The outline of the paper is as follows: in Section 2 we give a concise review of the intensity-based models proposed so far, and we explain the motivation of our approach. Section 3 presents the main results of the approximations to the defaultable bond price, the credit default swap spread and the price of a bond option. Finally, Section 4 illustrates the calibration of the model on data and the conclusions drawn from that are summarized on Section 5.

2. Background and Motivation

For the setup of the problem we employ the usual intensity-based framework for the modeling of the default intensity process. While it is not the most general framework for intensity modeling, since it does not allow for contagion and frailty phenomena, it has been visited widely in the literature.

2.1. **Brief Review.**

Driven by the growth in the credit derivatives market, the modeling of credit risk and credit defaults has seen a remarkable surge during the last decade. Defaultable securities are derivatives that have their payoff linked to a firm's intrinsic risk of defaulting before meeting its financial obligations.

The first attempts to model credit risk were made in the mid 1970's by Merton (1974) and Black and Cox (1976). Merton adopted the model by Black and Scholes (1973) for the valuation of claims contingent on the evolution of an underlying stock price, to credit risk. This is the *structural* approach to credit risk modeling. In particular, the assets of a corporation are modeled by a diffusion process and the default happens at some fixed predetermined maturity time if the asset value at that time is smaller than a predefined default level. Black and Cox introduced the *first-passage* framework that extended Merton's model to allow for default at any time before the maturity time.

The failure to capture market observed short-term yield spreads with default times that are predictable in the diffusion framework led to the introduction of a different class of credit risk models during the 1990's. Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Madan and Unal (1998), Lando (1998), Schönbucher (1998a) Schönbucher (1998b), Duffie and Singleton (1999), and Bielecki et al. (2004) studied so-called *intensity-based* models where the default arrives according to a *conditionally Poisson process* (or *Cox process*). These models resolve the problem of non-trivial short-term yield spreads that appeared in their structural framework counterparts and provide us with convenient pricing tools because of the advantageous doubly-stochastic structure.

A major benefit of intensity-based models is the direct modeling and calibration of the risk of default. In structural models, the modeling of the assets of the underlying firm value and debt value requires knowledge of the firm's financial situation. On the contrary, models under the intensity-based setup instead use the hazard rate and implied probability of default data from credit markets.

Furthermore, in the intensity-based framework the pricing of defaultable securities is an extension of the pricing methodology of their default-free counterparts, for which there are many tractable models. The dynamics of the processes needed for pricing require at least two-factor models, since we need to describe the evolution of the short rate of interest and the stochastic intensity of default. The class of models we work with is described in Section 3.1.

2.2. **The Doubly-Stochastic Poisson Framework.** Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a complete probability space. We assume that there exist two stochastic processes $r = (r_t)_{t\geq 0}$ and $\lambda = (\lambda_t)_{t\geq 0}$ with continuous paths that represent the evolution of the basic ingredients of our model. The former process is known as the *short rate of interest*, and the latter as the *stochastic default rate* or *stochastic intensity of default*. We also define the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ given by

$$\mathcal{F}_t := \sigma\{r_s,\, \lambda_s;\ 0 \leq s \leq t\},$$

that describes the history of the processes r and λ.

We use a doubly-stochastic Poisson process (or *Cox process*, or *conditionally Poisson process*) to describe the arrivals of default events. A doubly-stochastic Poisson process is a generalization of a time-inhomogeneous Poisson process. In particular, if $N = (N(t))_{t\geq 0}$ is a Poisson process with unit rate that is also independent of the process λ, the process $\tilde{N}$ with

$$\tilde{N}_t := N\left(\int_0^t \lambda_s\, ds\right),$$

is called doubly-stochastic Poisson process with *intensity process* λ. We define the *default time* τ as the time of first jump of the process $\tilde{N}$. The probability that the default time will be greater than t, conditional on the path of $(\lambda_s)_{0\leq s\leq t}$ is

$$\mathbb{P}\{\tau > t \mid (\lambda_s)_{0\leq s\leq t}\} = \exp\left(-\int_0^t \lambda_s\, ds\right).$$

For the details of the existence and construction of the doubly-stochastic Poisson process we refer to Brémaud (1981).

Intensity-based models provide closed-form pricing expressions for basic defaultable securities such as bonds, bond options and single-name credit default swaps. There have been a few suggestions in the literature for specific models for r and λ, and the pricing functions stemming from them. Duffie and Singleton (1997), Duffee (1999) and Duffie and Liu (2001) proposed either multidimensional affine processes for the state variables that drive the processes r and λ or affine combinations thereof.

One of the inherent drawbacks of the multidimensional affine models is that they do not allow for complete generality in the specification of the underlying processes. As is usually the case, if closed-form expressions for the pricing functions are needed we assume mean-reverting processes for the short rate of interest r and the intensity of default λ such as Ornstein-Uhlenbeck or square-root diffusions (CIR type). Then we obtain these expression only under restrictive assumptions, such as independence, or positive-only correlation between the two processes. Additionally, the intensity of default needs to remain positive at all times, thus the choice of Gaussian processes for the intensity process λ is not valid. For more on these shortcomings we refer to Chapter 7 of Schönbucher (2003) and Chapter 5 of Lando (2004).

In what follows we illustrate a method to relax the usual assumptions imposed on multi-dimensional affine models, and obtain a pricing tool that offers flexibility in yield curve and credit default swap spread fitting. We adopt a similar setup to that of Fouque et al. (2003a) for multiscale perturbation methods for stochastic volatility in equity models, as well as Cotton et al. (2004) for the application of stochastic volatility models to interest-rate derivatives. Fouque et al. (2006) apply multiscale methods to price defaultable securities with stochastic volatility in the structural framework.

3. Multiscale Intensity Models

In this section we provide the main results of the paper. We introduce the main model that is based on the Vasicek model for the short rate of interest. We present the asymptotic approximation of the pricing functions for the defaultable bond, the credit default swap, and the option on a defaultable bond.

3.1. Model.

We assume the existence of a probability measure $\mathbb{P}^\star$, equivalent to $\mathbb{P}$, which we use for pricing. In what follows we will present the stochastic processes r and λ under the market-determined pricing measure, ignoring their representation under the "real-world" measure $\mathbb{P}$.

A convenient practice is to model the interest rate r as a mean-reverting process. Usual practices are to use the models suggested by either Vasicek (1977) or Cox et al. (1985) (or extensions of them) because of their closed-form expressions for the price functions and also due to their simplicity in calibration. Many other term-structure models—and the prices of default-free derivatives under them—have been studied in the literature, and their advantages and shortcomings have been thoroughly discussed. For a survey see Rogers (1995) while for a more complete collection of interest rate models, their analysis, and extensions we refer to Brigo and Mercurio (2001).

The following system of stochastic differential equations (SDEs) describes the dynamics of the model:

$$dr_t = \alpha(\bar{r} - r_t)dt + \sigma dW_t^{(0)}, \tag{3.1}$$

$$\lambda_t = f(Y_t, Z_t),$$

$$dY_t = \frac{1}{\varepsilon}(m - Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\,dW_t^{(1)}, \tag{3.2}$$

$$dZ_t = \delta\, c(Z_t)\,dt + \sqrt{\delta}\, g(Z_t)\,dW_t^{(2)}, \tag{3.3}$$

where the Wiener process $W = (W^{(0)}, W^{(1)}, W^{(2)})$ has covariance matrix $\Sigma\Sigma^T$ with

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \tilde{\rho}_{12} & \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2} \end{pmatrix}.$$

All the correlation coefficients ρ_1, ρ_2, $\tilde{\rho}_{12}$ are in $(-1, 1)$.

For modeling purposes, we link the evolution of the stochastic default rate λ to two driving processes $Y = (Y_t)_{t\geq 0}$, $Z = (Z_t)_{t\geq 0}$ through a bounded, smooth, and strictly positive function $f : \mathbb{R}^2 \to \mathbb{R}_+$.

The first driving process for the intensity of default, Y, is modeled as a mean-reverting Ornstein-Uhlenbeck process. The ε in (3.2) is a small, strictly positive, parameter that scales the reversion time of the process to its long-term mean m. The rate of mean reversion is $1/\varepsilon$. Recall that processes of such form are Gaussian, and that the invariant distribution of Y, that is, the distribution of $Y_\infty = \lim_{t\to\infty} Y_t$ is Gaussian with law

$$\mathbb{P}^\star\{Y_\infty \in dy\} = \Psi(y)\,dy,$$

where

$$\Psi(y) = \frac{1}{\sqrt{2\pi\nu^2}} \exp\left(-\frac{(y-m)^2}{2\nu^2}\right), \quad y \in \mathbb{R}. \tag{3.4}$$

Notice that under the above formulation the rate of mean reversion $1/\varepsilon$ of the process Y does not appear in its invariant distribution.

The process Z of the intensity of default is modeled as a general diffusion with the assumption that the Lipschitz and growth conditions for the drift and the volatility coefficients c and g are satisfied, so that it admits a unique strong solution. The δ parameter in the SDE (3.3) above for Z controls the speed of fluctuation of Z in the following sense. Let the process $Z^{(1)}$ satisfy (3.3) with $\delta = 1$. Then, we have

$$Z_t \overset{d}{=} Z^{(1)}_{\delta t}, \quad t \geq 0,$$

where the equality is in distribution.

We work in the regime of the parameters ε and δ such that

$$0 < \varepsilon \ll 1, \qquad 0 < \delta \ll 1.$$

These determine the speed of the evolution of each process: *fast* evolution for Y and *slow* evolution for Z, with respect to the time horizon T.

The SDE (3.1) corresponds to an Ornstein-Uhlenbeck process, adopted by Vasicek (1977) as a short-rate model for term-structure modeling. It has the favorable property that is mean-reverting, in the sense that the drift parameter changes accordingly so that it tries to force each trajectory towards the *level of mean-reversion* $\bar{r}$. The strictly positive parameter σ is the diffusion coefficient of the process r.

As mentioned already the distribution of the random variable r_t is Gaussian, which is one of the drawbacks of the Vasicek model since there is always positive probability that the short rate of interest will be negative. This comes in contradiction with the usual no-arbitrage consideration of term-structure models but is often overlooked by practitioners in light of the numerical tractability of this process.

3.2. **Defaultable Bond Price.** Consider a defaultable zero-coupon bond with maturity date T and par value \$1. To account for recovery, we assume that if the bond defaults prior to maturity, it recovers a constant fraction $1-q$ of the pre-default value with $q \in (0,1]$. The no-arbitrage price of such a bond at time $t \leq T$ is

$$\begin{aligned} P(t;T) = \mathbb{E}^\star \Bigg[& \exp\left(-\int_t^T r_s\,ds\right) \mathbf{1}_{\{\tau > T\}} \\ & + \exp\left(-\int_t^\tau r_s\,ds\right) \mathbf{1}_{\{\tau \leq T\}} (1-q) P(\tau -;T) \mid \mathcal{F}_t \vee \sigma\{\tilde{N}_s;\ 0 \leq s \leq t\} \Bigg], \end{aligned}$$

where τ is the time of the first jump of the doubly-stochastic Poisson process (default time). This recovery assumption is known as the *recovery of market value* with *rate of recovery* $1-q$, and we refer to q as the *loss fraction.* Duffie and Singleton (1999) and Schönbucher (1998b) showed that the recovery rate enters the pricing expression as

$$P(t, r_t, Y_t, Z_t; T) = \mathbb{E}^\star \left[\exp\left(-\int_t^T (r_s + q f(Y_s, Z_s))\,ds\right) \mid r_t, Y_t, Z_t \right], \tag{3.5}$$

on $\{\tau > t\}$ where we also used the Markov property of the three-dimensional process (r, Y, Z).

The process r as described in (3.1) is not bounded from below, but it is straightforward to show that the expectation in (3.5) is indeed finite.[1] Then the bond price function $P(t, x, y, z; T)$ satisfies the Feynman-Kac partial differential equation (PDE) problem

$$\begin{aligned} \mathcal{L}^{\varepsilon,\delta} P(t,x,y,z;T) &= 0, \quad t < T, \\ P(T,x,y,z;T) &= 1, \end{aligned} \tag{3.6}$$

where

$$\mathcal{L}^{\varepsilon,\delta} := \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3, \tag{3.7}$$

[1]Since $r = (r_t)_{t \geq 0}$ is a Gaussian process, the random variable $\int_t^T r_s\,ds$ has moments of all orders. Also the Laplace transform of the positive random variable $\int_t^T f(Y_s, Z_s)\,ds$ is finite, and the rest follows from the inequality $|ab| \leq a^2/2 + b^2/2$, for $a, b \in \mathbb{R}$.

and $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ are defined as

$$\mathcal{L}_0 := \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{3.8}$$

$$\mathcal{L}_1 := \rho_1 \sigma \nu \sqrt{2} \frac{\partial^2}{\partial x \partial y},$$

$$\mathcal{L}_2 := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \alpha(\bar{r} - x)\frac{\partial}{\partial x} - (x + qf(y,z)) \cdot ,$$

where the $\cdot$ in the last term of $\mathcal{L}_2$ denotes the identity operator. The operator $\varepsilon^{-1}\mathcal{L}_0$ is the infinitesimal generator of the process Y, the operator $\mathcal{L}_1$ contains the correlation between the fast factor Y and the short rate r, and $\mathcal{L}_2$ is the Vasicek generator for the price function P corresponding to (3.1), with potential term $x + qf(y,z)$.

The operators $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ associated with the slow-varying process $Z = (Z_t)_{t \geq 0}$ are defined as

$$\mathcal{M}_1 := \rho_2 \sigma g(z) \frac{\partial^2}{\partial x \partial z},$$

$$\mathcal{M}_2 := \frac{1}{2} g(z)^2 \frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}, \tag{3.9}$$

$$\mathcal{M}_3 := \rho_{12} \nu \sqrt{2} g(z) \frac{\partial^2}{\partial y \partial z}. \tag{3.10}$$

The operator $\mathcal{M}_1$ contains the correlation between the short rate process r and the slow factor Z, and $\delta\mathcal{M}_2$ is the infinitesimal generator of the process Z. Finally, the operator $\mathcal{M}_3$ comes from the correlation between the processes Y and Z. The correlation coefficient ρ_{12} appearing in the expression for $\mathcal{M}_3$ is the instantaneous correlation of the Wiener processes $W^{(1)}$ and $W^{(2)}$ which is written in terms of the other correlation coefficients as $\rho_{12} := \rho_1\rho_2 + \tilde{\rho}_{12}\sqrt{1-\rho_1^2}$.

Recall that in the absence of default risk, that is $f \equiv 0$, the price of such a bond at time t is the usual exponential-affine expression in the state variable r_t:

$$A(T-t)\exp(-B(T-t)r_t).$$

The terms A and B satisfy the ordinary differential equations (ODEs)

$$B' + \alpha B = 1, \tag{3.11}$$

$$-\frac{A'}{A} + \frac{1}{2}\sigma^2 B^2 - \alpha \bar{r} B = 0, \tag{3.12}$$

that along with the initial conditions

$$A(0) = 1, \qquad B(0) = 0,$$

yield the expressions

$$B(s) = \frac{1 - e^{-\alpha s}}{\alpha}, \quad s \geq 0, \tag{3.13}$$

$$A(s) = \exp\left(-\left(\bar{r} - \frac{\sigma^2}{2\alpha^2}\right)(s - B(s)) - \frac{\sigma^2}{4\alpha}B(s)^2\right), \quad s \geq 0. \tag{3.14}$$

For more on the affine representations of bond prices we refer to Brigo and Mercurio (2001).

3.2.1. *Asymptotic Expansion.* We construct an asymptotic expansion for the price of the bond, P and we adopt the notation convention $P_{j,k}$ for the $\varepsilon^{j/2}\delta^{k/2}$-order term ($j,\, k = 0,1,2,\ldots$). We shall refer to $P_{0,0} \equiv P_0$ simply as the *leading order term*, and to the higher order terms as the *perturbation* or *correction terms.*

The price function will first be expanded in the slow scale in half-powers of δ, and then for each of these terms we will construct a fast scale expansion in half-powers of ε. The approximation remains the same if we expand in the fast scale first and then in the slow, but we choose the aforementioned route because the calculations are slightly simpler. We write then

$$P(t,x,y,z;T) = P_0^\varepsilon(t,x,y,z;T) + \sqrt{\delta}\, P_1^\varepsilon(t,x,y,z;T) + \delta\, P_2^\varepsilon(t,x,y,z;T) + \cdots, \quad t \le T, \tag{3.15}$$

$$P_k^\varepsilon(t,x,y,z;T) = P_{0,k}(t,x,y,z;T) + \sqrt{\varepsilon}\, P_{1,k}(t,x,y,z;T) + \varepsilon\, P_{2,k}(t,x,y,z;T) + \cdots, \quad t \le T, \tag{3.16}$$

for $k = 0,1,2,\ldots$. The convergence of the approximation (3.15) to $P(t,x,y,z;T)$ defined in (3.5) for fixed t,x,y,z as $\varepsilon,\delta \to 0$ is given in Theorem 3.1.

The PDE problems that P_0^ε and P_1^ε satisfy are defined by

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\varepsilon = 0, \quad t < T, \qquad P_0^\varepsilon(T,x,y,z;T) = 1, \tag{3.17}$$

and

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^\varepsilon = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3\right) P_0^\varepsilon, \quad t < T, \qquad P_1^\varepsilon(T,x,y,z;T) = 0, \tag{3.18}$$

which are obtained by inserting (3.15) into (3.6) and comparing terms in δ^0 and $\sqrt{\delta}$ respectively.

Next we insert the fast-scale expansion (3.16) with $k = 0$ for P_0^ε into (3.17) and collecting terms in ε^{-1} gives

$$\mathcal{L}_0 P_0 = 0.$$

This is a homogeneous ODE in y, and its solutions either have exponential growth at infinity or are independent of y. We therefore construct our approximation so that P_0 is independent of y, i.e., $P_0 = P_0(t,x,z;T)$.

The terms in $\varepsilon^{-1/2}$ lead to

$$\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0.$$

Since $\mathcal{L}_1$ takes derivatives in y, $\mathcal{L}_1 P_0 = 0$ hence the PDE becomes the ODE $\mathcal{L}_0 P_{1,0} = 0$ and we again choose $P_{1,0} = P_{1,0}(t,x,z;T)$.

The terms independent of ε give

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0, \tag{3.19}$$

since $\mathcal{L}_1 P_{1,0} = 0$. This is a Poisson equation for $P_{2,0}$ in y, which has a solution only if the source term $\mathcal{L}_2 P_0$ is centered with respect to the invariant distribution of Y. This solvability condition in also known as the *Fredholm alternative*. In other words, we require

$$\langle \mathcal{L}_2 \rangle P_0(t,x,z;T) = 0, \quad t < T, \tag{3.20}$$

where $\langle \mathcal{L}_2 \rangle$ corresponds to the operator $\mathcal{L}_2$ after averaging with respect to the invariant distribution of the process Y, or

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \alpha(\bar{r} - x)\frac{\partial}{\partial x} - (x + q\langle f \rangle(z)) \cdot \, .$$

Here, $\langle \cdot \rangle$ denotes integration with respect to the invariant distribution of Y (see (3.4)),

$$\langle f \rangle(z) := \int_{\mathbb{R}} f(y,z)\Psi(y)\, dy.$$

Along with the terminal condition $P_0(T,x,z;T) = 1$ the PDE problem (3.20) defines the leading order term.

The terms in $\sqrt{\varepsilon}$ give

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0,$$

which is a Poisson equation for $P_{3,0}$ whose solvability condition is

$$\langle \mathcal{L}_2 \rangle P_{1,0} = -\langle \mathcal{L}_1 P_{2,0} \rangle.$$

From (3.19) and (3.20) we have

$$\mathcal{L}_0 P_{2,0} = -\mathcal{L}_2 P_0 = -(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0,$$

and so the solvability condition becomes

$$\langle \mathcal{L}_2 \rangle P_{1,0} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0.$$

Let us define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ to be a solution to the Poisson equation

$$\mathcal{L}_0 \varphi(y,z) = f(y,z) - \langle f \rangle(z), \tag{3.21}$$

and we also define the operator $\mathcal{A}$ as

$$\mathcal{A} := \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \right\rangle,$$

which, when it is applied to functions that depend only on t, x, and z, can be written explicitly as

$$\mathcal{A} = -\rho_1 \sigma \nu q \sqrt{2} \left\langle \varphi_y \right\rangle (z) \frac{\partial}{\partial x}. \tag{3.22}$$

We arrived to this by using the definition of the operator $\mathcal{L}_1$, the solution φ to the Poisson equation (3.21), since $\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle = q(\langle f \rangle(z) - f(y,z)) \cdot$.

This leads us to define $\tilde{P}_{1,0} := \sqrt{\varepsilon}\, P_{1,0}$ as the solution of

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \tilde{P}_{1,0}(t,x,z;T) &= \sqrt{\varepsilon}\, \mathcal{A} P_0(t,x,z;T), \quad t < T, \\ \tilde{P}_{1,0}(T,x,z;T) &= 0. \end{aligned} \tag{3.23}$$

Next, we insert the expansion (3.16) with $k = 1$, namely

$$P_1^{\varepsilon} = P_{0,1} + \sqrt{\varepsilon}\, P_{1,1} + \varepsilon\, P_{2,1} + \cdots \, ,$$

into (3.18) and collect terms in like powers of ε. Similar arguments as before lead us to define $\tilde{P}_{0,1} := \sqrt{\delta}\, P_{0,1}$ as the solution of

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \tilde{P}_{0,1}(t,x,z;T) &= -\sqrt{\delta}\, \mathcal{M}_1 P_0(t,x,z;T), \quad t<T, \\ \tilde{P}_{0,1}(T,x,z;T) &= 0. \end{aligned} \tag{3.24}$$

Notice that the asymptotic approximation up to order $\mathcal{O}(\varepsilon,\delta)$ does not depend on the current value of the process Y, i.e. on $\{Y_t = y\}$. Instead, only parameters of the process Y enter the pricing function via the distribution of Y_∞. This is an important feature of the solutions that we take advantage of in the calibration of the pricing functions in Section 4.

Below we summarize this asymptotic pricing function for the defaultable zero-coupon bond and give explicit formulas. We also state the accuracy result for the bond price and sketch its proof.

3.2.2. *Bond Price and Bond Yield Approximation.* Let us define

$$\bar{\lambda}(z) := q\langle f\rangle(z), \tag{3.25}$$

$$V_1(z) := \sqrt{\varepsilon}\,\rho_1\nu q\sqrt{2}\langle\varphi_y\rangle(z), \tag{3.26}$$

$$V_2(z) := \sqrt{\delta}\,\rho_2 g(z) q\langle f\rangle_z(z), \tag{3.27}$$

$$h_1(s) := \frac{\sigma}{\alpha}(B(s)-s), \tag{3.28}$$

$$h_2(s) := \frac{\sigma}{2\alpha^2}(2+\alpha s)s - \frac{\sigma}{\alpha^2}(1+\alpha s)B(s), \tag{3.29}$$

with B as in (3.13), and $\langle f\rangle_z(z) := \frac{\partial}{\partial z}\langle f\rangle(z)$.

The solutions to the PDE problems (3.20), (3.23), and (3.24) determine the following approximation terms

$$P_0(t,x,z;T) = A(T-t)\exp\left(-B(T-t)x - \bar{\lambda}(z)(T-t)\right), \tag{3.30}$$

$$\tilde{P}_{1,0}(t,x,z;T) = V_1(z)h_1(T-t)P_0(t,x,z;T), \tag{3.31}$$

$$\tilde{P}_{0,1}(t,x,z;T) = V_2(z)h_2(T-t)P_0(t,x,z;T), \tag{3.32}$$

with A defined in (3.14). The leading order term is the default-free zero-coupon bond price assuming the model (3.1) for the short rate process further discounted by the "average credit spread" $\bar{\lambda}$ evaluated at the time t value of the slowly-varying process Z, i.e., at $Z_t = z$.

The correction terms, $\tilde{P}_{1,0}$ and $\tilde{P}_{0,1}$, are products of the leading order term with time-dependent functions and the parameters V_1 and V_2. The parameter V_1 contains the effect of the correlation ρ_1 between the fast intensity factor Y and the short rate r, and similarly V_2 contains the effect of ρ_2 between the slow intensity factor Z and the short rate r. Both are proportional to the loss fraction q and the square root of ε and δ respectively.

We denote the approximation up to order $\mathcal{O}(\varepsilon,\delta)$ of the price at time t of a defaultable zero-coupon bond with maturity T by $P^{\varepsilon,\delta}$ and is given by

$$P^{\varepsilon,\delta}(t,x,z;T) = P_0(t,x,z;T) + \tilde{P}_{1,0}(t,x,z;T) + \tilde{P}_{0,1}(t,x,z;T), \quad t\le T. \tag{3.33}$$

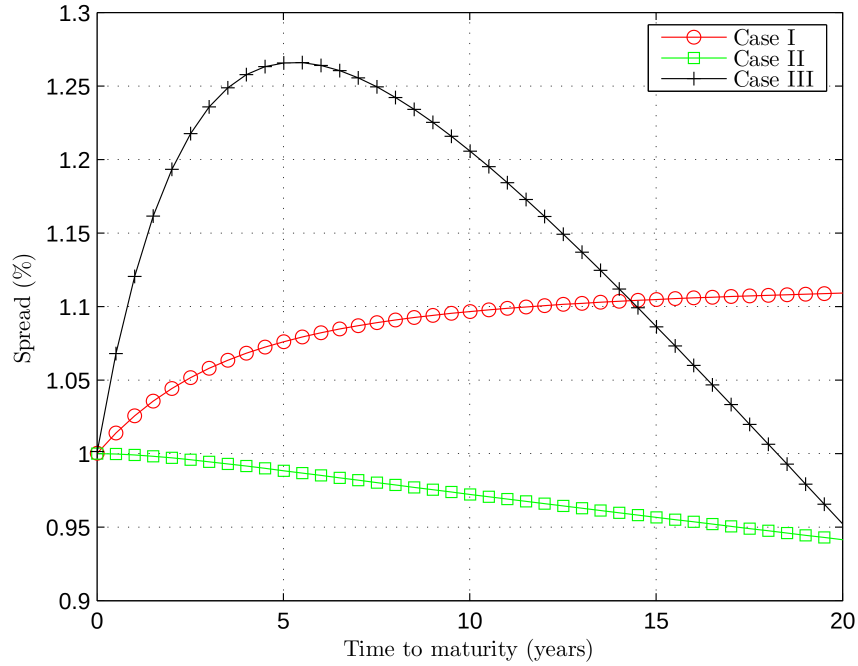

FIGURE 3.1. *The effect of the group parameters* V_1 *and* V_2 *on the credit spreads derived from the approximation to the defaultable zero-coupon bond. The three spread curves were drawn for the combinations* $V_1 = 0.02$, $V_2 = 0$ *(Case I),* $V_1 = 0$, $V_2 = 0.001$ *(Case II), and* $V_1 = 0.1$, $V_2 = 0.01$ *(Case III). We set* $\bar{\lambda} = 0.01$ *for all curves and the parameters for the short-rate process are* $\alpha = 0.5$, $\bar{r} = 0.05$, $\sigma = 0.03$, *and* $r_0 = x = 0.045$.

The yield curve corresponding to the approximate defaultable bond price is the mapping $s \mapsto R(\cdot, \cdot, \cdot; s)$ with

$$\begin{aligned} R(\bar{\lambda}(z), V_1(z), V_2(z); s) &= -\frac{1}{s} \log P^{\varepsilon,\delta}(0, x, z; s) \\ &= -\frac{1}{s} \log\left(1 + h_1(s)V_1(z) + h_2(s)V_2(z)\right) - \frac{1}{s} \log P_0(0, x, z; s), \end{aligned} \tag{3.34}$$

for $s > 0$.

Similarly, the yield spread curve or *credit spread* curve of the defaultable bond is defined as the excess spread of the yield of the defaultable bond over the equivalent default-free security, in this case a treasury bond. The spread curve is then, simply, the mapping $s \mapsto R(\cdot, \cdot, \cdot; s) - R(0, 0, 0; s)$ for $s \in \mathbb{R}_+$.

The asymptotic approximation of the bond price offers a variety of spread curve shapes, as can be seen from Figure 3.1. This is particularly important when calibrating the model on spread curves of distressed firms, such as firms with low credit ratings where the probability of an imminent default is considerably higher. Notice the hump-shaped spread curve for Case III, which matches a typical spread curve for structural models as in the Merton (1974) model, but with non-zero spread at short maturities. We elaborate on the spread curves and the effects of the group parameters V_1 and V_2 on them in Section 4 where we also perform a calibration exercise for two investment grade companies.

3.2.3. *Accuracy of the Asymptotic Approximation.* Under the expansion described in Section 3.2.1, the asymptotic approximation of the bond price given by the expression (3.33) has an error of order $\mathcal{O}(\varepsilon,\, \delta)$. We state this formally below.

Theorem 3.1. *For fixed $0 < t < T$, fixed $x,\, y,\, z \in \mathbb{R}$, and for every $\varepsilon \leq 1$, $\delta \leq 1$, there exists a positive constant $C < \infty$ that depends on (t, x, y, z) but not on ε and δ such that*

$$|P(t, x, y, z; T) - P^{\varepsilon,\delta}(t, x, z; T)| \leq C(\varepsilon + \delta).$$

Proof. The first step of the proof is to control the unbounded potential term in $\mathcal{L}_2$, since the Ornstein-Uhlenbeck process r may go to $-\infty$. To that end, we make the transformation

$$P(t, x, y, z; T) = A(T - t)e^{-B(T-t)x} F(t, y, z; T), \quad t \leq T.$$

Using the ODEs (3.12) and (3.11) for A and B, we find that F solves the following PDE problem

$$\begin{aligned} \widetilde{\mathcal{L}}^{\varepsilon,\delta} F &= 0, \quad t < T, \\ F(T, y, z; T) &= 1, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \widetilde{\mathcal{L}}^{\varepsilon,\delta} &:= \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\widetilde{\mathcal{L}}_1 + \widetilde{\mathcal{L}}_2 + \sqrt{\delta}\,\widetilde{\mathcal{M}}_1 + \delta\,\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3, \\ \widetilde{\mathcal{L}}_1 &:= -B(T - t)\rho_1\sigma\nu\sqrt{2}\frac{\partial}{\partial y}, \\ \widetilde{\mathcal{L}}_2 &:= \frac{\partial}{\partial t} - qf(y, z)\cdot\,, \\ \widetilde{\mathcal{M}}_1 &:= -B(T - t)\rho_2\sigma g(z)\frac{\partial}{\partial z}, \end{aligned}$$

and $\mathcal{L}_0$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are defined in (3.8), (3.9), and (3.10) respectively.

Now, it suffices to show that for fixed y and z in $\mathbb{R}$ and for fixed $0 < t < T$ the approximation of F (shown below) is of order $\mathcal{O}(\varepsilon,\, \delta)$, since the pre-factor $A(T - t)e^{-B(T-t)x}$ is independent of $(\varepsilon,\, \delta)$.

Similar to the approximation for the bond price P we can construct an expansion for F, denoted $F^{\varepsilon,\delta}$ and given explicitly by

$$F^{\varepsilon,\delta}(t, y, z; T) = F_0(t, y, z; T) + \sqrt{\varepsilon}\, F_{1,0}(t, y, z; T) + \sqrt{\delta}\, F_{0,1}(t, y, z; T), \quad t \leq T.$$

Analogous to (3.30), (3.31), and (3.32) we have

$$\begin{aligned} F_0(t, z; T) &= \exp(-\bar{\lambda}(z)(T - t)), \\ \sqrt{\varepsilon}\, F_{1,0}(t, z; T) &= V_1(z)h_1(T - t)F_0(t, z; T), \\ \sqrt{\delta}\, F_{0,1}(t, z; T) &= V_2(z)h_2(T - t)F_0(t, z; T), \end{aligned}$$

where V_1, V_2, h_1, and h_2 were defined in (3.26), (3.27), (3.28), and (3.29) respectively.

We further define the higher-order terms $F_{1,1}$, $F_{2,0}$, $F_{2,1}$, and $F_{3,0}$ to be solutions of

$$\left\langle \widetilde{\mathcal{L}}_2 \right\rangle F_{1,1} = \widetilde{\mathcal{A}} F_{0,1} - \widetilde{\mathcal{M}}_1 F_{1,0}, \tag{3.36}$$

$$\mathcal{L}_0 F_{2,0} = -\left(\widetilde{\mathcal{L}}_2 - \left\langle \widetilde{\mathcal{L}}_2 \right\rangle\right) F_0, \tag{3.37}$$

$$\mathcal{L}_0 F_{2,1} = -\left(\widetilde{\mathcal{L}}_2 - \left\langle \widetilde{\mathcal{L}}_2 \right\rangle\right) F_{0,1}, \tag{3.38}$$

$$\mathcal{L}_0 F_{3,0} = -\left(\widetilde{\mathcal{L}}_1 F_{2,0} - \left\langle \widetilde{\mathcal{L}}_1 F_{2,0} \right\rangle\right) - \left(\widetilde{\mathcal{L}}_2 - \left\langle \widetilde{\mathcal{L}}_2 \right\rangle\right) F_{1,0}, \tag{3.39}$$

where

$$\widetilde{\mathcal{A}} := \left\langle \widetilde{\mathcal{L}}_1 \mathcal{L}_0^{-1} \left(\widetilde{\mathcal{L}}_2 - \left\langle \widetilde{\mathcal{L}}_2 \right\rangle\right) \right\rangle = \rho_1 \sigma \nu q \sqrt{2} \langle \varphi_y \rangle (z) B(T-t) \cdot \ .$$

These PDEs are motivated by inserting the higher expansion for $F^{\varepsilon,\delta}$ into the PDE (3.35) and collecting the terms of order $\sqrt{\varepsilon\delta}$, ε, $\varepsilon\sqrt{\delta}$, and $\varepsilon^{3/2}$, respectively.

Next we introduce the higher order approximation for F,

$$\begin{aligned}
\widehat{F^{\varepsilon,\delta}} &:= F^{\varepsilon,\delta} + \varepsilon\,(F_{2,0} + \sqrt{\varepsilon}\, F_{3,0}) + \sqrt{\delta}\,(\sqrt{\varepsilon}\, F_{1,1} + \varepsilon\, F_{2,1}) \\
&= F_0 + \sqrt{\varepsilon}\, F_{1,0} + \varepsilon\, F_{2,0} + \varepsilon^{3/2} F_{3,0} + \sqrt{\delta}(F_{0,1} + \sqrt{\varepsilon} F_{1,1} + \varepsilon F_{2,1}),
\end{aligned}$$

and we introduce the error term $Q^{\varepsilon,\delta}$ defined as

$$Q^{\varepsilon,\delta}(t,y,z;T) := \widehat{F^{\varepsilon,\delta}}(t,y,z;T) - F(t,y,z;T).$$

Applying the operator $\widetilde{\mathcal{L}}^{\varepsilon,\delta}$ on $Q^{\varepsilon,\delta}$ and using (3.35) we get

$$\begin{aligned}
\widetilde{\mathcal{L}}^{\varepsilon,\delta} Q^{\varepsilon,\delta} &= \widetilde{\mathcal{L}}^{\varepsilon,\delta} \widehat{F^{\varepsilon,\delta}} \\
&= \frac{1}{\varepsilon} \mathcal{L}_0 F_0 + \frac{1}{\sqrt{\varepsilon}} (\widetilde{\mathcal{L}}_1 F_0 + \mathcal{L}_0 F_{1,0}) + \mathcal{L}_0 F_{2,0} + \widetilde{\mathcal{L}}_1 F_{1,0} + \widetilde{\mathcal{L}}_2 F_0 \\
&\quad + \sqrt{\varepsilon}\,(\mathcal{L}_0 F_{3,0} + \widetilde{\mathcal{L}}_1 F_{2,0} + \widetilde{\mathcal{L}}_2 F_{1,0}) + \sqrt{\delta}\,\left[\frac{1}{\varepsilon}\mathcal{L}_0 F_{0,1} + \frac{1}{\sqrt{\varepsilon}}\,\left(\widetilde{\mathcal{L}}_1 F_{0,1} + \mathcal{L}_0 F_{1,1} + \mathcal{M}_3 F_0\right)\right] \\
&\quad + \sqrt{\delta}\,(\mathcal{L}_0 F_{2,1} + \widetilde{\mathcal{L}}_1 F_{1,1} + \widetilde{\mathcal{L}}_2 F_{0,1} + \widetilde{\mathcal{M}}_1 F_0 + \mathcal{M}_3 F_{1,0}) + \varepsilon\, Q_1^\varepsilon + \sqrt{\varepsilon\delta}\, Q_2^\varepsilon + \delta\, Q_3^\varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
Q_1^\varepsilon &:= \widetilde{\mathcal{L}}_2 F_{2,0} + \widetilde{\mathcal{L}}_1 F_{3,0} + \sqrt{\varepsilon}\, \widetilde{\mathcal{L}}_2 F_{3,0}, \\
Q_2^\varepsilon &:= \widetilde{\mathcal{L}}_2 F_{1,1} + \widetilde{\mathcal{L}}_2 F_{2,1} + \widetilde{\mathcal{M}}_1 F_{1,0} + \mathcal{M}_3 F_{2,0} + \sqrt{\varepsilon}\,(\widetilde{\mathcal{L}}_2 F_{2,1} + \widetilde{\mathcal{M}}_1 F_{2,0} + \mathcal{M}_3 F_{3,0}) \\
&\quad + \varepsilon\, \widetilde{\mathcal{M}}_1 F_{3,0}, \\
Q_3^\varepsilon &:= \widetilde{\mathcal{M}}_1 F_{0,1} + \mathcal{M}_2 F_0 + \mathcal{M}_3 F_{1,1} + \sqrt{\varepsilon}\,(\widetilde{\mathcal{M}}_1 F_{1,1} + \mathcal{M}_2 F_{1,0} + \mathcal{M}_3 F_{2,1}) \\
&\quad + \varepsilon\,(\widetilde{\mathcal{M}}_1 F_{2,1} + \mathcal{M}_2 F_{2,0}).
\end{aligned}$$

All the terms on the right-hand side of the expression for $\widetilde{\mathcal{L}}^{\varepsilon,\delta} Q^{\varepsilon,\delta}$, apart from the last three, are zero by the definition of the terms F_0, $F_{1,0}$, $F_{0,1}$, $F_{1,1}$, $F_{2,0}$, $F_{2,1}$, $F_{3,0}$ above and the PDEs (3.36), (3.37), (3.38), and (3.39).

Furthermore, at the maturity time T we have (suppressing the dependence on y and z at places)

$$\begin{aligned}
Q^{\varepsilon,\delta}(T;T) &= \widehat{F^{\varepsilon,\delta}}(T;T) - F(T;T) \\
&= \varepsilon\, F_{2,0}(T;T) + \varepsilon^{3/2}\, F_{3,0}(T;T) + \sqrt{\varepsilon\delta}\, F_{1,1}(T;T) + \varepsilon\sqrt{\delta}\, F_{2,1}(T;T) \\
&= \varepsilon \left(F_{2,0}(T;T) + \sqrt{\varepsilon}\, F_{3,0}(T;T)\right) + \sqrt{\varepsilon\delta}\left(F_{1,1}(T;T) + \sqrt{\varepsilon} F_{2,1}(T;T)\right) \\
&=: \varepsilon\, G_1(y,z;T) + \sqrt{\varepsilon\delta}\, G_2(y,z;T).
\end{aligned}$$

We write then the probabilistic representation for $Q^{\varepsilon,\delta}$ as

$$\begin{aligned}
&Q^{\varepsilon,\delta}(t,y,z;T) \\
&= \varepsilon\, \mathbb{E}^\star_{t,y,z}\left[e^{-\int_t^T qf(Y_s,Z_s)\,ds} G_1(Y_T,Z_T;T) - \int_t^T e^{-\int_t^u qf(Y_s,Z_s)\,ds} Q_1^\varepsilon(u,Y_u,Z_u)\,du\right] \\
&\quad + \sqrt{\varepsilon\delta}\, \mathbb{E}^\star_{t,y,z}\left[e^{-\int_t^T qf(Y_s,Z_s)\,ds} G_2(Y_T,Z_T;T) - \int_t^T e^{-\int_t^u qf(Y_s,Z_s)\,ds} Q_2^\varepsilon(u,Y_u,Z_u)\,du\right] \\
&\quad + \delta\, \mathbb{E}^\star_{t,y,z}\left[-\int_t^T e^{-\int_t^u qf(Y_s,Z_s)\,ds} Q_3^\varepsilon(u,Y_u,Z_u)\,du\right],
\end{aligned}$$

with $\mathbb{E}^\star_{t,y,z}[\cdot] = \mathbb{E}^\star[\cdot \mid Y_t = y,\, Z_t = z]$. For fixed $y,\, z \in \mathbb{R}$ and $t < T$ the terms Q_1^ε, Q_2^ε, and Q_3^ε are uniformly bounded by smooth functions of t, z independent of ε, δ which grow at most linearly in $|y|$. We arrive to this conclusion by the fact that f is bounded and so we can choose the solution φ to the Poisson equation (3.21) to be at most linearly growing in $|y|$ (see Fouque et al., 2003b, Lemma 4.3 and Appendix C). Additionally, as functions of z the higher order terms $F_{1,1}$, $F_{2,0}$, $F_{2,1}$, $F_{3,0}$ as defined in (3.36), (3.37), (3.38), and (3.39) are bounded because of our boundedness assumption on f, and because they satisfy ODEs in y with z just as a parameter. Further, $F_{1,1}$, $F_{2,0}$, $F_{2,1}$, $F_{3,0}$ can be chosen to grow at most linearly in $|y|$ because of their dependence on φ. For this reason, the terms G_1 and G_2 (which do not depend on t) are uniformly bounded in z and at most linearly growing in $|y|$.

Therefore, since for fixed $y,\, z \in \mathbb{R}$ and fixed $0 < t < T$ all the expectations above are uniformly bounded by some constants, we can write

$$\begin{aligned}
|F(t,y,z;T) - F^{\varepsilon,\delta}(t,z;T)| &= |F - \widehat{F^{\varepsilon,\delta}} + \widehat{F^{\varepsilon,\delta}} - F^{\varepsilon,\delta}| \\
&\le |F - \widehat{F^{\varepsilon,\delta}}| + |\widehat{F^{\varepsilon,\delta}} - F^{\varepsilon,\delta}| \\
&= |Q^{\varepsilon,\delta}| + \varepsilon\,|F_{2,0} + \sqrt{\varepsilon}\, F_{3,0}| + \sqrt{\varepsilon\delta}\,|F_{1,1} + \sqrt{\varepsilon}\, F_{2,1}| \\
&\le \varepsilon\, C_1 + \sqrt{\varepsilon\delta}\, C_2 + \delta\, C_3 \\
&\le C(\varepsilon + \delta),
\end{aligned}$$

for suitably defined constants C, C_1, C_2, and C_3 which depend on the parameters, but not on ε and δ. This completes the proof. □

3.3. **Credit Default Swap.** A credit default swap (CDS) contract is a derivative that provides insurance against the default of a *reference entity* which is usually a corporation or a sovereign (i.e., a national government). Below we provide a brief description of such a contract.

3.3.1. *Preliminaries.* In a single-name credit default swap the *protection buyer*, that is, the counterparty that receives a payoff if the reference entity defaults, pays a periodic premium to the *protection seller*. In return, the protection seller has to compensate the protection buyer in the case where the reference entity defaults prior to a pre-determined maturity time. The premiums the protection buyer pays to the protection seller are usually paid quarterly or semiannually until the maturity of the credit default swap contract, or until the time of the default event, whichever comes first.

To fix ideas, suppose that the CDS is written on a bond issued by the reference entity. In the case where the reference entity defaults before the maturity of the CDS contract, we assume that the protection seller compensates the protection buyer with a *cash settlement*. In particular, the protection seller will make a cash payment to the protection buyer equal to the difference of the notional amount of the bond and its post-default market value which is usually determined by polling several dealers. Unlike the recovery of market value introduced above, here we assume that in default the bond recovers $1-q$ of its face value—which is known as *recovery of face value*—and the protection seller provides the remaining proportion q of the face value to the protection buyer.

While there are other types of settlement used in practice, we will not go into details here. For a detailed specification of the repayment methods at default, more information on what constitutes a default event, usual practices for settlements in the financial industry, and other legal issues we refer to the books by Duffie and Singleton (2003), Schönbucher (2003), and Lando (2004).

The pricing of a credit default swap amounts to determining the *CDS spread* which determines the amount paid by the protection buyer to the protection seller on each payment date. Assume that there are M such scheduled periodic payments and let $\mathcal{T}$ be the *payment tenor*, used to denote the sequence of these payment dates T_m, $m = 1, \dots, M$ or in other words $\mathcal{T} = (T_1, \dots, T_M)$, with $T_1 < \dots < T_M$. We refer to T_M as the *maturity* of the CDS contract.

The CDS spread is quoted in basis points or one hundredth of one percent (i.e., 1/10000) and thus the payment of the protection buyer to the protection seller at each payment date is the product of the CDS spread and the face value of the bond which is designated in the CDS contract. We will also make the simplifying assumptions that

(i) the bond coupon dates match the payment dates of the CDS, and
(ii) if a default occurs the settlement takes place at a coupon date following the default but we do not consider the accrued interest of the intermediate period.

Instead of determining the CDS spread for a credit default swap that would be active immediately we will price a *forward CDS* (forward contract on a CDS). A forward CDS is a CDS contract between a protection buyer and a protection seller with payment tenor $\mathcal{T}$ that is active after some initial time T_0 (the *effective date*, where $0 \le T_0 < T_1$). In the case where the credit event happens prior to the effective date the forward CDS is worthless and no payments are made from either counterparty.

Since the CDS contract is made up from two distinct counterparties' payoffs, its pricing follows from no-arbitrage arguments based on the pricing of each counterparty's position. We will determine the payment of the protection buyer, c^{pb}, and that of the protection seller, c^{ps}, at some time t with $t \leq T_0$. We denote the forward CDS spread at time t with effective date T_0 and payment tenor $\mathcal{T}$ as $c^{\mathrm{ds}}(t, T_0; \mathcal{T})$ and so the spread of a CDS is given by the spread of a forward CDS with immediate effective date, i.e., $c^{\mathrm{ds}}(t, t; \mathcal{T})$.

Let us consider the case of the protection buyer first. Such an individual will pay a premium at each payment date T_m $(m = 1, \ldots, M)$ until maturity or until default, whichever comes first, for every dollar of the face value of the bond. By recalling that τ is the time of the first jump of the doubly-stochastic Poisson process $\tilde{N}$ we can express this premium as

$$
\begin{aligned}
c^{\mathrm{pb}}(t, T_0; \mathcal{T}) &= \mathbb{E}^\star \left[\sum_{m=1}^{M} \exp\left(-\int_t^{T_m} r_s \, ds \right) \mathbf{1}_{\{\tau > T_m\}} c^{\mathrm{ds}}(t, T_0; \mathcal{T}) \mid \mathcal{F}_t \vee \sigma\{\tilde{N}_s;\ 0 \leq s \leq t\} \right] \\
&= c^{\mathrm{ds}}(t, T_0; \mathcal{T}) \sum_{m=1}^{M} \mathbb{E}^\star \left[\exp\left(-\int_t^{T_m} (r_s + \lambda_s) \, ds \right) \mid \mathcal{F}_t \right], \quad t < \tau. \qquad (3.40)
\end{aligned}
$$

We arrived at the expression (3.40) as in the defaultable bond pricing expression (3.5). Indeed, the conditional expectation that appears in (3.40) is simply the price of a zero-coupon bond at time t with zero recovery of market value (or zero recovery of face value) and maturity T_m, which we denote by $p(t; T_m)$ and is given by

$$
p(t; T_m) = \mathbf{1}_{\{t < \tau\}} \mathbb{E}^\star \left[\exp\left(-\int_t^{T_m} (r_s + \lambda_s) \, ds \right) \mid \mathcal{F}_t \right].
$$

The payment of the protection buyer is zero if $t \geq \tau$.

Under the cash settlement assumption we made, the present (time t) payment by the protection seller at the default time τ is

$$
\begin{aligned}
c^{\mathrm{ps}}(t, T_0; \mathcal{T}) &= \mathbb{E}^\star \left[\exp\left(-\int_t^{\tau} r_s \, ds \right) \mathbf{1}_{\{T_0 \leq \tau \leq T_M\}} q \mid \mathcal{F}_t \vee \sigma\{\tilde{N}_s;\ 0 \leq s \leq t\} \right] \\
&= \mathbb{E}^\star \left[\int_{T_0}^{T_M} \exp\left(-\int_t^{u} (r_s + \lambda_s) \, ds \right) q \lambda_u \, du \mid \mathcal{F}_t \right], \quad t < \tau.
\end{aligned}
$$

Similarly, the protection seller payment is zero on $\{t \geq \tau\}$.

The forward CDS spread is, by definition, the spread that equates the payments of the two counterparties, thus

$$
c^{\mathrm{ds}}(t, T_0; \mathcal{T}) = \frac{\mathbb{E}^\star \left[\int_{T_0}^{T_M} \exp\left(-\int_t^{u} (r_s + \lambda_s) \, ds \right) q \lambda_u \, du \mid \mathcal{F}_t \right]}{\sum_{m=1}^{M} p(t; T_m)}, \quad t < \tau.
$$

3.3.2. *Asymptotic Approximation of the CDS Spread.* The asymptotic approximation of the payment of the protection buyer is obtained from the corresponding defaultable bond expression given in Section 3.2.1. Under the asymptotic approximation (3.33) for the defaultable zero-coupon bond price, we will denote this price by $p^{\varepsilon,\delta}(t; T_m)$, or $p^{\varepsilon,\delta}(t, x, z; T_m)$ as needed, and it is equal to $P^{\varepsilon,\delta}$ in (3.33) with $q = 1$.

For the protection seller payment let

$$w(t,r_t,Y_t,Z_t;T)=\mathbb{E}^\star\left[\int_t^T \exp\left(-\int_t^u (r_s+f(Y_s,Z_s))\,ds\right) qf(Y_u,Z_u)\,du \mid r_t,Y_t,Z_t\right].$$

Then the protection seller payment is given by

$$c^{\mathrm{ps}}(t,T_0;\mathcal{T})=w(t,r_t,Y_t,Z_t;T_M)-w(t,r_t,Y_t,Z_t;T_0),\quad t<\tau\wedge T_0.$$

The inhomogeneous PDE problem that w satisfies is

$$\begin{aligned}\ell^{\varepsilon,\delta}w(t,x,y,z;T)+qf(y,z)&=0,\quad t<T,\\ w(T,x,y,z;T)&=0,\end{aligned}\tag{3.41}$$

where the operator $\ell^{\varepsilon,\delta}$ is defined similar to $\mathcal{L}^{\varepsilon,\delta}$in (3.7),

$$\ell^{\varepsilon,\delta}:=\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\ell_2+\sqrt{\delta}\mathcal{M}_1+\delta\mathcal{M}_2+\sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3.$$

The operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are defined in Section 3.2, and ℓ_2 corresponds to $\mathcal{L}_2$ with $q=1$ for the no-recovery case,

$$\ell_2=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}+\alpha(\bar r-x)\frac{\partial}{\partial x}-(x+f(y,z))\cdot\,.$$

Note the presence of the source term $qf(y,z)$ in the PDE (3.41).

We denote $w^{\varepsilon,\delta}$ the approximation to the pricing function w under our setup. Then using the same notation conventions as in the bond price case we write for the asymptotic expansion of the pricing function

$$\begin{aligned}w(t,x,y,z;T)&=w_0^\varepsilon(t,x,y,z;T)+\sqrt{\delta}\,w_1^\varepsilon(t,x,y,z;T)+\delta\,w_2^\varepsilon(t,x,y,z;T)+\cdots,\\ w_k^\varepsilon(t,x,y,z;T)&=w_{0,k}(t,x,y,z;T)+\sqrt{\varepsilon}\,w_{1,k}(t,x,y,z;T)+\varepsilon\,w_{2,k}(t,x,y,z;T)+\cdots,\end{aligned}$$

for $k=0,1,2,\ldots$. Setting again $\tilde w_{1,0}:=\sqrt{\varepsilon}\,w_{1,0}$, $\tilde w_{0,1}:=\sqrt{\delta}\,w_{0,1}$, the defining PDE problems for the leading order and perturbation terms are

$$\begin{aligned}\langle\ell_2\rangle w_0(t,x,z;T)+\bar\lambda(z)&=0,\quad t<T,\\ w_0(T,x,z;T)&=0,\end{aligned}\tag{3.42}$$

$$\begin{aligned}\langle\ell_2\rangle\tilde w_{1,0}(t,x,z;T)&=\sqrt{\varepsilon}\,\frac{1}{q}\mathcal{A}w_0(t,x,z;T),\quad t<T,\\ \tilde w_{1,0}(T,x,z;T)&=0,\end{aligned}\tag{3.43}$$

$$\begin{aligned}\langle\ell_2\rangle\tilde w_{0,1}(t,x,z;T)&=-\sqrt{\delta}\,\mathcal{M}_1w_0(t,x,z;T),\quad t<T,\\ \tilde w_{0,1}(T,x,z;T)&=0,\end{aligned}\tag{3.44}$$

where $\bar\lambda(z)$ is defined in (3.25), and the operator $\langle\ell_2\rangle$ is similar to $\langle\mathcal{L}_2\rangle$:

$$\langle\ell_2\rangle=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}+\alpha(\bar r-x)\frac{\partial}{\partial x}-(x+\langle f\rangle(z))\cdot\,.$$

Notice that in the source term of the PDE (3.43) the operator $\mathcal{A}$ is divided by the loss fraction, q, to account for the use of the operator ℓ_2 instead of $\mathcal{L}_2$ in the original PDE (3.41). This is due to the recovery of face value assumption made for the CDS contract.

Let $p_0(t,x,z;T)$ be equal to the leading order term $P_0(t,x,z;T)$ in (3.30) of the approximation of a defaultable zero-coupon bond for zero recovery of market value, or

$$p_0(t,x,z;T) = A(T-t)\exp\left(-B(T-t)x - \langle f\rangle(z)(T-t)\right).$$

The solutions of the PDEs (3.42), (3.43), and (3.44) are

$$w_0(t,x,z;T) = \bar{\lambda}(z)\int_0^{T-t} p_0(0,x,z;s)\,ds,$$

for the leading order term, and

$$\tilde{w}_{1,0}(t,x,z;T) = -\sigma V_1(z)\frac{\bar{\lambda}(z)}{q}\int_t^T\int_s^T B(v-s)p_0(t,x,z;v)\,dv\,ds, \tag{3.45}$$

$$\begin{aligned}\tilde{w}_{0,1}(t,x,z;T) = {} & \sigma V_2(z)\frac{\bar{\lambda}(z)}{q}\int_t^T\int_s^T (v-s)B(v-s)p_0(t,x,z;v)\,dv\,ds \\ & - \sigma V_2(z)\int_t^T\int_s^T B(v-s)p_0(t,x,z;v)\,dv\,ds,\end{aligned} \tag{3.46}$$

for the perturbation terms with the functions V_1 and V_2 defined in (3.26) and (3.27), respectively. The latter integrals cannot be calculated in closed form but can be handled numerically.

The asymptotic approximation up to order $\mathcal{O}(\varepsilon,\,\delta)$ for the price of the protection seller payment is

$$w^{\varepsilon,\delta}(t,x,z;T) = w_0(t,x,z;T) + \tilde{w}_{1,0}(t,x,z;T) + \tilde{w}_{0,1}(t,x,z;T). \tag{3.47}$$

Notice again that the approximation up to order $\mathcal{O}(\varepsilon,\,\delta)$ does not depend explicitly on the current level $Y_t = y$ but the parameters of Y enter through V_1 and V_2.

Unlike the bond and bond option price approximations (see Section 3.4.1 below) in the final expression for the CDS we have the presence of an additional q parameter in the expressions for $\tilde{w}_{1,0}$ and $\tilde{w}_{0,1}$ in (3.45) and (3.46). This is due to the recovery assumption made for the protection seller payment. Hence, we assume that the rate of recovery $1-q$ (or, equivalently, the loss fraction q) is known, to avoid estimating it separately. Typical values for the recovery fraction of the underlying bond value are in $[0.4, 0.5]$, or $q \in [0.5, 0.6]$.

Figure 3.2 shows four spread curves for different values of the group parameters V_1 and V_2. The curves take a variety of shapes which is necessary for optimal fit to data.

The order of approximation of the protection seller payment by $w^{\varepsilon,\delta}$ is $\mathcal{O}(\varepsilon,\,\delta)$ for fixed $x, y, z \in \mathbb{R}$ and fixed $t < T$. The argument is very similar as for the bond price given in Theorem 3.1 and we do not repeat it here.

3.4. **Option on a Defaultable Bond.** We are interested in approximating the price of a European call option on a defaultable bond using our asymptotic expansion method. Let T_0 be the maturity of the option, T_1 the maturity of the bond with $T_0 < T_1$, and K the strike price. Assuming that the bond has recovery of market value $1-q$ as before, the price of the

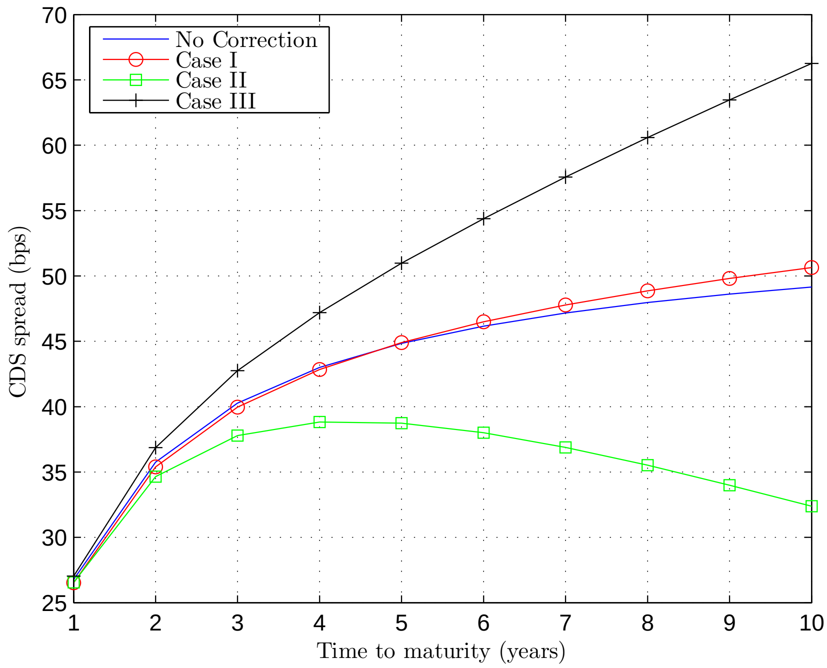

FIGURE 3.2. *CDS spread curves for different values of the group parameters* V_1 *and* V_2. *The unmarked line corresponds to the CDS spread curve with no correction effect* $(V_1 = V_2 = 0)$ *The other three lines were drawn for the parameter sets* $V_1 = -0.5$, $V_2 = 0$ *(Case I),* $V_1 = 0$, $V_2 = 0.01$ *(Case II), and* $V_1 = 0$, $V_2 = -0.01$ *(Case III). The parameter* $\bar{\lambda}$ *was set to 0.005 for all lines and the rest of the parameters were* $\alpha = 0.5$, $\bar{r} = 0.06$, $\sigma = 0.03$, $r_0 = x = 0.06$, *and* $q = 0.6$.

option at time $t \leq T_0$ is given by

$$(3.48) \quad u(t, r_t, Y_t, Z_t) \\ = \mathbb{E}^\star \left[\exp\left(-\int_t^{T_0} (r_s + qf(Y_s, Z_s))\, ds \right) (P(T_0, r_{T_0}, Y_{T_0}, Z_{T_0}; T_1) - K)^+ \mid r_t,\, Y_t,\, Z_t \right],$$

for $t < \tau$. When a default happens prior to T_0, the option expires worthless.

The Feynman-Kac PDE problem corresponding to the price u of such a derivative is the same as the one for the defaultable zero-coupon bond in (3.6) but with a different terminal condition, namely

$$\mathcal{L}^{\varepsilon,\delta} u(t, x, y, z) = 0, \quad t < T_0,$$
$$u(T_0, x, y, z) = (P(T_0, x, y, z; T_1) - K)^+,$$

where $\mathcal{L}^{\varepsilon,\delta}$ was defined in (3.7).

3.4.1. *Asymptotic Approximation of the Bond Option.* We look for an asymptotic expansion of the price of the bond option in half-powers of ε and δ similar to the bond and CDS cases above. We construct the expansion in the following form

$$u(t,x,y,z) = u_0^\varepsilon(t,x,y,z) + \sqrt{\delta}\, u_1^\varepsilon(t,x,y,z) + \delta\, u_2^\varepsilon(t,x,y,z) + \mathcal{O}(\delta^{3/2}),$$
$$u_k^\varepsilon(t,x,y,z) = u_{0,k}(t,x,y,z) + \sqrt{\varepsilon}\, u_{1,k}(t,x,y,z) + \varepsilon\, u_{2,k}(t,x,y,z) + \mathcal{O}(\varepsilon^{3/2}),$$

where $k = 0, 1, 2, \ldots$. We denote $u_0 := u_{0,0}$ the leading order term. Let H be the payoff function,

$$H(s) = (s-K)^+,$$

and combining the asymptotic approximation (3.33) with a Taylor expansion for H we write the terminal condition of the PDE for u as

$$\begin{aligned} &H(P^{\varepsilon,\delta}(T_0,x,z;T_1)) \\ &\quad\approx H(P_0(T_0,x,z;T_1)) + (\sqrt{\varepsilon}\, P_{1,0}(t,x,z;T_1) + \sqrt{\delta}\, P_{0,1}(t,x,z;T_1))H'(P_0(T_0,x,z;T_1)). \end{aligned} \tag{3.49}$$

The payoff function H is not smooth, since the first derivative has a discontinuity at K, or in other words

$$H'(x) = \mathbf{1}_{\{x \geq K\}}.$$

This affects the accuracy of the price approximation and we address this in Section 3.4.2. We use this expansion of the terminal condition to define the terminal conditions for the PDE problems of the leading order term u_0 and the correction terms $u_{1,0}$, $u_{0,1}$. The defining PDE problems for these are given by (using the shorthand notations $\tilde{u}_{1,0} := \sqrt{\varepsilon}\, u_{1,0}$ and $\tilde{u}_{0,1} := \sqrt{\delta}\, u_{0,1}$)

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_0(t,x,z) &= 0, \quad t < T_0, \\ u_0(T_0,x,z) &= (P_0(T_0,x,z;T_1) - K)^+, \end{aligned} \tag{3.50}$$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \tilde{u}_{1,0}(t,x,z) &= \sqrt{\varepsilon}\, \mathcal{A} u_0(t,x,z), \quad t < T_0, \\ \tilde{u}_{1,0}(T_0,x,z) &= \tilde{P}_{1,0}(T_0,x,z;T_1)\mathbf{1}_{\{P_0(T_0,x,z;T_1)\geq K\}}, \end{aligned} \tag{3.51}$$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \tilde{u}_{0,1}(t,x,z) &= -\sqrt{\delta}\, \mathcal{M}_1 u_0(t,x,z), \quad t < T_0, \\ \tilde{u}_{0,1}(T_0,x,z) &= \tilde{P}_{0,1}(T_0,x,z;T_1)\mathbf{1}_{\{P_0(T_0,x,z;T_1)\geq K\}}, \end{aligned} \tag{3.52}$$

where $\mathcal{A}$ is given in (3.22). These PDEs are derived in the same way as (3.20), (3.23), and (3.24). Their terminal conditions come from the expansion (3.49).

The leading order term is given by

$$u_0(t,x,z) = P_0(t,x,z;T_1)\Phi(d_1) - KP_0(t,x,z;T_0)\Phi(d_2), \tag{3.53}$$

with

$$d_{1,2} := \frac{\log(P_0(t,x,z;T_1)/P_0(t,x,z;T_0)) - \log K \pm \bar{\sigma}(t)^2/2}{\bar{\sigma}(t)},$$

$$\bar{\sigma}(t) := \sigma B(T_1 - T_0)\left(\frac{1-e^{-2\alpha(T_0-t)}}{2\alpha}\right)^{1/2}, \tag{3.54}$$

where Φ is the usual standard normal distribution function. This expression is similar to the price of an option on a default-free bond, when the short rate of interest process is driven by the Vasicek model. The difference here is that this option is written on a defaultable bond where the default risk is contained in P_0 through $\bar{\lambda}(z)$.

The expressions for the correction terms are complicated functions of the leading order term. However, the only dependence of $\tilde{u}_{1,0}$ and $\tilde{u}_{0,1}$ on the parameters of Y and Z is via the aggregate parameters V_1 and V_2 as shown below. This is precisely the same effect as in the asymptotic approximations of the bond price. In particular,

$$\tilde{u}_{1,0}(t,x,z) = \eta_1(t)V_1(z)P_0(t,x,z;T_1)\Phi(d_1) - h_1(T_0-t)V_1(z)KP_0(t,x,z;T_0)\Phi(d_2) + \tilde{P}_{1,0}(t,x,z;T_1)\Phi(d_3), \tag{3.55}$$

$$\tilde{u}_{0,1}(t,x,z) = \eta_2(t)V_2(z)P_0(t,x,z;T_1)\Phi(d_1) - h_2(T_0-t)V_2(z)KP_0(t,x,z;T_0)\Phi(d_2) + \tilde{P}_{0,1}(t,x,z;T_1)\Phi(d_3) + \eta_3(t)V_2(z)P_0(t,x,z;T_1)\Phi'(d_1) \tag{3.56}$$

where Φ' is the standard normal density function, and h_1, h_2 were defined in (3.28) and (3.29), respectively. The functions d_3, η_1, η_2, and η_3 are defined in Appendix A along with their derivation details. Once again, the approximation up to this order does not depend explicitly on the current value of $Y_t = y$.

The group parameters V_1 and V_2 have a complicated role in the approximation of the bond option price. In the two left panels of Figure 3.3, we plot the effect of the correction terms $\tilde{u}_{1,0}$ and $\tilde{u}_{0,1}$ on the bond option price with respect to either the bond's time to maturity, T_1 (upper left figure), or the bond option's time to maturity, T_0 (lower left figure). We borrow the numerical values of the parameters from the calibration example of the next Section. The bond prices were at-the-money (to leading-order term), or in other words the strike price at each bond option quote was taken as

$$K \equiv K(T_0,T_1) = P_0(T_0,x,z;T_1).$$

In both cases, the correction terms reduce the bond price and subsequently the bond option price, but the magnitude of this reduction is different for the two maturities. More specifically, when the option maturity, T_0, remains constant the correction effect becomes noticeable as the bond maturity, T_1, increases. On the other hand, when the option maturity, T_0, varies while keeping the bond maturity, T_1, constant the size of the correction is constant across T_0, thus making the two curves move in parallel as can be seen from the lower left figure.

The right panels plot the implied $\bar{\lambda}$ of the corresponding plots on the left. By *implied* $\bar{\lambda}$ of an option price $u^\star$ we mean the value of $\bar{\lambda}$ that makes the leading order term u_0 as defined in (3.53) match $u^\star$. We imply the average default rate $\bar{\lambda}$ through u_0 in (3.53) as it is the immediate extension of the corresponding formula for pricing options on default-free bonds.

Notice that the implied $\bar{\lambda}$ spikes when the maturities of the bond option, T_0, and the bond, T_1, approach each other. This is indicative of the very high average default intensity, $\bar{\lambda}$, that is required so that the effect of the correction terms $\tilde{u}_{1,0}$ and $\tilde{u}_{0,1}$ is offset by the leading order term u_0 on the approximate bond prices.

3.4.2. *On the Asymptotic Approximation with Irregular Payoffs.* The following theorem gives the order of approximation. We provide a sketch of the proof below.

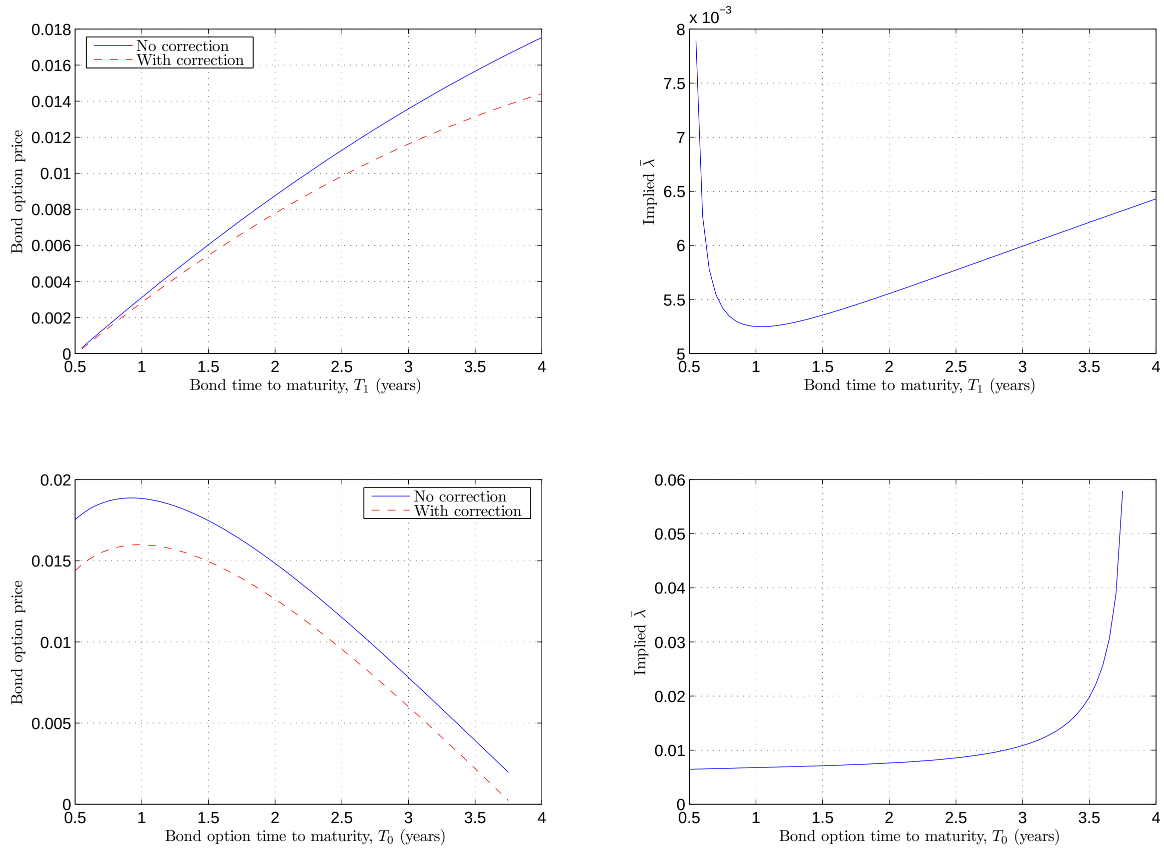

FIGURE 3.3. *The effect of bond time to maturity, T_1, (upper panels) and bond option time to maturity, T_0, (lower panels) on the bond option price (left panels) and implied $\bar{\lambda}$ parameter (right panels). For the upper panels we held $T_0 = 0.5$ whereas for the lower panels we set $T_1 = 4$. The bond prices were quoted at-the-money and we also set the various parameters from our calibration example for IBM Corp. in Section 4 as $\alpha = 0.0816$, $\bar{r} = 0.1658$, $\sigma = 0.0327$, $r_0 = x = 0.0205$, $\bar{\lambda} = 0.0038$, $V_1 = 0.0358$, and $V_2 = 0.0008$.*

Theorem 3.2. *For fixed $0 < t < T_0 < T_1$, fixed x, y, $z \in \mathbb{R}$, and $\varepsilon \leq 1$, $\delta \leq 1$, there exists a positive constant $C < \infty$ that depends on (t, x, y, z) but not on ε and δ such that*

$$|u - (u_0 + \tilde{u}_{1,0} + \tilde{u}_{0,1})| \leq C(\varepsilon|\log \varepsilon| + \delta). \tag{3.57}$$

Sketch of the proof. The order of this approximation cannot be justified in the same way as the bond price approximation in Theorem 3.1 due to the nonsmoothness of the payoff function H at K. It requires a regularization of the payoff function as in Fouque et al. (2003a) for equity call options with stochastic volatility and Cotton et al. (2004) for options on default-free bonds with stochastic volatility on a fast scale. The regularization technique is similar for options on defaultable bonds, yet more involved due to the presence of both singular and regular perturbations.

In the case where the payoff function H is continuous and piecewise smooth (as is the case with the bond option), we can approximate the price of such a security through a *regularized* *payoff* H^ζ, such that $|H(x) - H^\zeta(x)| = \mathcal{O}(\zeta)$, for every $x \in \mathbb{R}$ and $\zeta \leq 1$. If we denote by $\breve{u}_0$, $\breve{u}_{1,0}$, and $\breve{u}_{0,1}$ the analogous terms that correspond to u_0, $\tilde{u}_{1,0}$, and $\tilde{u}_{0,1}$ using the payoff H^ζ instead of H, then we can show that each $\breve{u}$ term approximates its corresponding u term to order $\mathcal{O}(\zeta)$.

Finally, we show through the finite exponential moments of the random variables $\int_t^{T_0} r_s\, ds$ and r_{T_0}, the aforementioned approximation argument for the regularized payoff H^ζ, and an additional approximation of the pricing function with *smooth* payoff H by the pricing function with payoff H^ζ that the approximation for the bond price by $u_0 + \tilde{u}_{1,0} + \tilde{u}_{0,1}$ (with the *non-smooth* payoff H) is indeed as in (3.57). We omit the lengthy derivation details here. □

4. Model Calibration & Empirical Evidence

In this section we test our multiscale model for the stochastic intensity of default process. We find from empirical evaluation that such a two-factor class of diffusion models is flexible, and the time-scale of the slow factor is on the order of three months (see below). Furthermore, we illustrate the performance of our modeling setup for the fitting of yield curves.

4.1. **Model Parametrization.** The calibration of the model amounts to estimating the parameters of the SDEs (3.1)–(3.3) so that the yield curve R of the approximation in (3.34) matches the corporate yield curve as closely as possible. The usual criterion to determine this "closeness" is the least-squares fitting and it is what we employ below.

A major advantage of the asymptotic approximation of the defaultable bond prices is the parameter reduction of the initial model (3.1)–(3.3). The parameters of the processes Y and Z appear in the yield curve expression only as factors in the aggregate functions V_1 and V_2 given in (3.26) and (3.27). For this reason, the estimation of every parameter of Y and Z is not necessary to quantify the effect of the correction—only certain products of them, namely the functions $\bar{\lambda}$, V_1, and V_2.

Moreover, the reduction in the parametric dependence of the pricing expressions does not detract from the interpretative ability of the model. The fast mean reversion of the process Y and the slow evolution of Z, with respect to time, allow for ample flexibility in calibrating across maturities of the yield spread (and CDS spread curve). In particular, for the yield curve of a defaultable bond, we can relate the behavior of

- the short end of the curve (small maturities) to the fast-scale parameter V_1 (more precisely the *slope* of the curve for small maturities),
- the long end of the curve (larger maturities) to the slow-scale parameter V_2, and
- the overall level (and thus the mid maturities) to $\bar{\lambda}$, which also allows for parallel shifts of the curve. This is the reason we interpret $\bar{\lambda}$ as an average credit spread of the defaultable bond as mentioned before.

Additionally, the asymptotic approximation of the price of an option written on a defaultable zero-coupon bond up to order $\mathcal{O}(\varepsilon, \delta)$ depends on the same parameters $\bar{\lambda}$, V_1, and V_2 (and, of course, on the parameters of the short rate model). Subsequently, the estimation

TABLE 4.1. *Summary for the estimated parameters α, $\bar{r}$, σ, and r_0 for the U.S. Treasury yield curve in the period January 2, 2004 to August 11, 2005.*

	Mean	StDev	Min	Max
α	0.0816	0.0296	0.0515	0.2037
$\bar{r}$	0.1658	0.0654	0.0284	0.3266
σ	0.0327	0.0165	0.0027	0.0708
r_0	0.0205	0.0109	0.0044	0.0393

of these parameters from the liquid market of bonds allows for an approximation of the corresponding bond option prices. The same is true as well for other credit derivatives on the same name—that is not to say though that having calibrated the model on corporate yield curves is equivalent to calibrating it on CDS curves. Recent studies have shown the existence of systematic parameters that affect the credit spreads from the corporate bonds and CDSs to behave differently (see Blanco et al., 2005; Longstaff et al., 2004).

Under our asymptotic approximation of the credit default swap spread curve the aggregate parameters $\bar{\lambda}$, V_1, and V_2 have again a similar effect on the different parts of the curve as for the bond yield curve.

4.2. **Calibration Method.** We regard the U.S. government Treasury yield curve as the risk-free rate of interest. As such, we calibrate the yield curve corresponding to the process r on the eight Treasury yield quotes: half-, one-, two-, three-, five-, seven-, ten-, and 20-year for each day between January 2, 2004 and August 11, 2005. From these, the estimates of the parameters $(\alpha, \bar{r}, \sigma, r_0)$ are obtained with a simple least-squares fitting and a summary table is provided on Table 4.1.

Since the beginning of 2004 the Federal Reserve started a series of gradual increases of the overnight bank lending rate for a quarter of a percent at a time. This had an immediate effect on the short end of the yield curve which increased as well. That policy continued throughout the period we examine, resulting in a "flattening" of the yield curve, or in other words a decreasing difference of the long and the short end of the yield curve. For this reason, we stopped our analysis on August 11, 2005. After that date the fitting of the treasury yield curve would require an interest-rate model with richer structure, which goes beyond the scope of the present exercise.

We demonstrate the calibration of the corporate yield curve on two investment grade companies: IBM Corporation (A+) and Wal-Mart Stores (AA). We perform least-squares estimations on $\bar{\lambda}$, V_1, and V_2 by solving for each of the 420 days between January 2, 2004

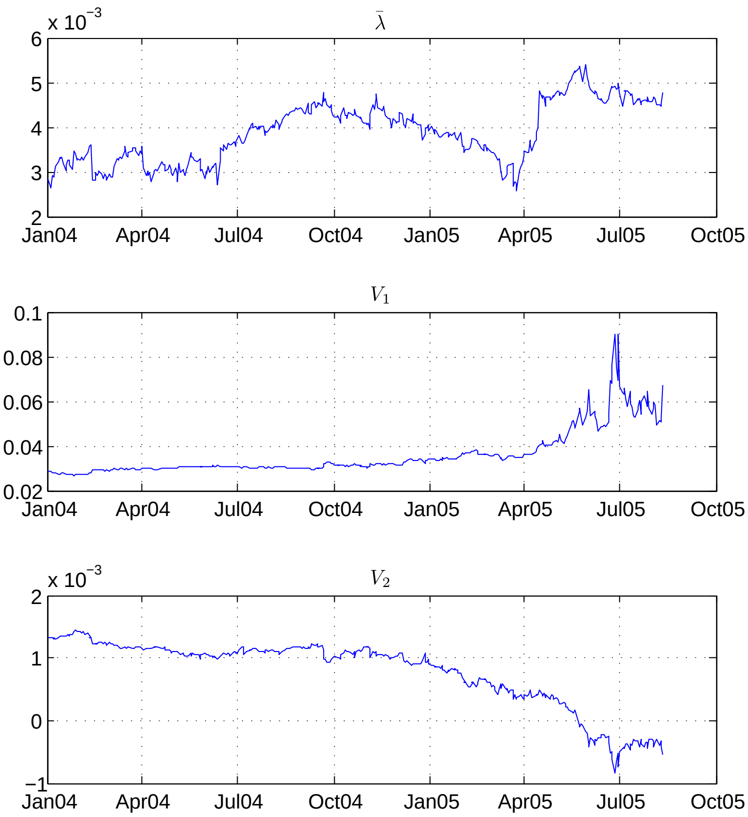

FIGURE 4.1. *Least-square estimates for the parameters $\bar{\lambda}$, V_1, and V_2 (top-down) for IBM (A+) for the 410 days between January 2, 2004 until August 11, 2005. The estimates for the group parameters V_1 and V_2 are very stable until April 2005. The bursting behavior of these estimates for the final part of the sample is attributed to the "flattening" of the Treasury yield curve.*

and August 11, 2005 the following three successive problems

$$\text{(I):} \qquad \widehat{\bar{\lambda}} := \arg\min_{\bar{\lambda}} \sum_{j=1}^{n} \left(R_{\mathrm{C}}(T_j^{\mathrm{C}}) - R\left(\bar{\lambda}, V_1^\star, V_2^\star; T_j^{\mathrm{C}}\right)\right)^2,$$

$$\text{(II):} \qquad \widehat{V}_1 := \arg\min_{V_1} \sum_{j=1}^{n} \left(R_{\mathrm{C}}(T_j^{\mathrm{C}}) - R\left(\widehat{\bar{\lambda}}, V_1, V_2^\star; T_j^{\mathrm{C}}\right)\right)^2,$$

$$\text{(III):} \qquad \widehat{V}_2 := \arg\min_{V_2} \sum_{j=1}^{n} \left(R_{\mathrm{C}}(T_j^{\mathrm{C}}) - R\left(\widehat{\bar{\lambda}}, \widehat{V}_1, V_2; T_j^{\mathrm{C}}\right)\right)^2,$$

where $R_{\mathrm{C}}(T_j^{\mathrm{C}})$ is the corporate yield curve quote corresponding to T_j^{C} remaining years-to-maturity and $V_1^\star$, $V_2^\star$ are constants in $\mathbb{R}$. The number n is the number of bonds that are used to construct the corporate yield curve for each company.

Under this calibration setup the average credit spread is fitted first, while the remaining part of the spread is explained by the other two group parameters. The order in which we execute problems (II) and (III) has a small effect on the estimates of V_1 and V_2. Had we reversed their order (and used $V_1^\star$ in (II) and $\widehat{V}_2$ in (III)), the focus of the calibration would

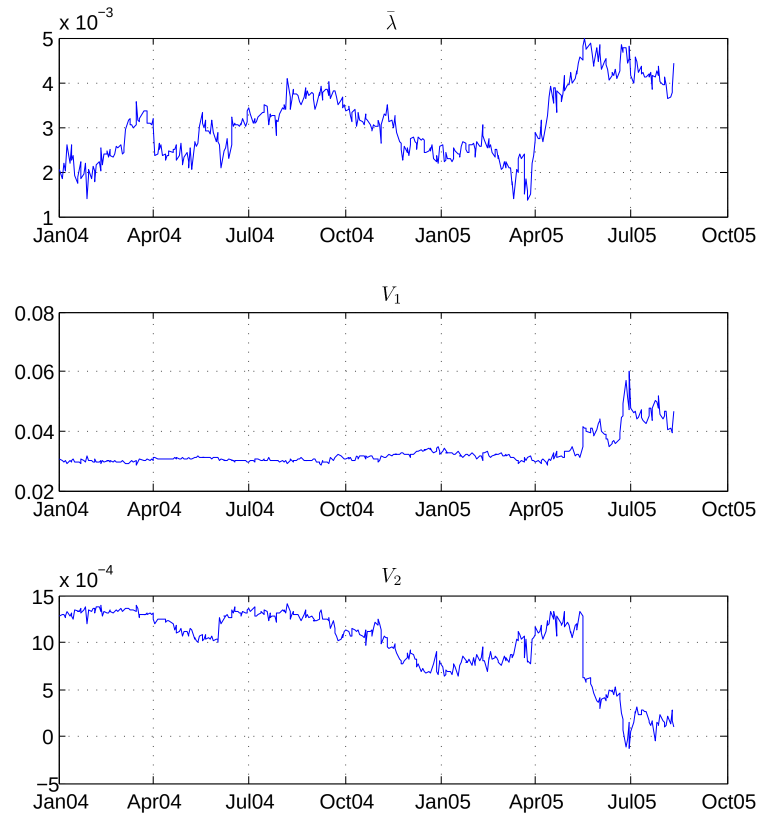

FIGURE 4.2. *Least-square estimates for the parameters $\bar{\lambda}$, V_1, and V_2 (top-down) for Wal-Mart (AA) from the 410 days between January 2, 2004 until August 11, 2005. Again, the estimates for the group parameters V_1 and V_2 are stable until about April 2005. The overall level of the average credit spread $\bar{\lambda}$ is lower than the corresponding level of IBM, that is in accordance with the higher rating of Wal-Mart (compare to Figure 4.1).*

change accordingly to improve the fitting of the yield curve on the short end of the maturity spectrum. In any way, the root-mean-squared-error (RMSE) does not change significantly between the two methods. The RMSE on a given day is defined as

$$\sqrt{\frac{1}{n-3}\sum_{j=1}^{n}\left(R_{\mathrm{C}}(T_j^{\mathrm{C}}) - R\left(\widehat{\bar{\lambda}}, \widehat{V}_1, \widehat{V}_2; T_j^{\mathrm{C}}\right)\right)^2}.$$

The starting guesses $V_1^\star$ and $V_2^\star$ in problems (I) and (II) were chosen recursively so that the average RMSE of the 420 days is minimized.

The stability of the group parameters V_1 and V_2 can be seen in Figures 4.1 and 4.2. These lend support and evidence that the separation of scales assumption we made throughout is a valid one. In particular, the parameters $\bar{\lambda}$, V_1, and V_2 are functions of the slowly-varying, with respect to time, process Z and as such they evolve likewise. By visual examination of the patterns for $\bar{\lambda}$ in Figures 4.1 and 4.2 the slow scale appears to exhibit a characteristic time scale on the order of three months.

TABLE 4.2. *Summary for the estimated group parameters $\bar{\lambda}$, V_1, and V_2 and their RMSEs for IBM (A+) and Wal-Mart (AA).*

	IBM				Wal-Mart			
	Mean	StDev	Min	Max	Mean	StDev	Min	Max
$\bar{\lambda}$	0.0038	0.0007	0.0025	0.0054	0.0030	0.0008	0.0013	0.0050
V_1	0.0358	0.0105	0.0259	0.0895	0.0326	0.0051	0.0281	0.0602
V_2	0.0008	0.0006	−0.0009	0.0015	0.0010	0.0004	−0.0001	0.0014
RMSE	0.0612	0.0484	0.0067	0.2027	0.0362	0.0230	0.0069	0.1232

Another aspect of the flexibility of the modeling setup is the yield curve fitting. Almost all of the fits were very close on both the short and the long ends of the yield curve having average RMSEs of 6.1 and 3.6 basis points for IBM and Wal-Mart, respectively as Table 4.2 indicates.

5. Conclusions

We have studied a computationally efficient approximation for single-name credit derivatives with stochastic interest-rate and intensity process. The two-factor multiscale approach is flexible for calibration from market yield spreads. The fast and slow factors in the intensity are well-suited to capturing the short and long ends respectively of the corporate yield curves.

The calibrated parameters can then be used in our closed-form approximation for options on defaultable bonds. Future work is to extend this approach to multi-name intensity models for valuation of basket credit derivatives such as collateralized debt obligations (CDOs) and single-tranche CDOs.

Appendix A. Bond Option Calculations

The solutions of the PDE problems for the leading order terms of either the bond (see (3.30)), or the bond option (see (3.53)) are simple extensions of their default-free counterparts. For detailed calculations, see for example Brigo and Mercurio (2001).

For the bond option correction terms, we solve the inhomogeneous PDEs (3.51) and (3.52). For instance, in the PDE (3.51) for $\tilde{u}_{1,0}$ the operator $\langle \mathcal{L}_2 \rangle$ corresponds to the infinitesimal generator of the short rate process r in (3.1) with the potential term $x + q\langle f\rangle(z)$, where $z \in \mathbb{R}$ is only a fixed constant. Then, using the expression for the operator $\mathcal{A}$ in (3.22) and the group parameter V_1 as in (3.26), the probabilistic representation of the correction term $\tilde{u}_{1,0}$ for the bond option price as defined in (3.51) is

$$
\begin{aligned}
\text{(A.1)}\quad & \tilde{u}_{1,0}(t,x,z) \\
&= \mathbb{E}^\star\left[\exp\left(-\int_t^{T_0}(r_s + q\langle f\rangle(z))\,ds\right)\tilde{P}_{1,0}(T_0, r_{T_0}, z; T_1)\mathbf{1}_{\{P_0(T_0, r_{T_0}, z; T_1)\geq K\}} \mid r_t = x\right] \\
&\quad + \mathbb{E}^\star\left[\int_t^{T_0}\exp\left(-\int_\theta^{T_0}(r_s + q\langle f\rangle(z))\,ds\right)\sigma V_1(z)\frac{\partial}{\partial x}u_0(\theta, r_\theta, z)\,d\theta \mid r_t = x\right].
\end{aligned}
$$

The first expectation corresponds to the terminal condition of the PDE. It can be calculated by the forward measure idea due to Jamshidian (1989), where we define the Radon-Nikodým process $\xi = (\xi_t)_{0\le t\le T_1}$, via

$$\xi_{T_1} := \frac{\exp\left(-\int_0^{T_1}(r_s + q\langle f\rangle(z))\,ds\right)}{\mathbb{E}^\star\left[\exp\left(-\int_0^{T_1}(r_s + q\langle f\rangle(z))\,ds\right)\right]},$$
$$\xi_t := \left.\frac{d\mathbb{P}^1}{d\mathbb{P}^\star}\right|_{\mathcal{F}_t} := \mathbb{E}^\star\left[\xi_{T_1} \mid \mathcal{F}_t\right],$$

use the probabilistic expression for $\tilde{P}_{1,0}$ (or the probabilistic representation for P_0 because of (3.31)), and the law of iterated expectations to get

$$\begin{aligned}
&\mathbb{E}^\star\left[\exp\left(-\int_t^{T_0}(r_s + q\langle f\rangle(z))\,ds\right)\tilde{P}_{1,0}(T_0, r_{T_0}, z; T_1)\mathbf{1}_{\{P_0(T_0,r_{T_0},z;T_1)\ge K\}} \mid r_t = x\right]\\
&\quad= \mathbb{E}^\star\left[V_1(z)h_1(T_1-T_0)\exp\left(-\int_{T_0}^{T_1}(r_s + q\langle f\rangle(z))\,ds\right)\mathbf{1}_{\{P_0(T_0,r_{T_0},z;T_1)\ge K\}} \mid r_t = x\right]\\
&\quad= \tilde{P}_{1,0}(t,x,z;T_1)\mathbb{P}^1\left\{P_0(T_0, r_{T_0}, z; T_1) \ge K \mid r_t = x\right\}.
\end{aligned}$$

For the probability of the last equation we observe that the random variable r_{T_0} given $r_t = x$ is normally distributed under the measure $\mathbb{P}^1$, and thus the probability is explicitly given by $\Phi(d_3)$, where Φ denotes the standard normal distribution function and

$$\begin{aligned}
d_3 &:= \frac{\log A(T_1-T_0) - \log K - (T_1-T_0)\bar{\lambda}(z) - B(T_1-T_0)\mu(t)}{\bar{\sigma}(t)},\\
\mu(t) &:= e^{-\alpha(T_0-t)}x + \left(\alpha\bar{r} - \frac{\sigma^2}{\alpha}\right)B(T_0-t) - \sigma^2\left(B(T_1-T_0) - \frac{1}{\alpha}\right)\left(\frac{1-e^{-2\alpha(T_0-t)}}{2\alpha}\right).
\end{aligned}$$

This justifies the third term in the right-hand side of (3.55) for $\tilde{u}_{1,0}$.

Denote the second expectation in the expression for $\tilde{u}_{1,0}$ in (A.1) by $v(t,x,z)$ and this will satisfy the inhomogeneous PDE

$$\begin{aligned}
\langle\mathcal{L}_2\rangle v(t,x,z) &= -\sigma V_1(z)\frac{\partial}{\partial x}u_0(t,x,z), \quad t < T_0,\\
v(T_0,x,z) &= 0.
\end{aligned}$$

Here z is again a fixed constant. To solve this problem we take advantage of the fact that u_0 satisfies a very similar PDE as v (without the source term). To wit, we make the ansatz

$$v(t,x,z) = D_1(T_0-t)\frac{\partial}{\partial x}u_0(t,x,z) + D_2(T_0-t)u_0(t,x,z), \quad 0 < t < T_0,$$

and we notice that the operator $\langle\mathcal{L}_2\rangle$ has the following commutation relationships with $D_1\partial/\partial x$ and $D_2\cdot$,

$$\begin{aligned}
\langle\mathcal{L}_2\rangle D_1\frac{\partial}{\partial x} &= -\left(\frac{\partial}{\partial t}D_1\right)\frac{\partial}{\partial x} + D_1\left(\alpha\frac{\partial}{\partial x} + \cdot\right) + D_1\frac{\partial}{\partial x}\langle\mathcal{L}_2\rangle,\\
\langle\mathcal{L}_2\rangle D_2 &= -\left(\frac{\partial}{\partial t}D_2\right)\cdot + D_2\langle\mathcal{L}_2\rangle.
\end{aligned}$$

When the last terms of these operators are applied to u_0 they cancel because of (3.50). Coefficient matching gives ODEs for D_1 and D_2, which after solving and combining with the expression for the *greek* of the leading order term gives us

$$v(t,x,z) = \eta_1(t)V_1(z)P_0(t,x,z;T_1)\Phi(d_1) - V_1(z)h_1(T_0-t)KP_0(t,x,z;T_0)\Phi(d_2),$$

with

$$\eta_1(t) := \frac{\sigma}{\alpha}\left(e^{-\alpha(T_1-T_0)}B(T_0-t) - (T_0-t)\right),$$

which is indeed equal to the first two terms of $\tilde{u}_{1,0}$ in (3.55).

The derivation for the expression of $\tilde{u}_{0,1}$ in (3.56) is similar to the one for $\tilde{u}_{1,0}$ above and we do not repeat it. The presence of the second order derivative in the operator $\mathcal{M}_1$ in the PDE (3.52) adds extra terms in the ansatz for the corresponding homogeneous PDE. This ansatz for the new term is

$$\tilde{D}_1(T_0-t)\frac{\partial^2}{\partial x\partial z}u_0(t,x,z) + \tilde{D}_2(T_0-t)\frac{\partial}{\partial x}u_0(t,x,z) + \tilde{D}_3(T_0-t)\frac{\partial}{\partial z}u_0(t,x,z) \\ + \tilde{D}_4(T_0-t)u_0(t,x,z), \quad 0<t<T_0,$$

and by using the additional commutative properties of the operator $\langle\mathcal{L}_2\rangle$ with $\tilde{D}_1\frac{\partial^2}{\partial x\partial z}$ and $\tilde{D}_3\partial/\partial z$, (where we denote $\bar{\lambda}_z := \partial\bar{\lambda}/\partial z$)

$$\begin{aligned}\langle\mathcal{L}_2\rangle\tilde{D}_1\frac{\partial^2}{\partial x\partial z} &= -\left(\frac{\partial}{\partial t}\tilde{D}_1\right)\frac{\partial^2}{\partial x\partial z} + \tilde{D}_1\left[\alpha\frac{\partial^2}{\partial x\partial z} + \frac{\partial}{\partial z} + \bar{\lambda}_z\frac{\partial}{\partial x}\right] + \tilde{D}_1\frac{\partial^2}{\partial x\partial z}\langle\mathcal{L}_2\rangle,\\ \langle\mathcal{L}_2\rangle\tilde{D}_3\frac{\partial}{\partial z} &= -\left(\frac{\partial}{\partial t}\tilde{D}_3\right)\frac{\partial}{\partial z} + \bar{\lambda}_z\tilde{D}_3\cdot + \tilde{D}_3\frac{\partial}{\partial z}\langle\mathcal{L}_2\rangle,\end{aligned}$$

we can solve the resulting ODEs for the $\tilde{D}_k$, $k=1,\dots,4$, terms:

$$\begin{aligned}-\tilde{D}_1' + \alpha\tilde{D}_1 + \rho_2\sigma g(z) &= 0,\\ -\tilde{D}_2' + \alpha\tilde{D}_2 + \bar{\lambda}_z\tilde{D}_1 &= 0,\\ -\tilde{D}_3' + \tilde{D}_1 &= 0,\\ -\tilde{D}_4' + \tilde{D}_2 + \bar{\lambda}_z\tilde{D}_3 &= 0,\end{aligned}$$

with $\tilde{D}_1(0) = \tilde{D}_2(0) = \tilde{D}_3(0) = \tilde{D}_4(0) = 0$. The associated derivatives of the leading order term are

$$\begin{aligned}u_{0,x}(t,x,z) &= -B(T_1-t)P_0(t,x,z;T_1)\Phi(d_1) + B(T_0-t)KP_0(t,x,z;T_0)\Phi(d_2),\\ u_{0,z}(t,x,z) &= -\bar{\lambda}_z(z)(T_1-t)P_0(t,x,z;T_1)\Phi(d_1) + \bar{\lambda}_z(z)(T_0-t)KP_0(t,x,z;T_0)\Phi(d_2),\\ u_{0,xz}(t,x,z) &= \bar{\lambda}_z(z)(T_1-t)B(T_1-t)P_0(t,x,z;T_1)\Phi(d_1)\\ &\quad - \bar{\lambda}_z(z)(T_0-t)B(T_0-t)KP_0(t,x,z;T_0)\Phi(d_2)\\ &\quad + \frac{1}{\bar{\sigma}(t)}\bar{\lambda}_z(z)(T_1-T_0)(B(T_1-t)-B(T_0-t))P_0(t,x,z;T_1)\Phi'(d_1),\end{aligned}$$

with $\bar{\sigma}$ defined in (3.54).

Putting everything together, we arrive to the expression (3.56) with the functions η_2 and η_3 given by

$$\eta_2(t) := \frac{\sigma}{\alpha}\left[B(T_1 - T_0)\left(T_1 - T_0 + \frac{1}{\alpha}\right) - B(T_1 - t)\left(T_1 - t + \frac{1}{\alpha}\right) + \frac{T_0 - t}{\alpha} + (T_0 - t)(T_1 - t) - \frac{(T_0 - t)^2}{2}\right],$$

$$\eta_3(t) := (T_1 - T_0)B(T_0 - t)\left(\frac{1 - e^{-2\alpha(T_0 - t)}}{2\alpha}\right)^{-1/2}.$$

Notice that $\eta_3(t) \to 0$, as $t \to T_0$.

Finally, we derive the bond price approximations in a simpler manner than those of the bond option because of the zero terminal condition for the PDEs of the perturbation terms. Furthermore, the CDS protection seller payment approximations can also be solved in the same way as we did with the bond option asymptotic approximations.

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