A Feedback Model for the Financialization of Commodity Markets

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Abstract

Recent empirical studies find evidence that commodity prices have become more correlated with financial markets since the early 2000s. This increased correlation is called the financialization of commodity markets and is conjectured to be due to the influx of external (portfolio optimizing) traders through commodity index funds, for instance. We build a feedback model to try and capture some of these effects, in which traditional economic demand for a commodity, oil say, is perturbed by the influence of portfolio optimizers. We approach the full problem of utility maximizing with a risky asset whose dynamics are impacted by trading through a sequence of problems that can be reduced to linear PDEs, and we find correlation effects proportional to the long or short positions of the investors, along with a lowering of volatility.

1 Introduction

1.1 Background and motivation

Recent empirical studies, for example [38, 7], have documented the "financialization" of commodity and energy futures markets due to an influx of external traders through investment vehicles such as commodity index funds or ETFs. They report that price movements of goods such as oil which, prior to the last decade, were mainly governed by supply and demand of users of the commodity, now exhibit much greater correlation with the movements of equity markets.

In the wake of the recent financial crisis, increased attention is being paid by politicians and regulators to the consequences of securitizing and trading derivatives on nontraditional investment assets. One result of the active trading of a securitized asset may be a fundamental shift in the economics that drives the price of the underlying. This change could be explained by professional investors beginning to trade actively in commodities for the purposes of portfolio diversification and speculation. At the beginning of the last decade, such trading activity in commodities increased dramatically, for example by hedge funds, and coincided with an increase in their correlation with the stock market. Causation between the change in trading activity and the change in the nature of commodity prices is still a subject of debate, as there are various global economic factors that may have played a role. While a purely mathematical model will not be able to separate out these possible effects, a possible explanation can be offered by analyzing the price impact of portfolio optimizers in a simple and idealized framework.

We build a model in which the demand of utility maximizing traders is introduced into an environment of a fundamentals-driven commodity price. We find the explicit form of the perturbed commodity price in the absence of investor feedback effects, and we develop observations on the changes in volatility, and correlation with stocks, that the demand of the portfolio optimizers creates.

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1.2 Literature review

Our paper is related to two different strands of literature: the literature of commodity modeling and in particular the financialization of commodity market, and the literature on the price impact and feedback effect of large traders.

Commodity modeling and financialization The modeling of commodity prices is discussed extensively in various sources. The books [12, 16, 30] give a primarily economic background on factors that drive prices for commodities and their derivatives. Benth *et al.* [4] details different models driven by time-inhomogeneous jump processes for electricity spot price dynamics, and the recent book of Swindle [37] provides a comprehensive overview of energy, commodities, and their derivatives. In a well known paper, Schwartz [33] fits parameters to a geometric Ornstein-Uhlenbeck model for the price of various commodities based on empirical data. The classic portfolio optimization analysis of Merton [26] has been extended by various authors to deal with mean-reverting processes. Jurek and Yang [21] derive explicit solutions to an optimal portfolio allocation and consumption problem for a portfolio optimizer seeking to profit from a mean-reverting pairs trade which follows an arithmetic Ornstein-Uhlenbeck process. Benth and Karlsen [5] solve the two-asset Merton problem for a risk-free asset and a risky asset with a geometric Ornstein-Uhlenbeck price process. These works provide a basis of comparison for the optimal stock/commodity allocation in our more complicated model.

Study of the financialization of commodity markets is somewhat new. In a recent paper, Tang and Xiong [38] discuss the history and development of the financialization of commodities as a result of increased index investing activity in the past decade. They find evidence of increased exposure of commodities prices to shocks in other asset classes via regression analysis on empirical data. A report published by FTI UK Holdings Limited [24] on the impact of speculation in commodity markets weighs the merits of speculative trading as providing liquidity for parties that need to hedge against the potential instability that speculation can create in a market. Brunetti and Büyükşahin [6] find, in a forecasting sense, that speculators are not causing any price movement, and moreover, speculative trading activity reduces volatility levels. In a subsequent paper, Büyükşahin and Robe [8] show that increased participation by hedge funds that trade in both equity and commodity markets could strengthen the correlation between the rates of return on commodities and equities rises. More recently, Silvennoinen and Thorp [34] and Henderson, Pearson, and Wang [18] provide further evidence for the financialization of commodity prices. There is a fast-growing literature of financialization of commodity markets, and we refer to Gilbert [17], Irwin and Sanders [19], Kaufmann [22], Mayer [25] and Singleton [35].

Price impact and feedback The problem of large agents having price impact in small or illiquid markets is one that will be at play in our model, as the introduction of speculators will have an impact on the commodity price. Çetin, Jarrow and Protter [9] build a model of liquidity risk in which a stochastic supply curve affects participants in a market that have large trade sizes. Bank and Kramkov [2] develop a game-theoretic large trader liquidity model. Jonsson and Keppo [20] treat a model where the portfolio position of a large agent has a particular impact on the price of call options. Bank and Baum [1] derive a general framework for dynamic liquidity effects of large traders, where assumptions about imperfect liquidity of an asset cause large purchases of the asset to affect its price.

The feedback effect of option hedging strategies in continuous time models was considered in a number of papers in the 1990s. Specifically, Frey and Stremme [14], Schönbucher and Wilmott [39], Sircar and Papanicolaou [32], and Platen and Schweizer [31] all studied the impact of option hedging strategies on stock prices. These models typically begin with reference traders who produce reference price dynamics such as geometric Brownian motion, but the equilibrium prices are then perturbed by the presence of noise traders who are hedging derivatives. These have the effect of increasing market volatility because, for example, hedging a short call option position entails selling stock when the stock price goes down, and therefore the presence of a significant number of hedgers has destabilizing price impact. This analysis explains to some extent the finding of the Brady Report into the 1987 crash, which attributed some cause to the presence of program trading for hedging option positions.

The feedback effect of portfolio optimizers was analyzed by Nayak and Papanicolaou [28]. Their reference setting consists of a stock following the geometric Brownian Motion, and a portfolio optimizer

having the power utility. They explicitly analyze the feedback effects on the price process when the relative influence of portfolio optimizers is small, and they analyze the system numerically under more general assumptions. Their main conclusion is that rational trading from solving a Merton portfolio optimization problem is stabilizing, and therefore lowers volatility, in contrast to what was found for option hedging strategies.

In a different but related problem, Christensen *et al.* [10] employ an equilibrium approach where the asset price is determined endogenously using a market clearing condition. With exponential utility, they solve for each investor's optimal investment strategy and find the equilibrium dynamics.

Models for financialization The present paper builds a simple feedback model to capture some of the empirical effects documented as the financialization of commodity market. We review several alternative avenues explored in the literature.

Sockin and Xiong [36] build a one-period feedback model for the financialization of commodities emphasizing the consequence of information frictions and production complementarity. They find that an increase in commodity futures prices may drive up producers' commodity demand and thus the spot price. In their model, the futures price is used as a proxy for global economic strength and other producers' production decisions, and in certain circumstances this information effect can dominate the cost effect and result in a positive demand elasticity. Basak and Pavlova [3] explore the effects of financialization in a model that features institutional investors alongside traditional futures markets participants, but they focus on the case where the institutional investors are evaluated relative to a benchmark index.

In a different context, Cont and Wagalath [11] illustrated how feedback effects due to distressed selling of mutual funds lead to endogenous correlations between asset classes. However, price impacts due to distressed selling are exogenously given by a block-shaped order book model and the funds follows a passive buy-and-hold strategy unless the fund value drops below certain threshold. In a static setting, Leclercq and Praz [23] consider an equilibrium based model that emphasizes the role of information aggregation of the commodity futures market. They demonstrate that speculation in futures markets facilitates hedging by suppliers, and hence decreases expected spot prices and increases the correlation between the financial and commodity markets.

1.3 Organization

In Section 2 we introduce a feedback model and derive an iterative sequence of problems that increasingly incorporate the feedback effect and capture the impact of financialization. In Section 3 we derive the HJB equation from dynamic programming principle for the stage-k problem and show that the HJB equation can be linearized with a power transformation. In Section 4 we present the numerical solutions to the first couple of stages and quantify the induced correlation between the equity and commodity markets. In Appendix B we return to the feedback model of Nayak and Papanicolaou [28] and present an explicit solution to the first stage in the feedback sequence which greatly facilitates the model analysis. Appendix C provides a tractable but less realistic model with exponential utility, where the full fixed point problem can be reduced to solution of a system of ordinary integro-differential equations. We conclude in Section 6.

2 Feedback Model with Market Users and Portfolio Optimizers

We build a model leading to the price of a commodity in which there are two main distinct groups creating demand: *market users* and *portfolio optimizers*. The price is determined from a supply-demand market clearing condition.

2.1 Reference model with market users

Market users trade in the commodity market for direct industrial use or for hedging their operational exposures. We assume their demand for commodity is driven by a stochastic incomes process I_t which,

roughly speaking, captures economic growth, and can be thought of as determining the amount of capital available to the market users for purchasing the commodity.

For simplicity of exposition and explicit calculations, we will take (I_t) to be a geometric Ornstein-Uhlenbeck process described by

$$\frac{dI_t}{I_t} = a(m - \log I_t)\,dt + b\,dW_t^c,$$

where $a, b > 0$ and W^c is a standard Brownian motion. This captures in a simple way periods of geometric growth along with mean-reversion or stochastic cyclicality.

Given a commodity price Y_t, the demand $D(Y_t, I_t)$ from market users is increasing in I_t and decreasing in Y_t. Again for simplicity and explicitness, we use the isoelastic demand function

$$D(Y_t, I_t) = \frac{I_t^\lambda}{Y_t}, \tag{1}$$

where $\lambda > 0$. This kind of demand structure in a continuous-time model is used for instance in [14, 32, 28]. We also assume a fixed constant supply A of the commodity available for trading at each time period. That is, we ignore growth or decline of supply over the short-run.

In the *reference model* in which there are only market users (or reference traders), we label the price process $Y_t = Y_t^{(0)}$. The market clearing condition $D(Y_t, I_t) = A$ gives

$$Y_t^{(0)} = \frac{I_t^\lambda}{A},$$

showing that the reference commodity price $Y^{(0)}$ dynamics is also a geometric Ornstein-Uhlenbeck process, which is the commonly-employed Schwartz [33] one-factor model of mean-reverting commodities prices:

$$\frac{dY_t^{(0)}}{Y_t^{(0)}} = a\left(\tilde{m} - \log Y_t^{(0)}\right)dt + \lambda b\,dW_t^c, \tag{2}$$

where

$$\tilde{m} = \lambda m - \log A + \frac{1}{2a}\lambda(\lambda - 1)b^2. \tag{3}$$

2.2 Incorporating portfolio optimizers

The portfolio optimizers, on the other hand, have no direct operational or hedging interest in the commodity, but seek to invest in the commodity market so as to maximize their expected utility at a fixed terminal horizon T. We assume for tractability the constant relative risk-aversion (CRRA) utility function with risk aversion $\gamma > 0$:

$$U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \qquad \gamma \neq 1. \tag{4}$$

In addition to the commodity market, the portfolio optimizers can invest in the risk-free money-market account with constant interest rate r, and a single representative stock index which follows the geometric Brownian motion

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t^s, \tag{5}$$

where W_t^s is a standard Brownian motion independent of W_t^c. By this choice, we have assumed that the pre-financialized commodity price $Y^{(0)}$ in (2) is independent of the equity market, so later feedback correlation is in comparison to this case.

Denoting by θ_t the investment in the commodity market by the portfolio optimizers, the aggregate demand for commodity is given by $D(Y_t, I_t) + \tilde{\epsilon}\theta_t/Y_t$, where $\tilde{\epsilon} > 0$ parametrizes the relative size of the portfolio optimizers compared to the market users. The market-clearing condition is

$$D(Y_t, I_t) + \tilde{\epsilon}\frac{\theta_t}{Y_t} = A,$$

which leads to

$$Y_t = \frac{I_t^\lambda}{A} + \epsilon\theta_t, \tag{6}$$

where $\epsilon = \tilde{\epsilon}/A$. This causes the equilibrium price process for Y_t to deviate from the geometric Ornstein-Uhlenbeck process $Y^{(0)}$.

The portfolio optimizers' position θ_t is modeled as coming from an expected utility maximizing criterion. However, unlike in the classical Merton problem, their actions impact the commodity they trade through (6), and we describe the extent of this feedback as related to the degree to which they are aware of their own price influence. Because they impact the commodity price Y and their trades are governed by portfolio diversification concerns, θ_t is affected by the stock index price S_t and this induces correlation between commodity and equity returns, that is, financialization of the commodity price. The goal is to try and quantify this effect.

In our model we do not get into the details of how commodities are traded, for instance through commodity index funds or ETFs, but instead think of all these investments as linked to futures contracts which are themselves linked to the physical delivery of the commodity which is necessarily finite. The grouping of the traders into two large groups means that each group has a significant price impact due to the finiteness at each time of the supply A available for trading.

2.3 Fixed point characterization of problem

We say that a pair $(\widehat{\pi}^*, \widehat{\theta}^*)$, respectively denoting the fractions of wealth invested in the stock and commodity markets, is an equilibrium solution to our *utility maximization problem with feedback* if the following are satisfied:

1. The stock price S_t is given by (5) and the commodity price Y_t is determined by the market clearing condition

$$D(Y_t, I_t) + \tilde{\epsilon}\frac{\widehat{\theta}_t X_t}{Y_t} = A, \tag{7}$$

where X_t is the controlled wealth process with strategies $(\widehat{\pi}^*, \widehat{\theta}^*)$ such that

$$\frac{dX_t}{X_t} = \frac{\widehat{\pi}_t^*}{S_t}\, dS_t + \frac{\widehat{\theta}_t^*}{Y_t}\, dY_t + r\left(1 - \widehat{\pi}_t^* - \widehat{\theta}_t^*\right) dt.$$

2. The pair $(\widehat{\pi}^*, \widehat{\theta}^*)$ maximizes the expected utility of terminal wealth Z_T

$$\sup_{(\widehat{\pi},\widehat{\theta})\in\mathcal{A}} \mathbb{E}\left[U(Z_T)|\mathcal{F}_t\right], \tag{8}$$

under the budget constraint

$$\frac{dZ_t}{Z_t} = \frac{\widehat{\pi}_t}{S_t}\, dS_t + \frac{\widehat{\theta}_t}{Y_t}\, dY_t + r\left(1 - \widehat{\pi}_t - \widehat{\theta}_t\right) dt.$$

The set of admissible strategies $\mathcal{A}$ will be given explicitly in Section 3.1 when we define the value function and derive the associated HJB equation.

2.4 Feedback iteration

We propose a *feedback iteration* to capture the successive improvement of trading strategies due to the increasing awareness of self-impact by the commodity traders. This approach follows an iterative chain of reasoning which is best illustrated in the simple setting of the famous "guessing 2/3 of the average" game.

Interlude - guessing 2/3 of the average

In this game, a number of players are asked to pick a number between 0 and 100, with the winner of the game being the one that is closest to 2/3 times the average number picked by all players.

We can solve this game iteratively as follows:

Stage-0 A typical player ignores the other players and choose a random number between 0 and 100.

Stage-1 He realizes that if the other players are following the *stage-0* strategy, the average is about 50; he can take advantage of this and update his guess to be 100/3.

Stage-2 He refines on the *stage-1* strategy and notice that if the other players are following the *stage-1* strategy, then the average is 100/3 and he should update his guess to be 200/9.

This chain of reasoning goes on: at *stage-k*, any individual player anticipates that the other players are following the *stage-*$(k-1)$ strategy, and take into account their aggregate effect (the sample average in this setting) when determining the *stage-k* strategy. As $k \to \infty$, the only rational guess is zero and this is called the Nash equilibrium.

Empirical studies show that, however, people do not behave as this simple model predicts. For instance, Nagel [27] conducted an experiment in which students were asked to guess what 2/3 of the average of their guesses will be, within limits of 0 and 100. She found that the average guess for the groups was around 35 and very few students chose 0. Moreover, it was observed that many students chose 33 and 22, which are respectively 2/3 of the midpoint 50 and 2/3 of 2/3 of the midpoint. The number of steps of iterated reasoning most students seemed to be doing were between 0 and 3 rounds.

2.5 Stage-k portfolio optimization problem

In light of the guessing game, we model the aggregate portfolio optimizers as comprising a large number of commodity traders. Each individual trader is too small to affect the market price, but their aggregate demand does have an impact, which is enforced by the market-clearing constraint. We first analyze the stage-k problem and return to the explicit solution for stage-0 in Section 3.3. [1]

In general, the stage-k commodity price process $Y^{(k)}$ for $k \geq 1$ no longer follows a geometric Ornstein-Uhlenbeck process, as was the case in the reference model $k = 0$ in (2). We suppose the dynamics of $Y^{(k)}$ can be written as

$$\frac{dY^{(k)}}{Y^{(k)}} = P^{(k)}\,dt + Q^{(k)}\,dW_t^c + R^{(k)}\,dW_t^s, \tag{9}$$

for some coefficients $P^{(k)}, Q^{(k)}$, and $R^{(k)}$. The aggregate wealth process for the bulk of the portfolio optimizers is denoted X_t, and they employ the stage-$(k-1)$ strategy $(\pi^{(k-1)}, \theta^{(k-1)})$, where $\pi_t^{(k-1)}$ is the dollar amount held in the stock index S at time t, and $\theta^{(k-1)}$ is the dollar amount held in the commodity $Y^{(k)}$ at time t. Their self-financing aggregate wealth process X follows

$$\begin{aligned} dX_t &= \frac{\pi_t^{(k-1)}}{S_t}\,dS_t + \frac{\theta_t^{(k-1)}}{Y_t^{(k)}}\,dY_t^{(k)} + r(X_t - \pi_t^{(k-1)} - \theta_t^{(k-1)})\,dt \\ &= \left(rX_t + \pi_t^{(k-1)}(\mu - r) + \theta_t^{(k-1)}\left(P^{(k)} - r\right)\right)\,dt \\ &\quad + \theta_t^{(k-1)}Q^{(k)}\,dW_t^c + \left(\pi_t^{(k-1)}\sigma + \theta_t^{(k-1)}R^{(k)}\right)\,dW_t^s. \end{aligned} \tag{10}$$

For our inductive hypothesis, we suppose that $P^{(k)} = P^{(k)}(t, X_t, Y_t^{(k)})$ and similarly for $Q^{(k)}$ and $R^{(k)}$, which come from the solution of the stage-$(k-1)$ problem; and that $\pi_t^{(k-1)}$ and $\theta_t^{(k-1)}$ are Markovian controls of the form $\pi_t^{(k-1)} = \pi^{(k-1)}(t, X_t, Y_t^{(k)})$ and $\theta_t^{(k-1)} = \theta^{(k-1)}(t, X_t, Y_t^{(k)})$. We have suppressed the

[1] We emphasize that, as in the 2/3 averaging game, the sequence of stages the portfolio optimizers run through are reasoning steps that are not temporally implemented. The choice of how many stages to go is a particular proposal related to the awareness of feedback effect of the portfolio optimizers (or the students in the 2/3 averaging game.

argument $(t, X_t, Y_t^{(k)})$ in (10). Under these hypotheses, $(X_t, Y_t^{(k)})$ is a Markov process with respect to the filtration generated by (W^c, W^s).

In stage-0 the portfolio optimizers do not trade the commodity so we have that $\theta^{(-1)} = 0$ and $Y^{(0)}$ is given by the reference model (2) from which we see that

$$P^{(0)}(t,x,y) = a\,(\tilde{m} - \log y)\,, \quad Q^{(0)}(t,x,y) = \lambda b, \quad R^{(0)} = 0,$$

and in particular they do not depend on X_t.

At stage-k, all but one of the commodity traders follow the stage-$(k-1)$ strategy. We imagine that a single "smart" trader seeks to outperform the others by taking into consideration the price impact of their stage-$(k-1)$ strategy. We denote his stage-k portfolio by $(\pi^{(k)}, \theta^{(k)})$ and the self-financing condition determines the following wealth process Z_t for the "smart" trader:

$$\begin{aligned} dZ_t &= \frac{\pi_t^{(k)}}{S_t}\,dS_t + \frac{\theta_t^{(k)}}{Y_t^{(k)}}\,dY_t^{(k)} + r(Z_t - \pi_t^{(k)} - \theta_t^{(k)})\,dt \\ &= \left(rZ_t + \pi_t^{(k)}(\mu - r) + \theta_t^{(k)}\left(P^{(k)}(t, X_t, Y_t^{(k)}) - r\right)\right)\,dt \\ &\quad + \theta_t^{(k)} Q^{(k)}(t, X_t, Y_t^{(k)})\,dW_t^c + \left(\pi_t^{(k)}\sigma + \theta_t^{(k)} R^{(k)}(t, X_t, Y_t^{(k)})\right)\,dW_t^s. \end{aligned} \tag{11}$$

His goal is to maximize expected utility at the terminal time T. This leads to a Merton problem we must solve to determine the stage-k optimal portfolio.

2.6 Deriving the stage-$(k+1)$ dynamics from stage-k strategies

Given the stage-k optimal portfolio $\pi^{(k)}$ and $\theta^{(k)}$ of the "smart" trader, we determine its effect on the stage-$(k+1)$ commodity price process. As is well known and we will confirm in the next section, because of power utility, the optimal Merton strategies are of the form $\pi_t^{(k)} = \widehat{\pi}^{(k)}(t, X_t, Y_t^{(k)})Z_t$ and $\theta_t^{(k)} = \widehat{\theta}^{(k)}(t, X_t, Y_t^{(k)})Z_t$. That is to say, they are given as fractions of the current wealth Z_t where the fractions are determined by the current levels of X_t and $Y_t^{(k)}$.

After having solved for the stage-k optimal portfolio, the "smart" trader realizes that the other traders will follow the same reasoning and trade according to the stage-k strategy. The aggregate position on the commodity is then $\widehat{\theta}^{(k)}(t, X_t, Y_t^{(k+1)})X_t$, and the stage-$(k+1)$ market clearing constraint reads

$$Y_t^{(k+1)} = \frac{I_t^\lambda}{A} + \epsilon\widehat{\theta}^{(k)}(t, X_t, Y_t^{(k+1)})X_t. \tag{12}$$

We can determine the dynamics of the stage-$(k+1)$ commodity price process $Y_t^{(k+1)}$ by applying Itô's formula to (12) and matching coefficients of the dt, dW_t^c, and dW_t^s terms. We can then solve for $P^{(k+1)}$, $Q^{(k+1)}$, and $R^{(k+1)}$ in terms of $\pi^{(k)}$ and $\theta^{(k)}$.

Proposition 1. *The dynamics of the stage-$(k+1)$ commodity price process $Y^{(k+1)}$ is*

$$\frac{dY_t^{(k+1)}}{Y_t^{(k+1)}} = P^{(k+1)}(t, X_t, Y^{(k+1)})\,dt + Q^{(k+1)}(t, X_t, Y^{(k+1)})\,dW_t^c + R^{(k+1)}(t, X_t, Y^{(k+1)})\,dW_t^s,$$

where

$$\begin{aligned} Q^{(k+1)}(t,x,y) &= \frac{\lambda b(y - \epsilon x\widehat{\theta}^{(k)})}{y - \epsilon x(y\partial_y\widehat{\theta}^{(k)} + \widehat{\theta}^{(k)}x\partial_x\widehat{\theta}^{(k)} + (\widehat{\theta}^{(k)})^2)}, \\ R^{(k+1)}(t,x,y) &= \frac{\epsilon x\sigma\widehat{\pi}^{(k)}(x\partial_x\widehat{\theta}^{(k)} + \widehat{\theta}^{(k)})}{y - \epsilon x(y\partial_y\widehat{\theta}^{(k)} + \widehat{\theta}^{(k)}x\partial_x\widehat{\theta}^{(k)} + (\widehat{\theta}^{(k)})^2)}, \\ P^{(k+1)}(t,x,y) &= \frac{a(y - \epsilon x\widehat{\theta}^{(k)})\left(\tilde{m} - \log(y - \epsilon x\widehat{\theta}^{(k)})\right) + \epsilon x\left(P_1^{(k+1)}(t,x,y) + P_2^{(k+1)}(t,x,y)\right)}{y - \epsilon x(y\partial_y\widehat{\theta}^{(k)} + \widehat{\theta}^{(k)}x\partial_x\widehat{\theta}^{(k)} + (\widehat{\theta}^{(k)})^2)}, \end{aligned} \tag{13}$$

with

$$\begin{aligned}
P_1^{(k+1)}(t,x,y) &= \partial_t \widehat{\theta}^{(k)} + \frac{1}{2}((Q^{(k+1)})^2 + (R^{(k+1)})^2) y^2 \partial_{yy} \widehat{\theta}^{(k)} + \mu \widehat{\theta}^{(k)} \widehat{\pi}^{(k)} + r\widehat{\theta}^{(k)}(1 - \widehat{\pi}^{(k)} - \widehat{\theta}^{(k)}) \\
&\quad + \Big(((Q^{(k+1)})^2 + (R^{(k+1)})^2)\widehat{\theta}^{(k)} + \sigma R^{(k+1)} \widehat{\pi}^{(k)}\Big) y \partial_y \widehat{\theta}^{(k)} \\
P_2^{(k+1)}(t,x,y) &= \Big((\widehat{\theta}^{(k)} Q^{(k+1)})^2 + (\widehat{\pi}^{(k)}\sigma + \widehat{\theta}^{(k)} R^{(k+1)})^2 + \mu \widehat{\pi}^{(k)} + r(1 - \widehat{\pi}^{(k)} - \widehat{\theta}^{(k)})\Big) x \partial_x \widehat{\theta}^{(k)} \\
&\quad + \frac{1}{2}\Big((\widehat{\theta}^{(k)} Q^{(k+1)})^2 + (\widehat{\pi}^{(k)}\sigma + \widehat{\theta}^{(k)} R^{(k+1)})^2\Big) x^2 \partial_{xx} \widehat{\theta}^{(k)} \\
&\quad + \Big(\widehat{\theta}^{(k)}(Q^{(k+1)})^2 + (\widehat{\pi}^{(k)}\sigma + \widehat{\theta}^{(k)} R^{(k+1)}) R^{(k+1)}\Big) xy \partial_{xy} \widehat{\theta}^{(k)}
\end{aligned}$$

and $\tilde{m}$ is given by (3).

Proof. Apply Itô formula to the market clearing constraint (12) and substitute the dynamics of X_t using (10) with k replaced by $k+1$. Finally substitute I_t in terms of X_t and $Y_t^{(k+1)}$ using (12). □

Therefore, given $\theta^{(k)}$ and $\pi^{(k)}$, we can determine the stage-$(k+1)$ commodity price dynamics $P^{(k+1)}$, $Q^{(k+1)}$, and $R^{(k+1)}$. Roughly speaking, the coefficient $R^{(k+1)}$ determines the stock-commodity correlation. Notice that the expressions for P, Q, and R are valid only for small ϵ as their denominators may become zero. However, we will see in Section 4 that the solution to the stage-k problem is well-behaved since the underlying price processes never get to the problematic region due to a repulsive potential. We focus here on the isoelastic demand function to illustrate the main features of financialization in a specific setting, but remark that the same techniques can be applied to more general demand functions, provided that they are invertible.

3 HJB analysis

In this section, we use dynamic programming to derive an HJB PDE that determines the optimal strategies the "smart" trader follows in each stage of the feedback iteration. We show that it can be reduced to a linear PDE and give the explicit solution in stage-0.

3.1 Value function and HJB equation

The value function for the "smart" trader in stage-k described in Section 2.5 is defined by

$$V(t,x,y,z) = \sup_{(\pi^{(k)},\theta^{(k)})\in\mathcal{A}} \mathbb{E}\left[U(Z_T)|X_t = x, Y_t^{(k)} = y, Z_t = z\right],$$

where X_t follows (10), Y_t follows (9), and Z_t follows (11), and we have defined the set of admissible strategies $\mathcal{A}$ to contain adapted processes (π_t, θ_t) such that $\mathbb{E}\int_0^T |\pi_t|^2 + |\theta_t|^2\, dt < \infty$. See for instance [29, chapter 3]. Following the usual Bellman's principle, we obtain the stage-k HJB equation

$$V_t + \mathcal{L}_{\mathbf{x}} V + rzV_z + \sup_{\boldsymbol{\nu}\in\mathbb{R}^2}\left[\frac{1}{2}\boldsymbol{\nu}^T C_1 \boldsymbol{\nu} V_{zz} + \boldsymbol{\nu}^T(\mu_1 - r)V_z + \boldsymbol{\nu}^T\sigma_1\sigma_2^T\nabla_{\mathbf{x}}V_z\right] = 0,$$

for $t < T$ and $x, y, z > 0$. Here we have denoted the trading strategy by $\boldsymbol{\nu} = (\pi^{(k)}, \theta^{(k)})^T$, $\mathbf{x} = (x,y)^T$, $\nabla_{\mathbf{x}} = (\partial_x, \partial_y)^T$, and the drift vector and volatility matrix of the pair of tradeable assets $(S, Y^{(k)})$ by

$$\mu_1 = \begin{pmatrix} \mu \\ P^{(k)} \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & \sigma \\ Q^{(k)} & R^{(k)} \end{pmatrix}, \qquad C_1 = \sigma_1\sigma_1^T.$$

Also defined are

$$\sigma_2 = \begin{pmatrix} \theta^{(k-1)}Q^{(k)} & \theta^{(k-1)}R^{(k)} + \pi^{(k-1)}\sigma \\ Q^{(k)}y & R^{(k)}y \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} \theta^{(k-1)}P^{(k)} + \pi^{(k-1)}\mu + r(x - \theta^{(k-1)} - \pi^{(k-1)}) \\ P^{(k)}y \end{pmatrix},$$

as well as

$$\mathcal{L}_{\mathbf{x}} = \frac{1}{2}\sum_{i,j=1}^{2}(C_2)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \mu_2^T \nabla_{\mathbf{x}}, \qquad C_2 = \sigma_2 \sigma_2^T,$$

where we identify $(x_1, x_2) = (x, y)$. The terminal condition is $V(T, x, y, z) = U(z)$.

3.2 Analysis of the HJB equation

From the point of view of the "smart" trader, he is facing a complete market Merton problem, being able to trade two assets, the commodity and the stock, driven by two Brownian motions W^c and W^s. Therefore we expect that the HJB equation can be reduced to a linear equation via a Cole-Hopf type transformation. We show that it is indeed the case in the following.

Proposition 2. *The value function is given by*

$$V(t, \mathbf{x}, z) = \frac{z^{1-\gamma}}{1-\gamma}\left(G(t, \mathbf{x})\right)^{\gamma}, \tag{14}$$

where $G(t, \mathbf{x})$ solves the linear PDE problem

$$G_t + \mathcal{L}_{\mathbf{x}} G + \left(\frac{1-\gamma}{\gamma}\right)\left(\sigma_2 \sigma_1^{-1}(\mu_1 - r)\right)^T \nabla_{\mathbf{x}} G + \frac{\zeta}{\gamma} G = 0, \tag{15}$$

with terminal condition $G(T, \mathbf{x}) = 1$, where

$$\zeta = \left(r(1-\gamma) + \frac{1-\gamma}{2\gamma} M\right), \qquad M = (\mu_1 - r)^T C_1^{-1}(\mu_1 - r). \tag{16}$$

The optimal portfolio $\boldsymbol{\nu}_t^* = (\pi^{(k)}, \theta^{(k)})^T$ is given by

$$\boldsymbol{\nu}_t^* = \left(\frac{1}{\gamma} C_1^{-1}(\mu_1 - r) + (\sigma_2 \sigma_1^{-1})^T \frac{\nabla_{\mathbf{x}} G}{G}\right) Z_t. \tag{17}$$

Proof.

Optimization Assuming for now that V_{zz} is negative (that is, it inherits the concavity from the terminal condition), the supremum in the HJB equation is given by

$$\boldsymbol{\nu}^* = -\frac{1}{V_{zz}} C_1^{-1}\Big((\mu_1 - r) + \sigma_1 \sigma_2^T \nabla_{\mathbf{x}}\Big) V_z. \tag{18}$$

The HJB equation can then be written as

$$V_t + \mathcal{L}_{\mathbf{x}} V + rzV_z - \frac{1}{2V_{zz}}\Big(MV_z^2 + 2(\mu_1 - r)^T C_1^{-1}\sigma_1 \sigma_2^T (\nabla_{\mathbf{x}} V_z) V_z + (\sigma_1 \sigma_2^T \nabla_{\mathbf{x}} V_z)^T C_1^{-1} \sigma_1 \sigma_2^T \nabla_{\mathbf{x}} V_z\Big) = 0$$

with terminal condition $V(T, \mathbf{x}, z) = U(z)$, where M is defined in (16).

Separation of variables Making the transformation

$$V(t, \mathbf{x}, z) = \frac{z^{1-\gamma}}{1-\gamma} g(t, \mathbf{x})$$

results in

$$g_t + \mathcal{L}_{\mathbf{x}} g + \left(\frac{1-\gamma}{\gamma}\right)(\mu_1 - r)^T C_1^{-1}\sigma_1 \sigma_2^T \nabla_{\mathbf{x}} g + \zeta g + \frac{1-\gamma}{2\gamma g}(\sigma_1 \sigma_2^T \nabla_{\mathbf{x}} g)^T C_1^{-1}(\sigma_1 \sigma_2^T \nabla_{\mathbf{x}} g) = 0, \tag{19}$$

with terminal condition $g(T, \mathbf{x}) = 1$, where ζ is defined in (16).

Cole-Hopf transformation We make the power transformation $g = G^\delta$, previously introduced by Zariphopoulou [40], and observe that the nonlinear term in (19) becomes

$$\delta G^{\delta-1}\left[\delta\frac{1-\gamma}{2\gamma G}(\sigma_1\sigma_2^T\nabla_{\mathbf{x}}G)^T C_1^{-1}(\sigma_1\sigma_2^T\nabla_{\mathbf{x}}G)\right].$$

Using the definition of C_1 and C_2, we note that

$$(\sigma_1\sigma_2^T\nabla_{\mathbf{x}}G)^T C_1^{-1}(\sigma_1\sigma_2^T\nabla_{\mathbf{x}}G) = (\nabla_{\mathbf{x}}G)^T C_2(\nabla_{\mathbf{x}}G)$$

so the nonlinear term is

$$\delta G^{\delta-1}\left[\delta\frac{1-\gamma}{2\gamma G}(\nabla_{\mathbf{x}}G)^T C_2(\nabla_{\mathbf{x}}G)\right].$$

Meanwhile,

$$\mathcal{L}_{\mathbf{x}}G^\delta = \delta G^{\delta-1}\left[\frac{1}{2}\sum_{i,j=1}^{2}(C_2)_{ij}\frac{\partial^2 G}{\partial x_i\partial x_j} + \frac{1}{2}(\delta-1)\frac{(\nabla_{\mathbf{x}}G)^T C_2(\nabla_{\mathbf{x}}G)}{G}\right],$$

so the nonlinear terms cancel provided we choose

$$\frac{1}{2}(\delta-1) + \delta\frac{1-\gamma}{2\gamma} = 0 \quad \Longrightarrow \quad \delta = \gamma.$$

With this choice, we obtain the linear PDE for $G(t,\mathbf{x})$ given in (15). Inserting transformations (14) for V into (18) gives the optimal portfolio (17) in terms of G. □

Notice in particular that the optimal holdings in commodity and stock index are proportional to wealth, as we expect for power utility, and we write:

$$\pi_t^{(k)} = \widehat{\pi}^{(k)}(t,X_t,Y_t)Z_t, \qquad \theta_t^{(k)} = \widehat{\theta}^{(k)}(t,X_t,Y_t)Z_t,$$

where the functions $\widehat{\pi}^{(k)}(t,x,y)$ and $\widehat{\theta}^{(k)}(t,x,y)$ can be read from the components of $\boldsymbol{\nu}^*$ in (17).

Remark 1. *In Appendix B we study an application of our feedback model to the equity market where the stage-1 HJB equation can be solved analytically.*

3.3 Stage-0 PDE and explicit solution

As it turns out, the stage-0 PDE problem has an explicit solution. This is the Merton problem with geometric OU dynamics as studied in [5] and [21], for instance. From the market-clearing constraint (without any influence of portfolio optimizers), we see that the stage-0 commodity price dynamics (2) is simply a geometric Ornstein-Uhlenbeck process. That is, we have $P^{(0)} = a(\tilde{m} - \log y)$, $Q^{(0)} = \lambda b$, and $R^{(0)} = 0$.

Proposition 3. *The stage-0 value function is given by*

$$V(t,y,z) = \frac{z^{1-\gamma}}{1-\gamma}\exp\Big(f_0(t) + f_1(t)\log y + f_2(t)(\log y)^2\Big), \tag{20}$$

where

$$\begin{aligned}
f_2(t) &= \frac{a(1-\gamma)}{2\lambda^2b^2}\frac{\sinh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right)}{\sinh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right) + \sqrt{\gamma}\cosh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right)},\\
f_1(t) &= (1-\gamma)\frac{\frac{r-a\tilde{m}}{\lambda^2b^2}\sinh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right) + \sqrt{\gamma}\left(\frac{r}{\lambda^2b^2}-\frac{1}{2}\right)\left(\cosh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right)-1\right)}{\sinh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right) + \sqrt{\gamma}\cosh\left(\frac{a}{\sqrt{\gamma}}(T-t)\right)},\\
f_0(t) &= k(T-t) + \int_t^T\left\{\left(\frac{a\tilde{m}-(1-\gamma)r}{\gamma} - \frac{\lambda^2b^2}{2}\right)f_1(s) + \lambda^2b^2f_2(s) + \frac{\lambda^2b^2}{2\gamma}f_1^2(s)\right\}ds,
\end{aligned} \tag{21}$$

and

$$k = \frac{1-\gamma}{2}\left(\frac{(a\tilde{m}-r)^2}{\lambda^2 b^2 \gamma} + 2r + \frac{(\mu-r)^2}{\sigma^2\gamma}\right).$$

Proof. After separating out the wealth variable z and making the linearizing transformation, the stage-0 equation (15) for G is

$$G_t + \frac{1}{2}\lambda^2 b^2 y^2 G_{yy} + rxG_x + \frac{1}{\gamma}\Big(a(\tilde{m} - \log y) - (1-\gamma)r\Big)yG_y + \frac{\zeta}{\gamma}G = 0$$

with terminal condition $G(T,x,y) = 1$. Observe that the terminal condition does not depend on x, and that the term rxG_x drops out from the PDE if we look for a solution $G(t,y)$ as a function of t and y only. Intuitively it is clear that the stage-0 value function should not depend on the aggregate wealth of the speculative traders as they have no influence on the commodity price.

Next, we make the transformation $G(t,y) = H(t,u)$ where $u = \log y$. This results in

$$H_t + \frac{1}{2}\lambda^2 b^2 (H_{uu} - H_u) + \frac{1}{\gamma}\Big(a(\tilde{m}-u) - (1-\gamma)r\Big)H_u + \frac{\zeta}{\gamma}H = 0.$$

Recall that M is defined by

$$M = (\mu_1 - r)^T C_1^{-1}(\mu_1 - r) = \frac{\big(a(\tilde{m}-u)-r\big)^2}{\lambda^2 b^2} + \frac{(\mu-r)^2}{\sigma^2}.$$

We have a PDE of the form

$$H_t + (c_0 + c_1 u)H_u + \frac{1}{2}\lambda^2 b^2 H_{uu} + \frac{1}{\gamma}(c_2 + c_3 u + c_4 u^2)H = 0, \tag{22}$$

with terminal condition $H(T,u) = 1$, where c_0 through c_4 are constants given in Appendix A. As shown there, the solution is of the form

$$H(t,u) = \exp\left(\frac{1}{\gamma}\Big(f_0(t) + f_1(t)u + f_2(t)u^2\Big)\right), \tag{23}$$

where f_0, f_1, and f_2 satisfies a system of ordinary differential equations, whose solutions lead to (20). □

The optimal portfolio can be recovered from the value function using (17):

$$\widehat{\pi}^{(0)}(t,y) = \frac{\mu - r}{\sigma^2\gamma} \quad \text{and} \quad \widehat{\theta}^{(0)}(t,y) = \Big(F_1(t) + F_2(t)\log y\Big) \tag{24}$$

where

$$F_1(t) = \frac{1}{\gamma}\left(f_1(t) + \frac{1}{\lambda^2 b^2}(a\tilde{m} - r)\right) \quad \text{and} \quad F_2(t) = \frac{1}{\gamma}\left(2f_2(t) - \frac{a}{\lambda^2 b^2}\right).$$

We observe that the investment in the stock $\widehat{\pi}^{(0)}$ follows the fixed-mix strategy as in the Merton problem [26]. The fraction of wealth $\widehat{\theta}^{(0)}$ invested in the commodity, however, depends on both the remaining time to the investment horizon $T-t$ and the commodity price Y_t.

Remark 2. *The verification argument for candidate solutions of the HJB equation is classical and a rigorous treatment follows closely Benth and Karlsen [5, Section 4] and Zariphopoulou [40, Theorem 3.2].*

4 Numerical Results

For stage-1 and onwards, we have to resort to numerical methods to solve the PDE problems. As an illustration, we consider the following set of parameter values detailed in Table 1.

Table 1: Parameter values for numerical illustration.

Parameter	Significance	Value
λ	demand from market users	1.0
a	mean-reversion rate	0.3
r	risk-free rate	0.02
m	mean of commodity log-price	3.0
b	volatility of commodity price	0.3
μ	drift of stock price	0.08
σ	volatility of stock price	0.2
γ	coefficient of risk aversion	1.5
A	market supply	1.0
ϵ	relative size of portfolio optimizers	0.5
T	Investment horizon	2.0

4.1 Solution via finite difference

The stage-k PDE (15) can be written in expanded form (where we suppress the k superscripts)

$$\begin{aligned}0 = G_t &+ \frac{1}{2}(Q^2+R^2)y^2G_{yy} + \frac{1}{2}\Big((\widehat{\theta}R+\widehat{\pi}\sigma)^2+(\widehat{\theta}Q)^2\Big)x^2G_{xx} + \Big((Q^2+R^2)\widehat{\theta}+\sigma R\widehat{\pi}\Big)xyG_{xy}\\ &+ \frac{1}{\gamma}\Big(P-(1-\gamma)r\Big)yG_y + \frac{1}{\gamma}\Big(\widehat{\theta}(P-r)+\widehat{\pi}(\mu-r)+\gamma r\Big)xG_x\\ &+ \frac{1}{\gamma}\left(r(1-\gamma)+\frac{1-\gamma}{2\gamma}\frac{1}{\sigma^2Q^2}\Big(\sigma^2(P-r)^2-2\sigma R(\mu-r)(P-r)+(Q^2+R^2)(\mu-r)^2\Big)\right)G.\end{aligned}$$

After using the log transformation $u=\log x$ and $v=\log y$, we may write the discretized equation as

$$\begin{aligned}0 = &\frac{G^{n+1}_{i,j}-G^n_{i,j}}{\Delta t} + \frac{1}{2}\Big((Q^n_{i,j})^2+(R^n_{i,j})^2\Big)\left(\frac{G^{n+1}_{i,j+1}-2G^{n+1}_{i,j}+G^{n+1}_{i,j-1}}{(\Delta v)^2}-\frac{G^{n+1}_{i,j+1}-G^{n+1}_{i,j-1}}{2\Delta v}\right)\\ &+\frac{1}{2}\Big((\widehat{\theta}^n_{i,j}R^n_{i,j}+\widehat{\pi}^n_{i,j}\sigma)^2+(\widehat{\theta}^n_{i,j}Q^n_{i,j})^2\Big)\left(\frac{G^{n+1}_{i+1,j}-2G^{n+1}_{i,j}+G^{n+1}_{i-1,j}}{(\Delta u)^2}-\frac{G^{n+1}_{i+1,j}-G^{n+1}_{i-1,j}}{2\Delta u}\right)\\ &+\left(\Big((Q^n_{i,j})^2+(R^n_{i,j})^2\Big)\widehat{\theta}^n_{i,j}+\sigma R^n_{i,j}\widehat{\pi}^n_{i,j}\right)\left(\frac{G^{n+1}_{i+1,j+1}-G^{n+1}_{i-1,j+1}-G^{n+1}_{i+1,j-1}+G^{n+1}_{i-1,j-1}}{4\Delta u\Delta v}\right)\\ &+\frac{1}{\gamma}\Big(P^n_{i,j}-(1-\gamma)r\Big)\left(\frac{G^{n+1}_{i,j+1}-G^{n+1}_{i,j-1}}{2\Delta v}\right)+\frac{1}{\gamma}\Big(\widehat{\theta}^n_{i,j}(P^n_{i,j}-r)+\widehat{\pi}^n_{i,j}(\mu-r)+\gamma r\Big)\left(\frac{G^{n+1}_{i+1,j}-G^{n+1}_{i-1,j}}{2\Delta u}\right)\\ &+\frac{1}{\gamma}\left(r(1-\gamma)+\frac{1-\gamma}{2\gamma}\frac{1}{(\sigma Q^n_{i,j})^2}\Big(\sigma^2(P^n_{i,j}-r)^2-2\sigma R^n_{i,j}(\mu-r)(P^n_{i,j}-r)+\Big((Q^n_{i,j})^2+(R^n_{i,j})^2\Big)(\mu-r)^2\Big)\right)G^{n+1}_{i,j}.\end{aligned} \tag{25}$$

In the scheme, the first subscript denotes the u coordinate in the uniform (u,v)-grid, while the second corresponds to the v coordinate. The superscript represents the time step. We note that we are solving a terminal value problem: we start with $G^N_{i,j}=1$ for all i,j and step backward in time using (25) which is the explicit Euler scheme.

Truncation of domain We approximate the Cauchy problem (15) with one on a bounded domain $[0,T]\times[u_{min},u_{max}]\times[v_{min},v_{max}]$. We choose a uniform mesh

$$\mathcal{G}=\{(t^n,u_i,v_j): n=0,1,\dots,N,\ i=0,1,\dots,I,\ j=0,1,\dots,J\}$$

where $t^n=n\Delta t$, $u_i=u_{min}+i\Delta u$, and $v_j=v_{min}+j\Delta v$.

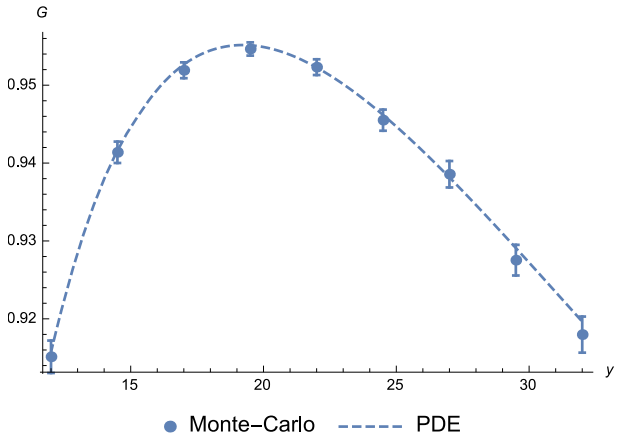


Figure 1: Comparison between Monte-Carlo and numerical PDE solutions to (15). The error bars represent ± 2 standard error of the Monte-Carlo estimates. Parameter values are as in Table 1.

Boundary conditions In the limit $x \to 0$, the price impact of portfolio optimizers vanishes because they have no capital to invest, and we should recover the stage-0 solution. We therefore use the analytic solution (20) for the stage-0 problem to impose a Dirichlet boundary condition at $x = 0$ (or equivalently $u \to -\infty$).

As for the other three boundaries, we derive approximate boundary conditions by the PDE itself, which are discretized by one-sided finite differences without requiring any additional information concerning the behavior of the solution for large x and for large/small y.

Notice that in (13), the denominators in the definitions of P, Q, and R can be zero for y sufficiently small or x sufficiently large. Therefore our model can breakdown when the commodity price is sufficiently low and the portfolio optimizers take over a sufficiently large share of market. However, we check that the truncation of domain does not result in significant error by solving the *linear* PDE (15) using Monte-Carlo simulations. In all instances we have tested, the difference between the PDE solutions and the Monte-Carlo estimates are within the Monte-Carlo standard error. See Figure 1 for instance where we compare the Monte-Carlo estimates with the numerical PDE solutions to (15).

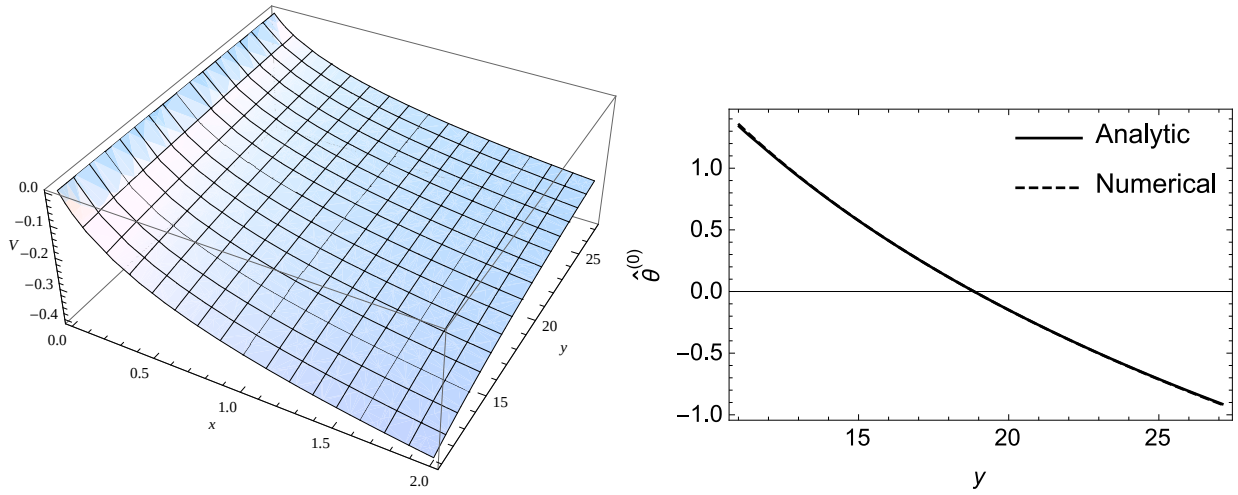
4.1.1 Stage-0 benchmark

Since we have an explicit solution to the stage-0 problem, it serves as a benchmark for testing our numerical scheme.

In Figure 2b we plot the optimal fraction of wealth invested in the commodity market, computed using (17) by differentiating the numerical solution to the *linear* equation (15). Also shown is the optimal portfolio $\widehat{\theta}^{(0)}$ from formula (24). Notice that the two curves are almost indistinguishable, and we check that the numerical solution has relative error less than 1.5% throughout the domain of interest. This validates the use of the numerical scheme in the following sections.

4.1.2 Stage-1 numerical solution

We now solve the stage-1 HJB equation numerically. The commodity price dynamics $P^{(1)}$, $Q^{(1)}$, and $R^{(1)}$ can be computed from the analytic solution to the stage-0 problem. Indeed, using that the stage-0 optimal



(a) Stage-0 value function.

(b) Stage-0 optimal commodity holding.

Figure 2: Analytic and numerical solutions to the stage-0 HJB equation.

portfolio (24) does not depend on x, equation (13) can be simplified considerably:

$$\begin{aligned} Q^{(1)}(t,x,y) &= \frac{\lambda b(y-\epsilon x\widehat{\theta}^{(0)})}{y-\epsilon x\left(y\partial_y\widehat{\theta}^{(0)}+(\widehat{\theta}^{(0)})^2\right)}, \\ R^{(1)}(t,x,y) &= \frac{\epsilon x\sigma\widehat{\pi}^{(0)}\widehat{\theta}^{(0)}}{y-\epsilon x\left(y\partial_y\widehat{\theta}^{(0)}+(\widehat{\theta}^{(0)})^2\right)}, \\ P^{(1)}(t,x,y) &= \frac{a\left(y-\epsilon x\widehat{\theta}^{(0)}\right)\left(\tilde{m}-\log(y-\epsilon x\widehat{\theta}^{(0)})\right)+\epsilon xF(t,x,y)}{y-\epsilon x\left(y\partial_y\widehat{\theta}^{(0)}+(\widehat{\theta}^{(0)})^2\right)}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} F(t,x,y) = \partial_t\widehat{\theta}^{(0)} &+ \frac{1}{2}\left((Q^{(1)})^2+(R^{(1)})^2\right)y^2\partial_{yy}\widehat{\theta}^{(0)}+\mu\widehat{\theta}^{(0)}\widehat{\pi}^{(0)} \\ &+ r\widehat{\theta}^{(0)}(1-\widehat{\pi}^{(0)}-\widehat{\theta}^{(0)})+\left(\left((Q^{(1)})^2+(R^{(1)})^2\right)\widehat{\theta}^{(0)}+\sigma R^{(1)}\widehat{\pi}^{(0)}\right)y\partial_y\widehat{\theta}^{(0)}. \end{aligned}$$

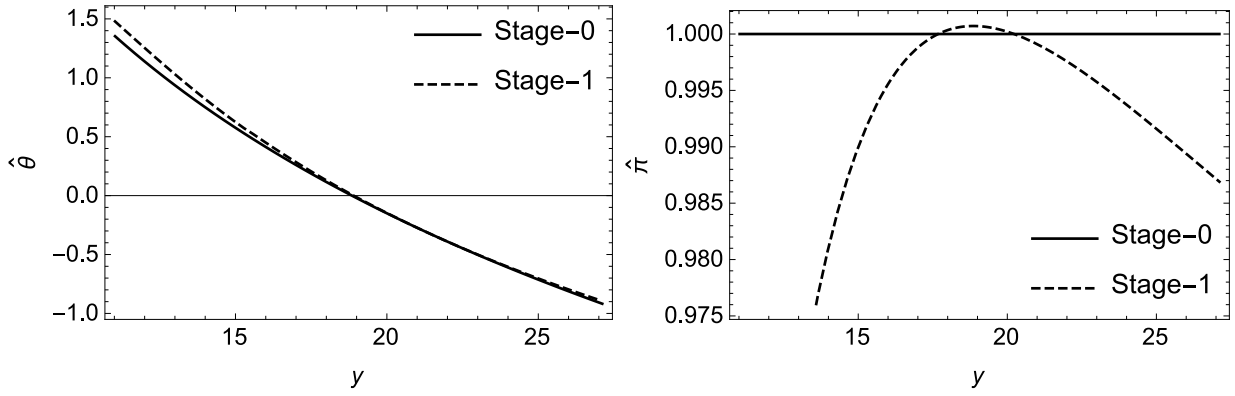
Using the explicit finite difference scheme (25), we can solve for the value function G. The optimal portfolio $(\pi^{(1)},\theta^{(1)})$ is then recovered from (17), see Figure 3 for the result. We observe that the stock holding $\pi^{(1)}$ now varies with the level of the commodity price y, where in the pre-financialization stage-0, it was independent.

4.1.3 Volatility and correlation

Ultimately we are interested in studying how price impact effects the volatility of the commodity price and its correlation with the stock price. To this end, we note that the stage-k volatility $\eta^{(k)}$ of the commodity price and the stage-k correlation $\rho^{(k)}$ with the stock price are given by

$$\eta^{(k)} = \sqrt{(Q^{(k)})^2+(R^{(k)})^2}, \qquad \rho^{(k)} = \frac{R^{(k)}}{\sqrt{(Q^{(k)})^2+(R^{(k)})^2}}.$$

In Figures 4a and 4b, we plot the stage-0, stage-1, and stage-2 volatilities and correlations as functions of $Y_0 = y$ with $X_0 = 1, T = 2$.



(a) Stage-1 optimal fraction of wealth invested in the commodity.

(b) Stage-1 optimal fraction of wealth invested in the stock.

Figure 3: Stage-0 and 1 optimal portfolio as functions of the current commodity level y.

Stage-0 We simply have

$$\eta^{(0)} = \lambda b \quad \text{and} \quad \rho^{(0)} = 0.$$

Stage-1 The stage-1 volatility and correlation can be computed explicitly using the explicit solution to the stage-0 problem. From (26), we see that

$$\begin{aligned} \eta^{(1)} &= \lambda b \left(1 + \frac{\epsilon X_t}{Y_t}\left(Y_t \partial_y \widehat{\theta} + \widehat{\theta}^2 - \widehat{\theta}\right) + \mathcal{O}(\epsilon^2)\right) \\ \rho^{(1)} &= \frac{\epsilon\sigma X_t \widehat{\pi}\widehat{\theta}}{\sqrt{\lambda^2 b^2 \left(Y_t - \epsilon X_t \widehat{\theta}_t\right)^2 + \left(\epsilon\sigma X_t \widehat{\pi}\widehat{\theta}\right)^2}} = \frac{\epsilon\sigma}{\lambda b}\frac{X_t}{Y_t}\widehat{\pi}\widehat{\theta} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{27}$$

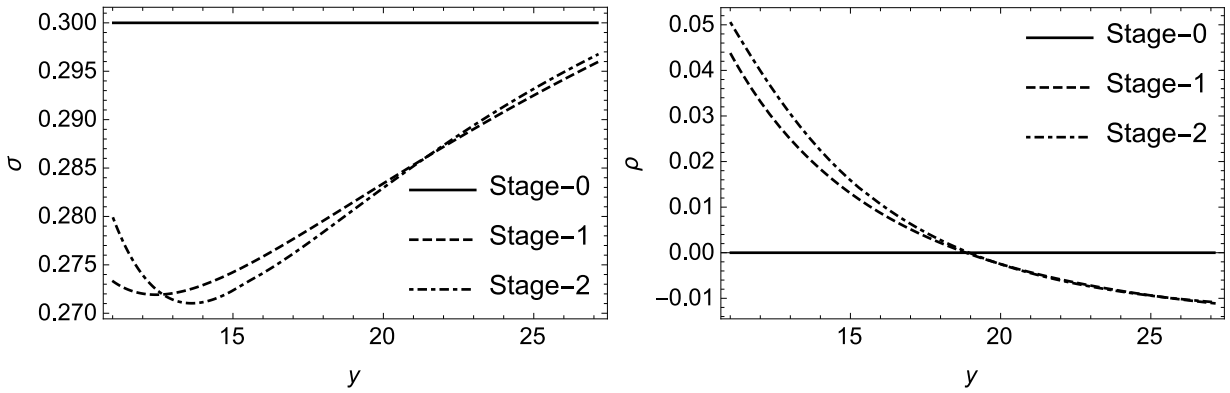
to leading order in ϵ, where $\widehat{\pi}$ and $\widehat{\theta}$ are the stage-0 optimal portfolio. Notice in particular that the sign of ρ_t is given by the sign of $\widehat{\theta}^{(0)}(t, Y_t^{(1)})$, as long as the Merton ratio $\widehat{\pi}^{(0)}$ is positive, or equivalently $\mu > r$. Therefore, when the commodity price is low (resp. high), portfolio optimizers are long (resp. short) the commodity and feedback correlation is positive (resp. negative).

Stage-2 From the (numerical) solution to the stage-1 portfolio optimization problem, we can determine the stage-2 commodity price dynamics using (13). This gives the stage-2 volatility of the commodity price as well as it correlation with the stock. We notice that the volatility of the commodity price reduces further in stage-2 (compared to stage-1) when the commodity is near its long-term mean; while the stage-2 correlation is greater when the commodity price is below its mean level, and approximately the same as in stage-1 when above.

4.2 Comparative Statics

We observe how $\rho^{(1)}(t, X_t, Y_t^{(1)})$ varies in the other model parameters, by modifying each of γ, ϵ, a, and b independently while holding the other parameters constant, see Figure 5. Since the instantaneous correlation depends on the commodity price $Y_t^{(1)}$, we consider a fixed low commodity price (50% of its mean, in blue) and a fixed high commodity price (150% of its mean, in red).

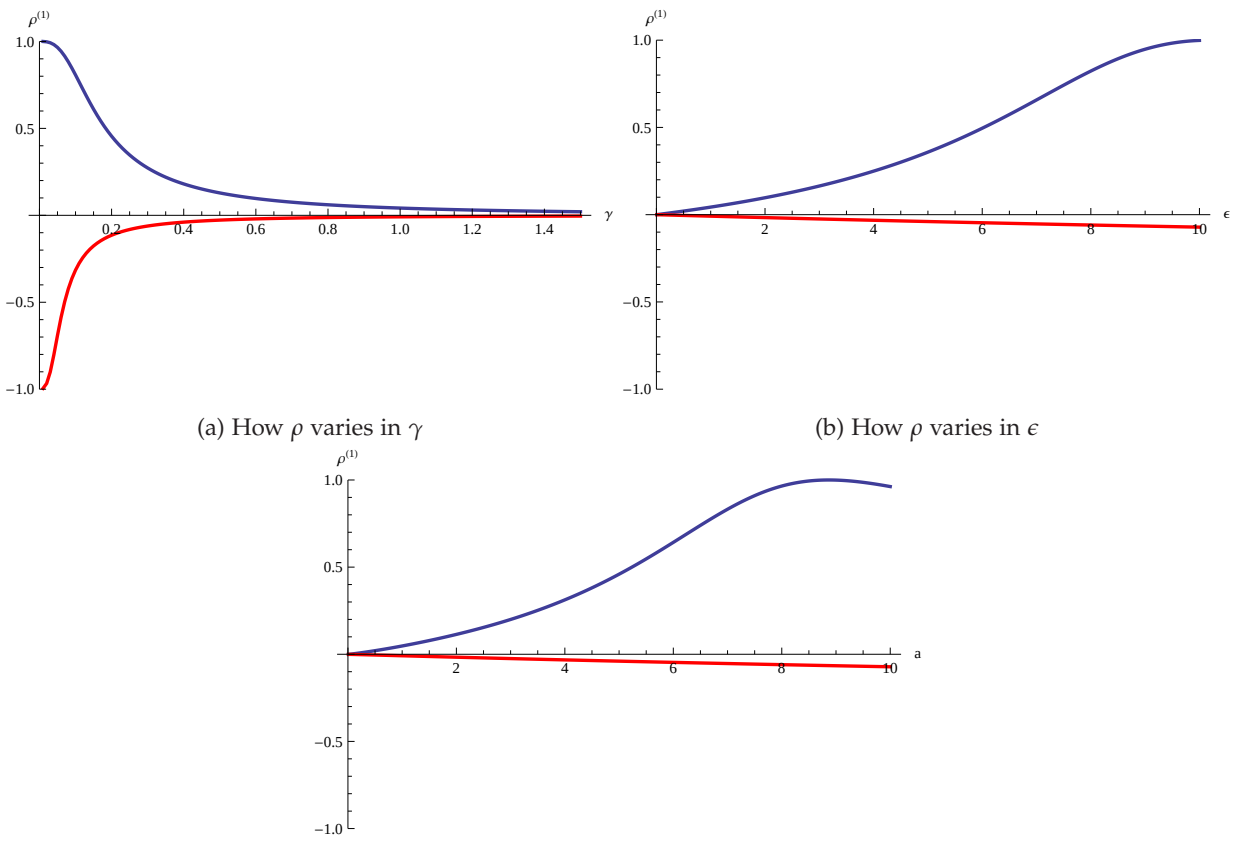
As we observed analytically earlier in this section, we see that when the commodity price is below (resp. above) its mean and the portfolio optimizers long (resp. short) the commodity, the correlation is positive (resp. negative). As the risk aversion γ increases, the correlation induced by the perturbation of the risk-averse portfolio optimizers tends to zero. A sufficiently risk-averse trader will invest in neither



(a) Volatility of the commodity price.

(b) Correlation between the commodity and stock.

Figure 4: Volatility and correlation at stage-0, stage-1, and stage-2.

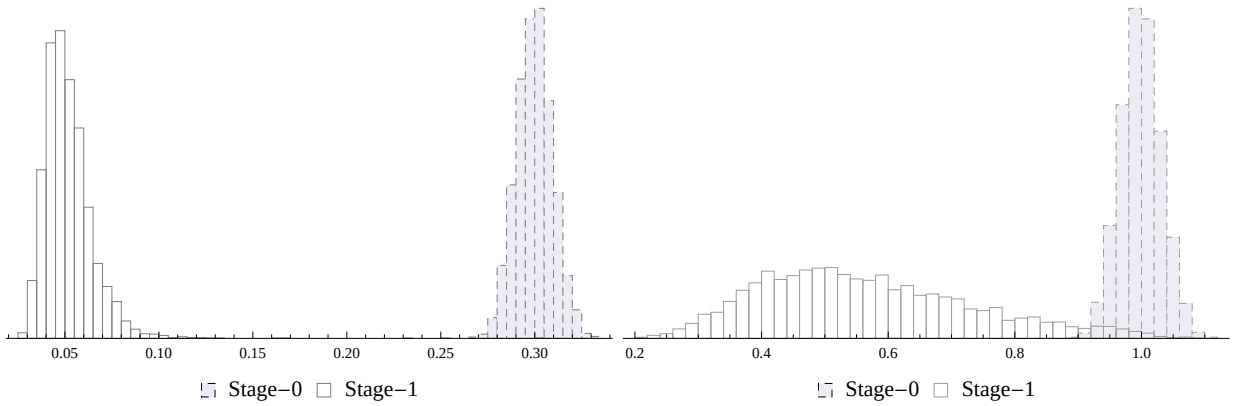


(a) How ρ varies in γ

(b) How ρ varies in ϵ

(c) How ρ varies in a

Figure 5: Comparative statics.



(a) Low commodity volatility $b = 0.3$.

(b) High commodity volatility $b = 1$.

Figure 6: Empirical volatility between commodity and stock at stage-0 and stage-1. Monte-Carlo simulation uses 5,000 sample paths and starting points $S_0 = 1$, $X_0 = 50$, and $Y_0 = e^3$.

the stock nor the commodity and thus will have no market impact. We also see that increasing ϵ increases the magnitude of the induced correlation. If the aggregate wealth of the portfolio optimizers is large enough, they overtake the market and the model breaks down. Interestingly, increasing the speed of mean-reversion a also increases the magnitude of the induced correlation. In the extreme case with a very large a, the portfolio optimizers will be able to achieve large and reliable gains in either a short or long position in the commodity whenever it deviates from its mean, and they choose a highly leveraged position in the commodity and it caused a high induced correlation.

4.3 Simulation

We can compute the empirical volatility and correlation arising from our feedback model. See Figures 6 and 7 for the daily sampled volatility and correlation over a 2-year horizon using 5,000 paths in two scenarios: low commodity volatility ($b = 0.3$) and high commodity volatility ($b = 1$). We see that financialization causes a decrease in commodity volatility in both scenarios. This is expected because the portfolio optimizers are buying low and selling high, and their trading has a stabilizing effect on the commodity price. Moreover, we see that the stage-0 correlation is sharply peaked at zero; while at stage-1, a relatively low (resp. high) commodity volatility b can induce a mostly negative (resp. positive) correlation between the commodity and stock.

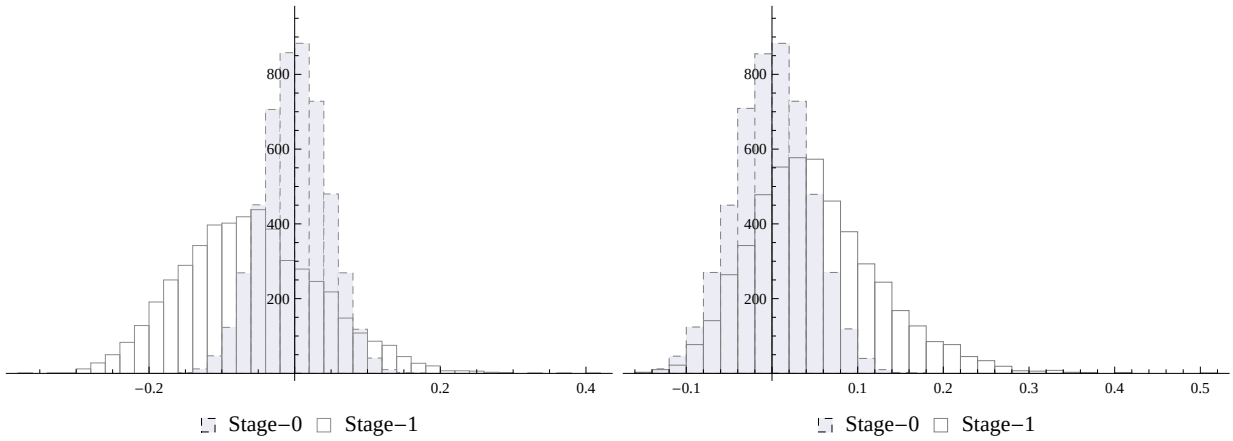
5 Empirical Analysis

5.1 Cross-correlations Between Commodities

We gathered daily prices of ten of the most heavily-traded commodity futures prices (see Table 2) over the period from 1990-2011.

In line with the hypotheses of Tang and Xiong [38], we choose 2004 as the dividing time-point between the unperturbed commodity price movement and the perturbed price movement due to the influx of index investors.

We find that the correlation between each commodity and the S&P 500 in the "non-indexed period" (1990-2004) is, averaging over the ten commodities, -0.0078435. In the "indexed period" (2004-2011), the average commodity-to-stock correlation is 0.1883. This dramatic increase supports the idea that commodities became more correlated with stocks after 2004.



(a) Low commodity volatility $b = 0.3$.

(b) High commodity volatility $b = 1$.

Figure 7: Empirical correlations between commodity and stock at stage-0 and stage-1. Monte-Carlo simulation uses 5,000 sample paths and starting points $S_0 = 1$, $X_0 = 50$, and $Y_0 = e^3$.

In comparing all possible pairs of cross-correlations between these commodities, we find that the average value of the correlation over all commodity pairs is 0.05988 in the non-indexed period and 0.1860 in the indexed period, which supports the idea of increased commodity cross-correlation as discussed in [38].

5.2 Parameter Estimation

For each of the ten commodities, using the historical data, we can estimate the parameters of the stage-0 price process through the use of maximum likelihood estimation. The continuous form of the price process is, by (2),

$$dY_t^{(0)} = a\left((\tilde{m} - \log(Y_t^{(0)})\right) Y_t^{(0)}\, dt + \lambda b Y_t^{(0)}\, dW_t. \tag{28}$$

By the non-closed-form approach detailed by Franco [13], we show, in Table 2, the maximum likelihood estimators for the parameters of this geometric Ornstein-Uhlenbeck process for each commodity, where a_{1990} denotes the best-fit value for the parameter a over the period beginning at 1990, and so on.

Table 2: Parameter estimates: the ten most heavily-traded commodity futures fitted to the geometric Ornstein-Uhlenbeck process. We note that the parameters are normalized daily.

Commodity	a_{1990}	a_{2004}	m_{1990}	m_{2004}	b_{1990}	b_{2004}
CBOT Corn Futures Prices	0.0023	0.0008	5.5204	6.5317	0.0116	0.0160
CSCE Cocoa Futures Prices	0.0024	0.0020	7.1868	7.8537	0.0169	0.0187
NYMEX Crude Oil Futures	0.0026	0.0028	3.1673	4.3921	0.0181	0.0190
NYCE Cotton Futures Prices	0.0018	0.0014	4.2041	4.3829	0.0141	0.0185
CSCE Coffee Futures Prices	0.0015	0.0013	4.6238	5.3168	0.0217	0.0158
NYMEX Natural Gas Futures Contract	0.0019	0.0031	1.2436	1.8634	0.0274	0.0265
CSCE Sugar No. 11 Futures Prices	0.0023	0.0013	2.1763	3.2023	0.0170	0.0190
CBOT Soybean Futures Prices	0.0018	0.0011	6.4326	7.0582	0.0102	0.0147
COMEX Silver Futures	0.0042	0.0005	6.1989	8.9836	0.0111	0.0179
CBOT Wheat Futures Price	0.0023	0.0015	5.8315	6.4878	0.0131	0.0180

We notice that most commodities experience a reduction in their best-fit value of a from the non-indexed period to the indexed period, though it increases for the two energy commodities. For each commodity, the best-fit value of the long-term mean log-price m increases, though not inconsistent with what would be expected due to inflation between the two periods. There is no clear trend in the best-fit values of b for the commodities over the two time periods.

5.3 Correlations In Extreme Price Cases

Table 3 shows the correlations between the commodities and the S&P 500. This illustrates the significant increase in correlation between commodities and stocks that we have tried to model.

Table 3: Correlations between the commodities and the S&P 500.

Commodity, correlated to S&P	*non-indexed period*	*indexed period*
CBOT Corn Futures Prices	0.0099	0.2231
CSCE Cocoa Futures Prices	-0.0182	0.1473
NYMEX Crude Oil Futures	-0.0546	0.3287
NYCE Cotton Futures Prices	0.0116	0.1511
CSCE Coffee Futures Prices	0.0290	0.1928
NYMEX Natural Gas Futures Contract	-0.0092	0.1035
CSCE Sugar No. 11 Futures Prices	-0.0082	0.1499
CBOT Soybean Futures Prices	0.0124	0.2074
COMEX Silver Futures	-0.0516	0.1848
CBOT Wheat Futures Price	0.0005	0.1943

5.4 Application: model calibration

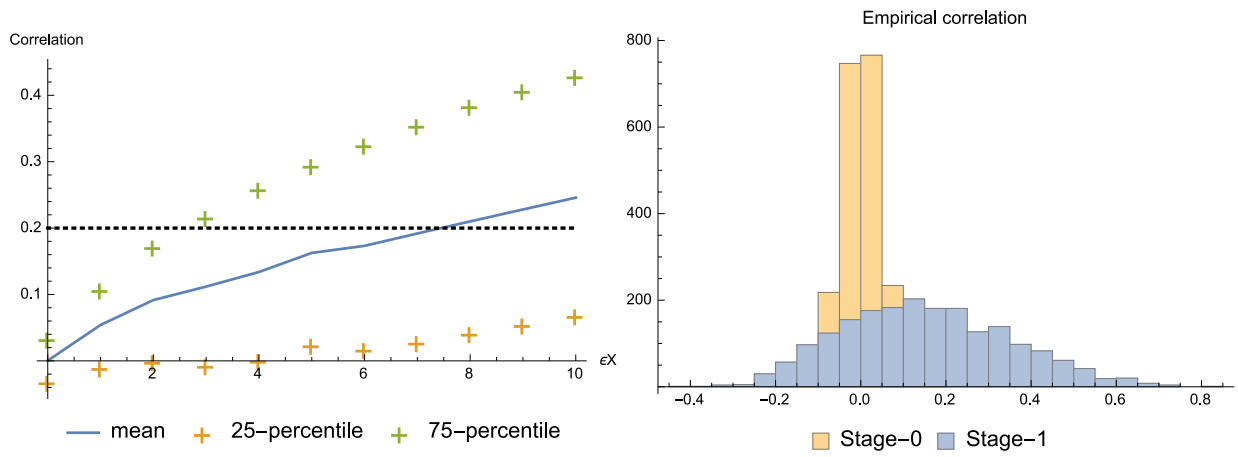
As an application of the proposed model, we perform a calibration exercise to demonstrate that our model can generate the size of empirical correlation typically observed in the commodity market as shown in Table 3. The pre-financialized commodity dynamics is taken to be the Schwartz one-factor model (2); using oil as an example, we take the market calibrated parameters found by Schwartz [33, Table 4] and estimate the size of portfolio optimizers in our model (using ϵX as a proxy) to generate realistic correlation observed in the commodity market.

Figure 8a shows the empirical correlation between the stage-1 commodity price and the stock price, for different values of ϵX. Using 1000 Monte-Carlo simulations, we show the average empirical correlation as well as the 25th and 75th percentiles. We see that with realistic market parameters, our model can generate the typical correlation level with the size of portfolio optimizers ϵX of order one. For instance, for $\epsilon X = 5$, a time series analysis of two-year daily data could reveal a correlation between 0.2% at the 25-percentile and 30% at the 75-percentile. Figure 8b shows the corresponding distribution of empirical correlation for $\epsilon X = 5$.

6 Conclusion

Despite the speculative nature of the portfolio optimizers in this model, they will frequently act to stabilize commodity prices through their trading. As in the simple economic argument of Friedman [15], the portfolio optimizers generally buy the commodity when the commodity price is below its mean and sell the commodity when the commodity price is above its mean, creating a demand effect which keeps the price nearer to its mean. We have shown that this volatility reduction occurs when the amount invested is somewhat near an unleveraged long position.

Correlation between commodities and the stock market is also of significant practical interest, and we have shown that the sign of the stage-1 induced correlation in our model will be the same as the sign



(a) Levels of empirical correlation.

(b) Distribution of empirical correlation.

Figure 8: Model calibration using Monte-Carlo simulations. Parameters for commodity dynamics are taken from Schwartz [33, Table 4]: $a = 0.301, b = 0.334, m = 3.093$. Other parameters are $A = 1, \gamma = 1, \lambda = 1, \mu = 0.1, r = 0, \sigma = 0.15, T = 2$.

of the fraction of wealth invested. This leads to a high positive correlation when the commodity price is unusually low, which is undesirable, but it also leads to a high negative correlation when the price is unusually high, which is desirable for investors as it will, in some sense, cause the commodity price to move in the opposite direction as the overall economy during the times when the commodity price is high.

Overall, through numerical simulations, for a few different batches of reasonable market parameters, the net effect of the introduction of the portfolio optimizers seems to be a reduction in commodity price volatility and an increase in correlation with the stock market.

A Solution to stage-0 PDE

The stage-0 equation (22) is

$$H_t + (c_0 + c_1 u)H_u + \frac{1}{2}\lambda^2 b^2 H_{uu} + (c_2 + c_3 u + c_4 u^2)H = 0$$

with terminal condition $H(T, u) = 1$, where c_0 through c_4 are given by

$$\begin{aligned} c_0 &= \frac{1}{\gamma}\Big(a\tilde{m} - (1-\gamma)r\Big) - \frac{1}{2}\lambda^2 b^2 \qquad c_1 = -\frac{a}{\gamma} \qquad c_2 = k \\ c_3 &= -\frac{1-\gamma}{\gamma}\frac{1}{\lambda^2 b^2}a(a\tilde{m} - r) \qquad c_4 = \frac{1-\gamma}{\gamma}\frac{a^2}{2\lambda^2 b^2}. \end{aligned}$$

Substituting the ansatz (23) into the above yields

$$f_0' + f_1' u + f_2' u^2 + (c_0 + c_1 u)(f_1 + 2f_2 u) + \frac{1}{2}\lambda^2 b^2 \left(2f_2 + \frac{1}{\gamma}(f_1 + 2f_2 u)^2\right) + (c_2 + c_3 u + c_4 u^2) = 0.$$

This yields three ordinary differential equations we have to solve to determine H:

$$f_0' + c_0 f_1 + \frac{1}{2}\lambda^2 b^2 (2f_2 + f_1) + c_2 = 0 \tag{29a}$$

$$f_1' + c_1 f_1 + 2c_0 f_2 + \frac{2}{\gamma}\lambda^2 b^2 f_1 f_2 + c_3 = 0 \tag{29b}$$

$$f_2' + \frac{2}{\gamma}\lambda^2 b^2 f_2^2 + 2c_1 f_2 + c_4 = 0 \tag{29c}$$

with terminal conditions $g_i(T) = 0$ for $i = 0, 1, 2$. These equations can be solved in closed form leading to (21); by inspection of the explicit formula, the solution does not explode in finite time provided that $\gamma > 0$.

B Application to Equity Market

In a related but simpler setting, Nayak and Papanicolaou [28] studied a feedback model for the equity market in which a single stock Y is traded by reference traders (who play the role of the commodity users in Section 2 of our analysis). The reference traders have a stochastic income process I_t. However, in the equity model, this is taken to be a geometric Brownian motion. Similar to the demand-supply analysis of Section 2.1, with an isoelastic demand function (1), when there are only reference traders, the stage-0 dynamics of the stock price $Y^{(0)}$ is also a geometric Brownian motion

$$\frac{dY_t^{(0)}}{Y_t^{(0)}} = \alpha_0\, dt + \sigma_0\, dW_t, \tag{30}$$

for some parameters α_0 and σ_0 and Brownian motion W, in contrast to (2) where this is an expOU in the commodity model. In addition, there are portfolio optimizers, who seek to maximize their expected CRRA utility (4) in a fixed horizon T. They will trade in the equity market and their demand will cause the stock price dynamics to deviate from (30).

B.1 Stage-0 portfolio optimization

Since the unperturbed stock price process (30) is the geometric Brownian motion, the stage-0 portfolio optimization is simply the Merton problem [26]. The optimal portfolio is given by a fixed-mix strategy

$$\theta_t^{(0)} = \widehat{\theta}_0 Z_t := \frac{\alpha_0 - r}{\sigma_0^2 \gamma} Z_t,$$

where Z_t is the wealth process of the portfolio optimizers.

B.2 Stage-1 portfolio optimization

As in Section 2.2, the stage-1 stock price dynamics is derived from the market clearing constraint, where we measure the relative size of the portfolio optimizers and reference traders by ϵ. The stage-1 drift α_1 and volatility σ_1 (analogs of $P^{(1)}$ and $Q^{(1)}$ in Section 2.5) are given by

$$\alpha_1(X_t, Y_t) = \frac{\alpha_0 \left(Y_t - \epsilon X_t \widehat{\theta}_0\right) + \epsilon X_t r \widehat{\theta}_0 (1 - \widehat{\theta}_0)}{Y_t - \epsilon X_t \widehat{\theta}_0^2}, \qquad \sigma_1(X_t, Y_t) = \frac{\sigma_0 (Y_t - \epsilon X_t \widehat{\theta}_0)}{Y_t - \epsilon X_t \widehat{\theta}_0^2}. \tag{31}$$

where X_t is the aggregate wealth of the portfolio optimizers who follows the stage-0 strategy.

We denote the stage-1 strategy by $\theta^{(1)} = \widehat{\theta}^{(1)} Z_t$, where Z_t is the wealth process of the "smart" traders. Analogous to Section 3.1, his value function V is determined by the HJB PDE problem

$$\begin{aligned} 0 = V_t + rzV_z + \Big(\widehat{\theta}_0(\alpha_1 - r) + r\Big)xV_x + \frac{1}{2}\widehat{\theta}_0^2\sigma_1^2 x^2 V_{xx} + \alpha_1 y V_y + \frac{1}{2}\sigma_1^2 y^2 V_{yy} \\ + \widehat{\theta}_0\sigma_1^2 xy V_{xy} + \sup_{\widehat{\theta}}\left(\widehat{\theta}\Big((\alpha_1 - r)zV_z + \sigma_1^2(\widehat{\theta}_0 xzV_{xz} + yzV_{yz})\Big) + \frac{1}{2}\widehat{\theta}^2\sigma_1^2 z^2 V_{zz}\right) \end{aligned} \tag{32}$$

with terminal condition $V(T, x, y, z) = U(z)$.

We observe that $\alpha_1 = \alpha_1(x, y)$ and $\sigma_1 = \sigma_1(x, y)$ are in fact functions only of the ratio $\xi = x/y$. We look for a similarity solution in the variable $\xi = x/y$. After separation of variables $V(t, x, y, z) = \frac{z^{1-\gamma}}{1-\gamma} G(t, \xi)^\gamma$, G solves

$$0 = G_t + \frac{1}{2}\sigma_1^2(1 - \widehat{\theta}_0)^2\xi^2 G_{\xi\xi} - \frac{1}{\gamma}\left(\alpha_1 - r - \gamma\sigma_1^2\right)(1 - \widehat{\theta}_0)\xi G_\xi + \frac{1-\gamma}{\gamma}\left(\frac{(\alpha_1 - r)^2}{2\gamma\sigma_1^2} + r\right)G \tag{33}$$

with terminal condition $G(T, \xi) = 1$. The key observation to solving (33) is that the stage-1 Sharpe ratio $(\alpha_1 - r)/\sigma_1$ is equal to its stage-0 counterpart $(\alpha_0 - r)/\sigma_0$ and, in particular, is independent of ξ. This allows us to deduce the analytic solution to (32) and give the following proposition.

Proposition 4. *The stage-1 value function V does not depend on the current stock price Y_t or the aggregate wealth process X_t:*

$$V(t, x, y, z) = \frac{z^{1-\gamma}}{1-\gamma}\exp\left((1-\gamma)\left(\frac{(\alpha_0 - r)^2}{2\gamma\sigma_0^2} + r\right)(T - t)\right).$$

The stage-1 optimal portfolio is given by the Merton ratio, evaluated at the stage-1 drift α_1 and volatility σ_1:

$$\widehat{\theta}^{(1)}(x, y) = \frac{(\alpha_1 - r)}{\gamma\sigma_1^2} = \frac{(\alpha_0 - r)}{\gamma\sigma_0}\frac{1}{\sigma_1} = \widehat{\theta}_0\left(\frac{y - \epsilon x\widehat{\theta}_0^2}{y - \epsilon x\widehat{\theta}_0}\right). \tag{34}$$

We have checked that (34) has excellent agreement with the order ϵ asymptotic expansion derived by Nayak and Papanicolaou [28] in the small feedback regime.

Remark 3. *It is also possible to show, by similar calculations and similarity solutions, that the stage-k optimal strategy $\widehat{\theta}^{(k)}$ depends only on the ratio $\xi_t = X_t/Y_t$.*

C Exponential Utility

In a tractable instance of our feedback model, with Ornstein-Uhlenbeck income process, linear demand function, and exponential utility $U(x) = -e^{-\gamma x}$ for the portfolio optimizer, we are able to characterize the full fixed point problem by a coupled system of integro-differential equations.

Specifically, we will take the income process (I_t) to be an Ornstein-Uhlenbeck process

$$dI_t = a(m - I_t)\,dt + b\,dW_t^c,$$

and use the linear demand function $D(Y_t, I_t) = I_t - Y_t$. In this simplified setting, the market clearing condition $D(Y_t, I_t) = A$ leads to

$$dY_t = a(\tilde{m} - Y_t)\,dt + b\,dW_t^c,$$

where we have defined $\tilde{m} = m - A$. We see that the pre-financialized commodity price Y_t is an Ornstein-Uhlenbeck process.

C.1 Stage-1 dynamics

We motivate the fixed-point characterization of the full feedback problem with the stage-0 optimal investment problem and the induced stage-1 dynamics. As in Section 2, there is a single representative stock index with dynamics (5). The portfolio optimizer allocates his wealth into investments in the stock and the commodity. The wealth process X_t follows

$$\begin{aligned} dX_t &= \frac{\pi_t}{S_t}\, dS_t + \frac{\theta_t}{Y_t}\, dY_t + r(X_t - \pi_t - \theta_t)\, dt \\ &= \left[\mu\pi_t + a(\tilde{m} - Y_t)\frac{\theta_t}{Y_t} + r(X_t - \pi_t - \theta_t)\right] dt + \sigma\pi_t dW_t^s + b\frac{\theta_t}{Y_t} dW_t^c. \end{aligned}$$

The value function for the exponential investor is the maximum expected terminal utility

$$v(t,x,y) = \sup_{\pi,\theta} \mathbb{E}\left[U(X_T) \middle| X_t = x, Y_t = y\right], \quad U(x) = -e^{-\gamma x}.$$

The usual dynamic programming principle leads to the HJB equation

$$0 = v_t + rxv_x + a(\tilde{m} - y)v_y + \frac{1}{2}b^2 v_{yy} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{v_x^2}{v_{xx}} - \frac{\left[(a\tilde{m} - (a+r)y)\, v_x + b^2 v_{xy}\right]^2}{2b^2 v_{xx}}, \tag{35}$$

with terminal condition $v(T,x,y) = -e^{-\gamma x}$.

Proposition 5. *The solution to the HJB equation* (35) *with terminal condition* $v(T,x,y) = -e^{-\gamma x}$ *is given by*

$$v(t,x,y) = -\exp\Big(A(t)x + B(t)y^2 + C(t)y + D(t)\Big), \tag{36}$$

where

$$A(t) = -\gamma e^{r(T-t)}, \quad B(t) = \frac{(a+r)^2}{4b^2 r}\left(1 - e^{2r(T-t)}\right), \quad C(t) = \frac{am(a+r)}{b^2 r}\left(e^{r(T-t)} - 1\right).$$

Proof. This follows easily by substituting the *ansatz* (36) and solving ODEs for the functions $A(t), B(t), C(t)$. □

The optimal investment in the commodity can be written in feedback form

$$\theta(t,x,y) = y\gamma e^{-r(T-t)}\left[\frac{a\tilde{m}}{b^2} + C(t) - \left(\frac{a+r}{b^2} - 2B(t)\right) y\right].$$

In particular, the number of shares θ/y demanded by the portfolio optimizer is affine in the stage-1 commodity price $Y_t^{(1)}$. The modified market clearing condition reads

$$\begin{aligned} A &= D(I_t, Y_t^{(1)}) + \epsilon\frac{\theta_t}{Y_t^{(1)}} \\ &= I_t - Y_t^{(1)} + \epsilon\gamma e^{-r(T-t)}\left[\frac{a\tilde{m}}{b^2} + C(t) - \left(\frac{a+r}{b^2} - 2B(t)\right) Y_t^{(1)}\right]. \end{aligned}$$

We therefore obtain

$$Y_t^{(1)} = \frac{I_t - A + \epsilon\gamma e^{-r(T-t)}\left(\frac{a\tilde{m}}{b^2} + C(t)\right)}{1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a+r}{b^2} - 2B(t)\right)} =: F_0(t) + F_1(t) I_t.$$

It follows that the stage-1 commodity dynamics is given by

$$dY_t^{(1)} = \left[amF_1(t) + F_0'(t) + F_0(t)\left(a - \frac{F_1'(t)}{F_1(t)}\right) - \left(a - \frac{F_1'(t)}{F_1(t)}\right) Y_t^{(1)}\right] dt + bF_1(t)\, dW_t^c. \tag{37}$$

In particular, the commodity price Y_t is still independent of the stock price S_t. Intuitively, portfolio optimizers with exponential utility do not induce financialization since their trading in the commodity markets is not affected by the wealth generated in the financial markets.

C.2 Full Feedback Fixed Point Problem

We now consider the fixed-point characterization of the full problem, where the price impact of the portfolio optimizers is fully incorporated in their trading decision, as described in Section 2.3. Motivated by the fact that the stage-1 commodity dynamics (37) is again an Ornstein-Uhlenbeck process, we postulate that the equilibrium commodity price with feedback described there will also be a time-inhomogeneous Ornstein-Uhlenbeck process

$$dY_t = a(t)(m(t) - Y_t)dt + b(t)\, dW_t^c. \tag{38}$$

The HJB equation for the value function corresponding to the utility maximization problem (8) now reads

$$0 = v_t + rxv_x + a(t)(m(t)-y)v_y + \frac{1}{2}b(t)^2 v_{yy} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{v_x^2}{v_{xx}} - \frac{\left[(a(t)m(t) - (a(t)+r)y)\, v_x + b(t)^2 v_{xy}\right]^2}{2b(t)^2 v_{xx}},$$

with terminal condition $v(T,x,y) = -e^{-\gamma x}$. The same *ansatz* (36) leads to the following system of ODEs:

$$\begin{cases} 0 = A'(t) + rA(t), & A(T) = -\gamma; \\ 0 = B'(t) + 2rB(t) - \frac{1}{2}\left(\frac{a(t)+r}{b(t)}\right)^2, & B(T) = 0; \\ 0 = C'(t) + rC(t) + \frac{a(t)m(t)}{b(t)^2}(a(t)+r), & C(T) = 0; \\ 0 = D'(t) + b(t)^2 B(t) + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{a(t)m(t)}{b(t)}\right)^2, & D(T) = 0. \end{cases}$$

The market clearing condition (7) reads

$$I_t - Y_t + \epsilon\gamma e^{-r(T-t)}\left[\frac{a(t)m(t)}{b(t)^2} + C(t) - \left(\frac{a(t)+r}{b(t)^2} - 2B(t)\right)Y_t\right] = A.$$

Proposition 6. *The full feedback equilibrium problem is characterized by the dynamics* (38)*, where a, m, b satisfy the following integro-differential equations*

$$\begin{aligned} 0 &= -\epsilon\gamma b(t)\left(a'(t) + (-a(t)+a+r)(a(t)+r)\right) + 2\gamma\epsilon(a(t)+r)b'(t) \\ &\quad + b(t)^3\left(2\gamma\epsilon\left(B(t)(-a(t)+a+r) + B'(t)\right) + (a(t)-a)e^{r(T-t)}\right), \\ 0 &= \left(1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a(t)+r}{b(t)^2} - 2B(t)\right)\right)b(t) - b, \\ 0 &= b(t)^3 e^{r(T-t)}(a(A-m)+m(t)) - \epsilon\gamma\left(b(t)\left(m(t)\left(a'(t) + (a+r-1)a(t) - r\right) + a(t)m'(t)\right)\right) \\ &\quad + \epsilon\gamma\left(2a(t)m(t)b'(t) - b(t)^3\left((a+r)C(t) + 2B(t)m(t) + C'(t)\right)\right), \end{aligned} \tag{39}$$

where

$$B(t) = -\frac{1}{2}\int_t^T e^{-2r(t-s)}\left(\frac{a(s)+r}{b(s)}\right)^2 ds, \quad C(t) = \int_t^T e^{-r(t-s)}\frac{a(s)m(s)(a(s)+r)}{b(s)^2}ds.$$

In particular, the equilibrium feedback commodity price Y_t is independent of the stock price S_t; there is no financialization under the feedback model with exponential utility.

Proof. An application of the Itô's lemma leads to the following differential algebraic equations:

$$\begin{aligned} 0 &= B'(t) + 2rB(t) - \frac{1}{2}\left(\frac{a(t)+r}{b(t)}\right)^2, \quad B(T) = 0, \\ 0 &= C'(t) + rC(t) + \frac{a(t)m(t)}{b(t)^2}(a(t)+r), \quad C(T) = 0, \\ a(t) &= a - \frac{F_1'(t)}{F_1(t)}, \quad b(t) = bF_1(t), \quad m(t) = \frac{amF_1(t) + F_0'(t) + F_0(t)\left(a - \frac{F_1'(t)}{F_1(t)}\right)}{a - \frac{F_1'(t)}{F_1(t)}}, \\ F_0(t) &= \frac{-A + \epsilon\gamma e^{-r(T-t)}\left(\frac{a(t)m(t)}{b(t)^2} + C(t)\right)}{1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a(t)+r}{b(t)^2} - 2B(t)\right)}, \quad F_1(t) = \frac{1}{1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a(t)+r}{b(t)^2} - 2B(t)\right)}. \end{aligned}$$

Solving the ODEs for B and C leads to proposition. □

Clearly the lack of induced correlation is due to the well-known limitations of exponential utility, whose tractability in many problems is undermined by paradoxical conclusions.

Finally, one can view the above proposition as the limit of the stage-k problem under exponential utility. Following the steps outlined in Section C.1, we can derive the stage-$(k+1)$ dynamics

$$dY_t^{(k+1)} = a_{k+1}(t)(m_{k+1}(t) - Y_t^{(k+1)})dt + b_k(t)\, dW_t^c. \tag{40}$$

where

$$\begin{aligned} b_{k+1}(t) &= bF_1^{(k)}(t), \qquad a_{k+1}(t) = a - \frac{\frac{d}{dt}F_1^{(k)}(t)}{F_1^{(k)}(t)}, \\ m_{k+1}(t) &= F_0^{(k)}(t) + \frac{amF_1^{(k)}(t) + \frac{d}{dt}F_0^{(k)}(t)}{a_{k+1}(t)}, \end{aligned} \tag{41}$$

with $b_0(t) \equiv b, a_0(t) \equiv a, m_0(t) \equiv \tilde{m}$, and

$$\begin{aligned} F_0^{(k)}(t) &= \frac{-A + \epsilon\gamma e^{-r(T-t)}\left(\frac{a_k(t)m_k(t)}{b_k(t)^2} + \int_t^T e^{-r(t-s)}\frac{a_k(s)m_k(s)(a_k(s)+r)}{b_k(s)^2}ds\right)}{1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a_k(t)+r}{b_k(t)^2} + \int_t^T e^{-2r(t-s)}\left(\frac{a_k(s)+r}{b_k(s)}\right)^2 ds\right)}, \\ F_1^{(k)}(t) &= \frac{1}{1 + \epsilon\gamma e^{-r(T-t)}\left(\frac{a_k(t)+r}{b_k(t)^2} + \int_t^T e^{-2r(t-s)}\left(\frac{a_k(s)+r}{b_k(s)}\right)^2 ds\right)}. \end{aligned} \tag{42}$$

While the convergence properties of the family (a_k, b_k, m_k) to (a, b, m) in Proposition 6 seems to be rather involved, we note that (by removing the k in (41) and (42)) the limit necessarily satisfies (39). So, if the feedback iteration scheme converges, it does so to the full feedback equilibrium.

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