

# General Black-Scholes models accounting for increased market volatility from hedging strategies

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## Abstract

Increases in market volatility of asset prices have been observed and analyzed in recent years and their cause has generally been attributed to the popularity of portfolio insurance strategies for derivative securities. The basis of derivative pricing is the Black-Scholes model and its use is so extensive that it is likely to influence the market itself. In particular it has been suggested that this is a factor in the rise in volatilities. In this work we present a class of pricing models that account for the feedback effect from the Black-Scholes dynamic hedging strategies on the price of the asset, and from there back onto the price of the derivative. These models do predict increased implied volatilities with minimal assumptions beyond those of the Black-Scholes theory. They are characterized by a nonlinear partial differential equation that reduces to the Black-Scholes equation when the feedback is removed.

We begin with a model economy consisting of two distinct groups of traders: Reference traders who are the majority investing in the asset expecting gain, and program traders who trade the asset following a Black-Scholes type dynamic hedging strategy, which is not known *a priori*, in order to insure against the risk of a derivative security. The interaction of these groups leads to a stochastic process for the price of the asset which depends on the hedging strategy of the program traders. Then following a Black-Scholes argument, we derive nonlinear partial differential equations for the derivative price and the hedging strategy. Consistency with the traditional Black-Scholes model characterizes the class of feedback models that we analyze in detail. We study the nonlinear partial differential equation for the price of the derivative by perturbation methods when the program traders are a small fraction of the economy, by numerical methods, which are easy to use and can be implemented efficiently, and by analytical methods. The results clearly support the observed increasing volatility phenomenon and provide a quantitative explanation for it.

**Keywords:** Black-Scholes model, Dynamic hedging, Feedback effects, Option pricing, Volatility.

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## 1 Introduction

One of modern financial theory's biggest successes in terms of both approach and applicability has been Black-Scholes pricing, which allows investors to calculate the 'fair' price of a derivative security whose value depends on the value of another security, known as the underlying, based on a small set of assumptions on the price behavior of that underlying. Indeed, before the method existed, pricing of derivatives was a rather mysterious task due to their often complex dependencies on the underlying, and they were traded mainly over-the-counter rather than in large markets, usually with high transaction costs. Publication of the Black-Scholes model [4] in 1973 roughly coincided with the opening of the Chicago Board of Trade and since then, trading in derivatives has become prolific.

Furthermore, use of the model is so extensive, that it is likely that the market is influenced by

it to some extent. It is this feedback effect of Black-Scholes pricing on the underlying's price and thence back onto the price of the derivatives that we study in this work. Thus we shall relax one of the major assumptions of Black and Scholes: that the market in the underlying asset is perfectly elastic so that large trades do not affect prices in equilibrium.

### 1.1 The Black-Scholes Model

The primary strength of the Black-Scholes model, in its simplest form, is that it requires the estimation of only one parameter, namely the market volatility of the underlying asset price (in general as a function of the price and time), without direct reference to specific investor characteristics such as expected yield, utility function or measures of risk aversion. Later work by Kreps [6] and Bick [2, 3] has placed both the classical and the generalized Black-Scholes formulations within the framework of a consistent economic model of market equilibrium with interacting agents having very specific investment characteristics.

In addition, the Black-Scholes analysis yields an explicit trading strategy in the underlying asset and riskless bonds whose terminal payoff is equal to the payoff of the derivative security at maturity. Thus selling the derivative and buying and selling according to this strategy 'covers' an investor against all risk of eventual loss, for a loss incurred at the final date as a result of one half of this portfolio will be exactly compensated by a gain in the other half. This replicating strategy, as it is known, therefore provides an insurance policy against the risk of holding or selling the derivative: it is called a dynamic hedging strategy (since it involves continual trading), where to hedge means to reduce risk. That investors can hedge and know how to do so is a second major strength of Black-Scholes pricing, and it is the proliferation of these hedging strategies causing a feedback onto the underlying pricing model that we shall study here.

The missing ingredients in this brief sketch of the Black-Scholes argument are, firstly, the existence of such replicating strategies, which is a problem of market completeness that is resolved in this setting by allowing continuous trading (or, equivalently, infinite trading opportunities); and secondly, that the price of running the dynamic hedging strategy should equal the price of the derivative if there is not to be an opportunity for some investor to make 'money for nothing', an arbitrage. Enforcement of no-arbitrage pointwise in time leads to the Black-Scholes pricing equation; more globally, it can be considered as a model of the observation that market trading causes prices to change so as to eliminate risk-free profit-making opportunities.

### 1.2 Feedback Effects

There has been much work in recent years to explain the precise interaction between dynamic hedging strategies and market volatility. As Miller [17] notes, "the widespread view, expressed almost daily in the financial press ... is that stock market volatility has been rising in recent years and that the introduction of low-cost speculative vehicles such as stock index futures and options has been mainly responsible." Modelling of this phenomenon typically begins with an economy of two types of investors, the first whose behavior upholds the Black-Scholes hypotheses, and the second who trade to insure other portfolios. Peters [18] has 'smart money traders' who invest according to value and 'noise traders' who follow fashions and fads; the latter overreact to news that may affect future dividends, to the profit of the former. Föllmer and Schweizer [9] have 'information traders' who believe in a fundamental value of the asset and that the asset price will take that value, and 'noise traders' whose demands come from hedging; they derive equilibrium diffusion models for the asset price based on interaction between these two.

Brennan and Schwartz [5] construct a single-period model in which a fraction of the wealth is held by an expected utility maximizing investor, and the rest by an investor following a simple portfolio insurance strategy that is *a priori* known to all. They assume the first investor has a CRRA utility function<sup>1</sup> and obtain between 1% and 7% increases in Black-Scholes implied market volatility<sup>2</sup> for values of the fraction of the market portfolio subject to portfolio insurance varying between 1% and 20%.

In similar vein, Frey and Stremme [11] present a discrete time and then a continuous time economy of reference traders (Black-Scholes upholders) and program traders (portfolio insurers). They derive an explicit expression for the perturbation of the Itô diffusion equation for the price of the underlying asset by feedback from the program traders' hedging strategies. In particular, they consider the case of a 'Delta Hedging' strategy for a European option whose price  $c(x, t)$  is given by a classical Black-Scholes formula, that is one with constant volatility  $\sigma$ . It is described by a function  $\phi(x, t)$  which is the number of units of the underlying asset to be held when the asset price is  $x$  at time  $t$ , and it is a result of Black-Scholes theory that this number is the 'delta' of the option price:  $\phi = \partial c / \partial x$ . Thus they examine the effect of  $\phi$  being completely specified but for the constant  $\sigma$ , and ask the questions i) how valid is it for the program traders to continue to use the classical Black-Scholes formula when they also know about feedback effects on the price of the underlying stock? and ii) given the single degree of freedom allowed for the choice of  $\phi$ , what is the best value to choose for  $\sigma$ ? The latter question leads them to a family of fixed-point problems and thence to lower and upper bounds for suitable  $\sigma$ . They report up to 5% increases in market volatility when program traders have a 10% market share. This framework is also studied by Schönbucher and Wilmott [22, 23] who analyse perturbations of asset prices by exogenously given Black-Scholes hedging strategies and, in particular, induced price jumps as expiration is approached.

A full description of the extent and type of program trading that occurs in practice is given by Duffee *et al.* [7]. They write that "the New York Stock Exchange (NYSE) has defined program trading as the purchase or sale of at least fifteen stocks with a value of the trade exceeding \$1 million. Program trading has averaged about 10 percent of the NYSE volume or 10 to 20 million shares per day in the last year-and-a-half that these data have been collected. This activity has fallen to about 3 percent of volume since the NYSE encouraged firms to restrict some program-related trades after October 1989." In addition, they report that 10 – 30% of all program trading occurs on foreign exchanges, particularly in London.

In Section 2, we follow the framework of the Frey-Stremme model, but consider the hedging strategy as unknown and derive equations for it using the modified underlying asset diffusion process. The key ingredients are a stochastic income process, a demand function for the reference traders, and a parameter  $\rho$  equal to the ratio of the number of options being hedged to the total number of units of the asset in supply. In addition to the equations arising from an arbitrary demand function and a general Itô income process, we give two families of pricing models that are consistent with the generalized and classical Black-Scholes pricing equations, and which reduce to these when the program traders are removed. Enforcing this consistency places a restriction on the form of the reference traders' demand function, which we derive.

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<sup>1</sup>A von Neumann-Morgenstern utility function with Constant Relative Risk Aversion is of the form  $u(x) = x^\gamma / \gamma$  for some  $\gamma > 0$ .

<sup>2</sup>In the sense of the instantaneous variance rate of the prices of the underlying asset resulting from their model.

We concentrate on the feedback model for a European call option consistent with classical Black-Scholes in Section 3, and give asymptotic results for when the volume of assets traded by the program traders is small compared to the total number of units of the asset. Here, the program traders cause a small perturbation to the classical Black-Scholes economy, and we measure the effects in terms of implied Black-Scholes volatility: the adjusted volatility parameter that should be used in the Black-Scholes formula to best approximate, in a sense that is made precise, the feedback-adjusted price of the option. We find that the market volatility does indeed increase as anticipated, and by a greater degree than found in the Brennan-Schwartz or Frey-Stremme studies. We also give an extension to the more realistic case when feedback comes from hedging a number of different options on the underlying asset, producing similar results.

In Section 4, we present general analytical results for an arbitrary European derivative security with a convex payoff function, again when program trading is small compared to reference trading. We find the model predicts higher implied market volatilities. Then in Section 5, we outline an efficient numerical scheme for solution of the model equations. We also present typical discrepancies between historical volatilities and feedback volatilities as predicted by our model. These illustrate that mispricings of up to 30% can occur if traders attempt to account for feedback-induced increased market volatility by constantly re-calibrating the Black-Scholes volatility parameter, instead of using the full theory.

If we assume that increased volatility is due primarily to feedback effects from program trading then we can use our model to estimate the fraction of the market that is being traded for portfolio insurance purposes. Jacklin *et al.* [15] argue that one of the causes of the crash of October 19, 1987 was information about the extent of portfolio insurance-motivated trading suddenly becoming known to the rest of the market. This prompted the realization that assets had been overvalued because the information content of trades induced by hedging concerns had been misinterpreted. Consequently, general price levels fell sharply. Similar conclusions are reached by Duffee *et al.* [7], Gennotte and Leland [12], and Grossman [13].

We summarize and give conclusions and plans for future work in Section 6.

## 2 Derivation of the Model

We state for reference the generalized<sup>3</sup> Black-Scholes pricing partial differential equation whose derivation is detailed in [8] for example, and then extend it to incorporate feedback effects from portfolio insurance.

### 2.1 The Generalized Black-Scholes Pricing Model

Suppose there is a model economy in which traders create a continuous-time market for a particular asset whose equilibrium price process is denoted by  $\{\tilde{X}_t, t \geq 0\}$ . There are two other securities in the economy: a riskless bond with price process  $\beta_t = \beta_0 e^{rt}$ , where  $r$  is the constant ‘spot’ interest rate, and a derivative security with price process  $\{\tilde{P}_t, t \geq 0\}$  whose payoff at some terminal date  $T > 0$  is contingent on the price  $\tilde{X}_T$  of the underlying asset on that date:  $\tilde{P}_T = h(\tilde{X}_T)$ , for some function  $h(\cdot)$ . The asset is assumed to pay no dividends in  $0 \leq t \leq T$ .

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<sup>3</sup>We use the word generalized in the sense that the underlying asset price is a general Itô process rather than the specific case of Geometric Brownian Motion as in the classical Black-Scholes derivation.

An Itô process for the price of the underlying is taken as given:

$$d\tilde{X}_t = \nu(\tilde{X}_t, t)dt + \lambda(\tilde{X}_t, t)d\tilde{W}_t, \quad (2.1)$$

where  $\{\tilde{W}_t, t \geq 0\}$  is a standard Brownian Motion on a probability space  $(\Omega, \mathcal{F}, P)$ , and  $\nu$  and  $\lambda$  satisfy Lipschitz and growth conditions sufficient for the existence of a continuous solution to (2.1). In practice, these functions are often calibrated from past data to make such a model tractable.

Then we suppose the price of the derivative is given by  $\tilde{P}_t = \tilde{C}(\tilde{X}_t, t)$  for some function  $\tilde{C}(x, t)$  that is assumed sufficiently smooth for the following to be valid. Well-known arguments involving construction of a self-financing replicating strategy in the underlying asset and bond lead to the generalized Black-Scholes partial differential equation for the function  $\tilde{C}(x, t)$

$$\frac{\partial \tilde{C}}{\partial t} + \frac{1}{2}\lambda(x, t)^2 \frac{\partial^2 \tilde{C}}{\partial x^2} + r \left( x \frac{\partial \tilde{C}}{\partial x} - \tilde{C} \right) = 0, \quad (2.2)$$

for each  $x > 0, t > 0$ . The terminal condition is  $\tilde{C}(x, T) = h(x)$ . Furthermore,  $\tilde{C}(x, t)$  is independent of the drift function  $\nu(x, t)$  in (2.1).

Later, we shall refer to the classical Black-Scholes equation which comes from taking the underlying's price to be a Geometric Brownian Motion,

$$d\tilde{X}_t = \bar{\mu}\tilde{X}_t dt + \sigma\tilde{X}_t d\tilde{W}_t, \quad (2.3)$$

for constants  $\bar{\mu}$  and  $\sigma$ . Substituting  $\lambda(x, t) \equiv \sigma x$  in (2.2), gives

$$\frac{\partial \tilde{C}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \tilde{C}}{\partial x^2} + r \left( x \frac{\partial \tilde{C}}{\partial x} - \tilde{C} \right) = 0, \quad (2.4)$$

which is known as the classical Black-Scholes equation. It does not feature the drift parameter  $\bar{\mu}$  in (2.3).

The dynamic hedging strategy to neutralize the risk inherent in writing<sup>4</sup> the derivative is, explicitly, to hold the amount  $\tilde{C}_x(\tilde{X}_t, t)$  of the underlying asset at time  $t$ , continually trading to maintain this number of the asset, and invest the amount  $\tilde{C}(\tilde{X}_t, t) - \tilde{X}_t \tilde{C}_x(\tilde{X}_t, t)$  in bonds at time  $t$ . The cost of doing this is  $\tilde{C}(x, t)$ , which is the 'fair' price of the derivative. Equivalently, holding the derivative and  $-\tilde{C}_x(\tilde{X}_t, t)$  of the underlying is a risk-free investment.

## 2.2 Framework of Model Incorporating Feedback

We next describe the framework of the continuous-time version of the model economy proposed by Frey and Stremme [11].

In this economy there are two distinct groups of traders: reference traders and program traders whose characteristics are outlined below. These investors create a continuous-time market for a particular asset whose equilibrium price process in this setting is denoted by  $\{X_t, t \geq 0\}$ . The price process of the bond is denoted  $\beta_t = \beta_0 e^{rt}$  as before, and the price of the derivative security is

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<sup>4</sup>The party that sells the derivative contract is called the writer, and the buyer is called the holder.

$\{P_t, t \geq 0\}$ .

We characterize the larger of the two groups of traders, the reference traders, as investors who buy and sell the asset in such a way that, were they the only agents in the economy, the equilibrium asset price would exactly follow a solution trajectory of (2.1). Furthermore, this price would be independent of the distribution of wealth amongst the reference traders, so we can consider all the reference traders' market activities by defining a single aggregate reference trader who represents the actions of all the reference traders together in the market. Alternatively, they can be described as traders who blindly follow this asset pricing model, and act accordingly. To derive the model for the asset price incorporating feedback effects, rather than taking  $X_t$  as given, we suppose the aggregate reference trader has two primitives:

1. an aggregate stochastic income (equal to the total income of all the reference traders) modelled by an Itô process  $\{Y_t, t \geq 0\}$  satisfying

$$dY_t = \mu(Y_t, t)dt + \eta(Y_t, t)dW_t, \quad (2.5)$$

where  $\{W_t\}$  is a Brownian motion, and  $\mu$  and  $\eta$  are exogenously given functions satisfying the usual technical conditions for existence and uniqueness of a continuous solution to (2.5). We note at this stage that the functions  $\mu(y, t)$  and  $\eta(y, t)$  will not appear in the options pricing equations that we shall derive, so that the incomes process, which is not directly observable, need not be known for our model.

2. a demand function  $\tilde{D}(X_t, Y_t, t)$  depending on the income and the equilibrium price process. Examples for  $\tilde{D}$  will be argued from a consistency criterion with the Black-Scholes model, as in the special case treated in the bulk of this paper.

The second group, consisting of program traders, is characterized by the dynamic hedging strategies they follow for purposes of portfolio insurance. Their sole reason for trading in the asset is to hedge against the risk of some other portfolio (eg. against the risk incurred in writing a European option on that asset). Their aggregate demand function is given by  $\phi(X_t, t)$  which is the amount of the asset those traders want to hold at time  $t$  given the price  $X_t$ ; we do not allow  $\phi$  to depend on knowledge of the representative reference trader's income process  $Y_t$  to which they have no access. We shall assume for the moment that the program traders are hedging against the risk of having written  $\zeta$  identical derivative securities, and indicate the extension to the general case where the insurance is for varying numbers of different derivatives in section (2.6). For convenience, we shall write  $\phi(X_t, t) = \zeta\Phi(X_t, t)$ , where  $\Phi$  can be thought of as the demand per security being hedged.

In this work, we do not assume that  $\Phi$  is given, but go on to derive equations it must satisfy.

### 2.3 Asset Price under Feedback

We now consider how market equilibrium and  $Y_t$  determine the price process of the asset,  $X_t$ . Let us assume that the supply of the asset  $S_0$  is constant and that  $\tilde{D}(x, y, t) = S_0 D(x, y, t)$ , so that  $D$  is the demand of the reference traders relative to the supply. Then we define the relative demand of the representative reference trader and the program traders at time  $t$  to be  $G(X_t, Y_t, t)$ , where

$$G(x, y, t) = D(x, y, t) + \rho\Phi(x, t), \quad (2.6)$$

and  $\rho := \zeta/S_0$  is the ratio of the volume of options being hedged to the total supply of the asset. The normalization by the total supply has been incorporated into the definition of  $D$ , and  $\rho\Phi$  is

the proportion of the total supply of stock that is being traded by the program traders.

Then, setting demand  $\equiv$  supply = 1 at each point in time to enforce market equilibrium gives

$$G(X_t, Y_t, t) \equiv 1, \quad (2.7)$$

which is the determining relationship between the trajectory  $X_t$  and the trajectory  $Y_t$  which is known. We assume that  $G(x, y, t)$  is strictly monotonic in its first two arguments, and has continuous first partial derivatives in  $x$  and  $y$ , so that we can invert (2.7) to obtain  $X_t = \psi(Y_t, t)$ , for some smooth function  $\psi(y, t)$ . This tells us that the process  $X_t$  must be driven by the same Brownian Motion as  $Y_t$ .

Using Itô's lemma, we obtain

$$dX_t = \left[ \mu(Y_t, t) \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial t} + \frac{1}{2} \eta^2(Y_t, t) \frac{\partial^2 \psi}{\partial y^2} \right] dt + \eta(Y_t, t) \frac{\partial \psi}{\partial y}(Y_t, t) dW_t, \quad (2.8)$$

and, differentiating the constraint  $G(\psi(y, t), y, t) \equiv 1$ , we obtain

$$\frac{\partial \psi}{\partial y} = - \frac{\partial G / \partial y}{\partial G / \partial x},$$

where  $\partial G / \partial x \neq 0$  by strict monotonicity. Thus the asset price process  $X_t$  satisfies (under the effects of feedback) the stochastic differential equation

$$dX_t = \alpha(X_t, Y_t, t) dt + v(X_t, Y_t, t) \eta(Y_t, t) dW_t, \quad (2.9)$$

where the adjusted volatility is

$$v(X_t, Y_t, t) = - \frac{D_y(X_t, Y_t, t) + \rho \Phi_y(X_t, t)}{D_x(X_t, Y_t, t) + \rho \Phi_x(X_t, t)}, \quad (2.10)$$

which, we note, is independent of the scaling by the supply.

The adjusted drift  $\alpha(X_t, Y_t, t)$  is

$$\alpha = - \left\{ \mu \frac{G_y}{G_x} + \frac{G_t}{G_x} + \frac{1}{2} \eta^2 \left[ \frac{G_{yy}}{G_x} - 2 \frac{G_{xy} G_y}{G_x^2} + \frac{G_y^2 G_{xx}}{G_x^3} \right] \right\}. \quad (2.11)$$

## 2.4 Modified Black-Scholes under Feedback

Next we examine how the modified volatility (which is now a function of  $\Phi$ ) effects the derivation of the Black-Scholes equation for  $P_t$  from the point of view of the program traders. We shall follow the Black-Scholes derivation of [8] with the crucial difference that the price of the underlying is driven not by (2.1), but by (2.9) which depends on a second Itô process  $Y_t$ . The calculations are similar to those for stochastic volatility models (see [14] and [8, Chapter 8]) in that such models typically start with a stochastic differential equation like (2.9) for the underlying asset price whose volatility is driven by an exogenous stochastic process given by an expression like (2.5), where  $Y_t$  would represent a non-traded source of risk.

Since the derivation will tell us how much of the asset the program traders should buy or sell to cover the risk of the derivative, we shall be able to obtain an expression for  $\Phi$  in terms of the



price of the derivative. Generalizing the usual argument, we suppose the price of one unit of the derivative security is given by  $P_t = C(X_t, t)$  for some sufficiently smooth function  $C(x, t)$ . Then we construct a self-financing replicating strategy  $(a_t, b_t)$  in the underlying asset and the riskless bond:

$$a_T X_T + b_T \beta_T = P_T,$$

with

$$a_t X_t + b_t \beta_t = a_0 X_0 + b_0 \beta_0 + \int_0^t a_s dX_s + \int_0^t b_s d\beta_s, \quad 0 \leq t \leq T, \quad (2.12)$$

and  $a_t = \Phi(X_t, t)$  because it is exactly the amount of the underlying asset that the trader must hold to insure against the risk of the derivative security, and it is therefore the time  $t$  demand for the asset per derivative security being hedged. Excluding the possibility of arbitrage opportunities gives

$$a_t X_t + b_t \beta_t = P_t, \quad 0 \leq t \leq T, \quad (2.13)$$

and by Itô's lemma and (2.12),

$$\begin{aligned} dP_t &= a_t dX_t + b_t d\beta_t \\ &= [a_t \alpha(X_t, Y_t, t) + b_t r \beta_t] dt + a_t v(X_t, Y_t, t) \eta(Y_t, t) dW_t. \end{aligned}$$

Comparing this expression with

$$dP_t = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial x} dX + \frac{1}{2} \eta^2 v^2 \frac{\partial^2 C}{\partial x^2} dt,$$

and equating coefficients of  $dW_t$  gives

$$a_t = \Phi(X_t, t) = \frac{\partial C}{\partial x}. \quad (2.14)$$

From (2.13),

$$b_t = \frac{P_t - a_t X_t}{\beta_t}, \quad (2.15)$$

and matching the  $dt$  terms in the two expressions for  $dP_t$  and using (2.14) and (2.15) we find

$$\frac{\partial C}{\partial t} + \alpha \frac{\partial C}{\partial x} + \frac{1}{2} \eta^2 v^2 \frac{\partial^2 C}{\partial x^2} = \alpha \Phi + r(C - x\Phi). \quad (2.16)$$

We now exploit the fact that the volatility term  $v$  comes from feedback from hedging strategies.

From (2.10), the adjusted volatility for  $X_t$  is a function of  $\Phi$  and its derivatives, so we can write

$$v(X_t, Y_t, t) = H \left( \frac{\partial \Phi}{\partial x}(X_t, t), \Phi(X_t, t), X_t, Y_t, t \right),$$

for some function  $H$ . Then from (2.14) and (2.16), the function  $C(x, t)$  must satisfy the nonlinear partial differential equation

$$\frac{\partial C}{\partial t} + \frac{1}{2} \eta^2 H^2 \left( \frac{\partial^2 C}{\partial x^2}, \frac{\partial C}{\partial x}, x, y, t \right) \frac{\partial^2 C}{\partial x^2} + r \left( x \frac{\partial C}{\partial x} - C \right) = 0, \quad (2.17)$$

for  $x, y > 0$ , and  $0 \leq t < T$  with

$$\begin{aligned} C(x, T) &= h(x), \\ \Phi(x, T) &= h'(x), \\ C(0, t) &= 0, \\ \Phi(0, t) &= 0. \end{aligned}$$

The dependence of  $H$  and  $\eta$  on  $y$  can be removed by inverting (2.6) to get a relation of the form  $y = \hat{\psi}(x, \rho\Phi(x, t), t)$ . We note that as a global partial differential equations problem, there is no guarantee that the equation has a solution for a given demand function. Such situations need to be tackled on a case-by-case basis for a given demand function. However, we shall study these equations in a *vicinity of the Black-Scholes equation* (2.2), and consider their global characteristics at a later stage.

We can rewrite the equations in terms of the relative demand function  $G = D + \rho\Phi$ . Since  $H = -G_y/G_x$ , we have

$$\frac{\partial C}{\partial t} + \frac{1}{2}\eta^2 \left( \frac{D_y}{D_x + \rho C_{xx}} \right)^2 \frac{\partial^2 C}{\partial x^2} + r \left( x \frac{\partial C}{\partial x} - C \right) = 0. \quad (2.18)$$

This equation is also given by Schönbucher and Wilmott [24] when  $Y_t$  is a standard Brownian motion. However, in this general form, the model depends on modelling of the unobservable process  $Y_t$  because of the  $\eta(y, t)$  factor. The final step in the modelling is to enforce consistency with the Black-Scholes model as  $\rho \rightarrow 0$  and, as we shall see in the next subsection, this imposes an important restriction on the demand function.

Another approach is that of Frey [10]<sup>5</sup> who examines the feedback effect of the option replicating strategy of a large trader who prices and hedges according to the value of a “fundamental state variable”, analogous to  $Y_t$  in the present analysis. Thus Frey performs similar calculations to those of this section using the portfolio  $\Pi$  evolving according to  $d\Pi = dC(Y_t, t) - \rho\Phi(Y_t, t)dX_t$ , and proceeds to eliminate  $X_t$  by the market-clearing condition. He obtains a quasilinear PDE for  $\Phi(y, t)$  (the hedging strategy) depending on observed values of the process  $Y_t$ , and proves existence and uniqueness of a solution.

In this paper, we view the feedback-perturbed process  $X_t$  as the observable (since feedback is occurring), and construct the model to reduce to Black-Scholes depending on the unperturbed  $X_t$  when there is no program trading. This is an essential step in our approach.

## 2.5 Consistency and Reduction to Black-Scholes

We now complete the model so that it will reduce to the generalized or classical Black-Scholes models in the absence of program traders. In this case, the no-arbitrage pricing argument starting with the function  $D(x, y, t)$  and the income process  $Y_t$  is carried out from the point of view of an arbitrary investor in the derivative security rather than a program trader, with the following modifications:  $\rho\Phi = 0$  in Section (2.3) so that (2.10) becomes

$$v_0(x, y, t) = -\frac{D_y(x, y, t)}{D_x(x, y, t)};$$

---

<sup>5</sup>This was pointed out to us by a referee.

then everything in Section (2.4) up to (2.16) goes through as before, omitting the identification  $a_t = \Phi$  after (2.12) and in (2.14). Equation (2.16) now becomes

$$C_t + \frac{1}{2}\eta^2(y, t)v_0^2(x, y, t)C_{xx} + r(xC_x - C) = 0. \quad (2.19)$$

We shall refer to (2.19) as the limiting partial differential equation of (2.18) when there are no program traders, and look for conditions on the reference traders' demand function  $D$  such that this reduces to the classical or generalized Black-Scholes equation.

### 2.5.1 Classical Black-Scholes under Feedback

We begin with the special case that reduces to the classical Black-Scholes equation (2.4) with volatility parameter  $\sigma$  to illustrate the method; the generalized version is a simple extension. We also suppose  $D$  does not explicitly depend on  $t$ , to simplify notation, and that reference traders have the rational characteristics  $D_x < 0$  (their demand for the asset decreases when its price rises) and  $D_y > 0$  (their demand increases with their income.)

**Proposition 1** *If the incomes process  $Y_t$  is a geometric Brownian motion with volatility parameter  $\eta_1$ , then the feedback model reduces to the classical Black-Scholes model in the absence of program traders if and only if the demand function is of the form*

$$D(x, y) = U(y^\gamma/x), \quad (2.20)$$

for some smooth increasing function  $U(\cdot)$ , where  $\gamma = \sigma/\eta_1$ .

*Proof.* By assumption,  $Y_t$  satisfies

$$dY_t = \mu_1 Y_t dt + \eta_1 Y_t dW_t, \quad (2.21)$$

for constants  $\mu_1$  and  $\eta_1$ . Then (2.19) reduces to (2.4) if and only if

$$\frac{1}{2}\eta_1^2 y^2 \left[ \frac{D_y(x, y)}{D_x(x, y)} \right]^2 = \frac{1}{2}\sigma^2 x^2.$$

Thus  $D$  must satisfy  $D_y/D_x = -\gamma x/y$ , where we have taken the negative square root because the left-hand side is negative under our hypotheses for rational trading. The general solution of this partial differential equation is  $D(x, y) = U(y^\gamma/x)$  for any differentiable function  $U(\cdot)$ . Finally, by direct differentiation,  $D_x < 0$  and  $D_y > 0$  for  $x, y > 0$  if and only if  $U'(\cdot) > 0$ .

Now, given a demand function of this form, we derive the pricing equation under feedback. The diffusion coefficient is

$$\frac{1}{2}\eta_1^2 y^2 \left[ \frac{D_y(x, y)}{D_x(x, y) + \rho\Phi_x(x, t)} \right]^2 = \frac{1}{2}\eta_1^2 y^2 \gamma^2 \left[ \frac{(y^{\gamma-1}/x)U'(y^\gamma/x)}{(y^\gamma/x^2)U'(y^\gamma/x) - \rho\Phi_x} \right]^2.$$

We can use the market-clearing equation  $U(Y_t^\gamma/X_t) = 1 - \rho\Phi(X_t, t)$  from (2.7) to eliminate  $y$ . Let  $V(\cdot)$  be the inverse function of  $U(\cdot)$ , whose existence is guaranteed by the strict monotonicity of  $U$ . Then substituting  $y^\gamma/x = V(1 - \rho\Phi)$  and also using  $\eta_1\gamma = \sigma$ , the diffusion coefficient becomes

$$\frac{1}{2}\sigma^2 x^2 \left[ \frac{V(1 - \rho\Phi)U'(V(1 - \rho\Phi))}{V(1 - \rho\Phi)U'(V(1 - \rho\Phi)) - \rho x\Phi_x} \right]^2, \quad (2.22)$$

and we obtain a family of nonlinear feedback pricing equations that are consistent with and, in the absence of program trading reduce to, the classical Black-Scholes equation:

$$C_t + \frac{1}{2} \left[ \frac{V(1 - \rho C_x) U'(V(1 - \rho C_x))}{V(1 - \rho C_x) U'(V(1 - \rho C_x)) - \rho x C_{xx}} \right]^2 \sigma^2 x^2 C_{xx} + r(x C_x - C) = 0. \quad (2.23)$$

We note that these are *independent of the parameters in the incomes process*  $Y_t$  and depend only on the function  $U$  and  $\sigma$ , the observable market volatility of the underlying asset.

Setting  $\rho = 0$  in (2.23) immediately recovers the classical Black-Scholes partial differential equation (2.4). Since  $\rho$  is roughly the fraction of the asset market held by the program traders, it is likely that it is a small number in practice. Thus as long as  $\Phi = C_x$  and  $C_{xx}$  remain bounded in magnitude by a reasonable constant, we can study (2.23) as a small perturbation of (2.4).

Schönbucher [22] also realises the need for consistency with (classical) Black-Scholes. He studied a linear (in  $x$  and  $y$ ) demand function, which is not admissible in our formulation, and forced consistency by allowing the drift and diffusion coefficients of the process  $Y_t$  in (2.5) to depend on  $X_t$ .

In the bulk of this paper, we shall study the particular model arising from taking  $U$  as linear:  $U(z) = \beta z$ ,  $\beta > 0$ , to infer qualitative properties of feedback from program trading. The equation becomes

$$\frac{\partial C}{\partial t} + \frac{1}{2} \left[ \frac{(1 - \rho \frac{\partial C}{\partial x})}{1 - \rho \frac{\partial C}{\partial x} - \rho x \frac{\partial^2 C}{\partial x^2}} \right]^2 \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + r \left( x \frac{\partial C}{\partial x} - C \right) = 0, \quad (2.24)$$

which is independent of  $\beta$ .

If we suppose that  $Y_t$  is the generalized Itô process (2.5) then (2.19) reduces to the generalized Black-Scholes equation (2.2) with volatility function  $\lambda(x, t)$  in the absence of program traders if and only if  $D$  is of the form  $D(x, y, t) = \tilde{U}(Q(y, t) - L(x, t), t)$  for some smooth function  $\tilde{U}$  that is strictly increasing in its first argument, where

$$Q(y, t) = \int^y \frac{dv}{\eta(v, t)}, \quad L(x, t) = \int^x \frac{dv}{\lambda(v, t)}.$$

This can be shown easily as before, this time by finding the general solution of

$$\frac{D_y}{D_x} = -\frac{\lambda(x, t)}{\eta(y, t)}.$$

If  $\tilde{V}(\cdot, t)$  denotes the inverse function of  $\tilde{U}(\cdot, t)$ , and  $\tilde{U}_1$  its derivative with respect to the first argument, then, by straightforward calculations, we find that the feedback pricing equation that reduces to the generalized Black-Scholes model when there are no program traders is

$$C_t + \frac{1}{2} \left[ \frac{\tilde{U}_1(\tilde{V}(1 - \rho C_x, t), t)}{\tilde{U}_1(\tilde{V}(1 - \rho C_x, t), t) - \rho \lambda(x, t) C_{xx}} \right]^2 \lambda(x, t)^2 C_{xx} + r(x C_x - C) = 0. \quad (2.25)$$

We note that this equation is independent of  $\mu(y, t)$  and  $\eta(y, t)$  in (2.5) and, as in the generalized Black-Scholes model, it is also independent of  $\nu(x, t)$  in (2.1).

## 2.6 Multiple Derivative Securities

We now consider the case where the program traders create the aggregate demand function  $\phi(x, t)$  as a result of hedging strategies for  $n$  different derivative securities with expiration dates  $T_i$ , and payoff functions  $h_i(x)$ ,  $i = 1, 2, \dots, n$ . Let  $\Phi_i(x, t)$  be the demand for the asset resulting from all the program traders' hedging strategies for insuring  $\zeta_i$  units of the  $i^{\text{th}}$  such option, so that

$$\phi(x, t) = \sum_{i=1}^n \zeta_i \Phi_i(x, t).$$

We assume the demand function of the reference traders is of the form  $U(y^\gamma/x)$  for consistency with classical Black-Scholes, and that  $U$  is linear to obtain an extension of (2.24). Then, if  $C_i(x, t)$  is the price of the  $i^{\text{th}}$  option, following a Black-Scholes argument for each option, we find

$$\Phi_i = \frac{\partial C_i}{\partial x}, \quad (2.26)$$

and

$$\begin{aligned} \frac{\partial C_i}{\partial t} + \frac{1}{2} \left[ \frac{\left(1 - S_0^{-1} \frac{\partial C^*}{\partial x}\right)}{1 - S_0^{-1} \frac{\partial C^*}{\partial x} - S_0^{-1} x \frac{\partial^2 C^*}{\partial x^2}} \right]^2 \sigma^2 x^2 \frac{\partial^2 C_i}{\partial x^2} + r \left( x \frac{\partial C_i}{\partial x} - C_i \right) &= 0, \quad t \leq T_i, \\ C_i(x, T_i) &= h_i(x), \\ C_i &= 0, \quad t > T_i, \end{aligned} \quad (2.27)$$

for each  $i = 1, 2, \dots, n$ , where<sup>6</sup>

$$C^*(x, t) = \sum_{j=1}^n \zeta_j C_j(x, t). \quad (2.28)$$

This is a system of  $n$  coupled nonlinear partial differential equations for the functions  $C_1(x, t), \dots, C_n(x, t)$ . The procedure can easily be applied to the case of a general demand function and  $Y_t$  a general Itô process to obtain a similar system of  $n$  equations for the functions  $C_1(x, t), \dots, C_n(x, t)$ , analogous to (2.18). The constants  $\zeta_i$  could be generalized to depend on time, or be random processes.

## 2.7 European Options Pricing and Smoothing Requirements

We focus on the problem of feedback caused by portfolio insurance against the risk of writing one particular type of European call option which gives the holder the right, but not the obligation, to buy the underlying asset at the strike price  $K$  at the expiration date  $T$ . The terminal payoff function is  $h(x) = (x - K)^+$ , and this was the original derivative security priced by Black and Scholes in [4], for which they obtained a closed-form solution known as the Black-Scholes formula:

$$C_{BS}(x, t) := xN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (2.29)$$

where

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (2.30)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \quad (2.31)$$

---

<sup>6</sup>We could also normalize the constants  $\zeta_i$  by  $\zeta := \sum_{j=1}^n \zeta_j$  and define  $\rho := \zeta/S_0$  as in the scalar case, but we shall keep the notation to a minimum.

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-s^2/2} ds.$$

As a terminal condition for the model in (2.24) that we shall study, this  $h(x)$  brings up the problem of the denominator of the diffusion coefficient,

$$\left[ \frac{(1 - \rho \frac{\partial C}{\partial x})}{1 - \rho \frac{\partial C}{\partial x} - \rho x \frac{\partial^2 C}{\partial x^2}} \right]$$

becoming zero. This is because<sup>7</sup>  $h''(x) = \delta(x - K)$ , so that at  $t = T$ , no matter how small  $\rho$  is, the denominator is negative in some neighborhood of  $K$ . Since we would expect the diffusion equation to smooth the terminal data as the equation is run backwards in time from  $T$ , the denominator will go through zero as  $C_x$  and  $C_{xx}$  become smaller, causing the equation to become meaningless.

To avoid this situation, which arises solely because of the kink in the option's payoff function, we impose a second consistency condition with the Black-Scholes model as the strike time is approached. That is, we ignore the feedback effects predicted by this theory as  $t \rightarrow T$  because of the oversensitivity of the Black-Scholes dynamic hedging strategies to price fluctuations around  $x = K$ , which is reflected by the fact that  $C_x^{BS}(x, t) \sim \mathcal{H}(x - K)$  and  $C_{xx}^{BS}(x, t) \sim \delta(x - K)$ , as  $t \rightarrow T$ . In practice, frenetic program trading close to expiration is tempered by transaction costs, which can be regarded as a natural smoother.

Technically, this means that we set the feedback price  $C$  equal to the Black-Scholes price  $C_{BS}$  in some small interval  $T - \varepsilon \leq t \leq T$ , and then run the full equation backwards from  $T - \varepsilon$  instead of  $T$ . It turns out that we can calculate  $\varepsilon$  in terms of  $\rho$  and  $\sigma$  to obtain sufficient smoothing of the data for well-posedness of the nonlinear partial differential equation. The smoothing parameter  $\varepsilon$  specified in this way, then completes our feedback pricing model.

Schönbucher and Wilmott [24, 23] analyse jumps in the asset price that can be induced by program traders following the hedging strategy from the Black-Scholes formula as  $t \rightarrow T$ . They also pose a nonlinear PDE from a linear (in  $x$  and  $y$ ) demand function with moving boundaries arising from the jump conditions in the case of the hedging strategy produced endogenously by the feedback theory.

### 2.7.1 The Smoothing Parameter

We define the smoothing parameter  $\varepsilon$  to be the minimum value of  $\hat{\varepsilon} > 0$  such that

$$\min_{x > 0} F_{BS}(x, T - \hat{\varepsilon}) = 0,$$

where

$$F_{BS}(x, t) := 1 - \rho \frac{\partial C_{BS}}{\partial x}(x, t) - \rho x \frac{\partial^2 C_{BS}}{\partial x^2}(x, t).$$

Substituting the Black-Scholes formula (2.29) and calculating the minimum for each fixed  $\hat{\varepsilon}$ , yields the following nonlinear algebraic equation satisfied by  $\varepsilon$ :

$$N(\sigma\sqrt{\varepsilon}) + \frac{e^{-\frac{1}{2}\sigma^2\varepsilon}}{\sigma\sqrt{2\pi\varepsilon}} = \frac{1}{\rho}.$$

---

<sup>7</sup>Here,  $\mathcal{H}(z)$  denotes the Heaviside function ( $\mathcal{H}(z) = 0$  if  $z < 0$  and  $\mathcal{H}(z) = 1$  if  $z > 0$ ), and  $\delta(z)$  is its derivative, the Delta function.

This can be solved numerically for given  $\rho$  and  $\sigma$ , and a plot of  $\sigma^2\varepsilon$  against  $\rho$  is shown in Figure (2.1).

We note that  $\varepsilon$  is independent of  $K$  and  $T$  (as well as  $r$ ), so that the smoothing does not depend

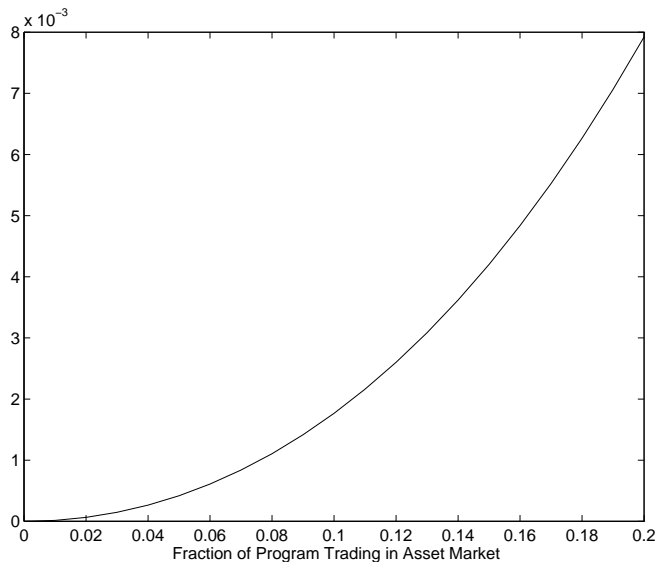


Figure 2.1: Dimensionless Smoothing Parameter  $\sigma^2\varepsilon$  as a function of  $\rho$

on the specifics of the options contract. Indeed, the same  $\varepsilon$  can be used in the feedback model for put option pricing (as can be seen from put-call parity for the Black-Scholes model), and also for the system of equations in (2.27) when there is a uniform distribution of different call options with the same strike time, but different strike prices (*ie.*  $\zeta_i \equiv \zeta$ .)

Furthermore, for small  $\rho$ ,  $\varepsilon \sim \text{const.}\rho^2$ , which indicates that when program trading is small, smoothing is a minor modification to our model away from the strike time. Our numerical experiments indicate that feedback prices away from  $T$  are quite insensitive to the choice of  $\varepsilon$  (even more so the further  $t$  is from  $T$ ).

### 2.7.2 Smoothing by Distribution of Strike Prices

In practice, the specifics of options contracts being hedged by other program traders might not be known to each bank that is trading to insure its portfolio. It could, therefore, be estimated by a smooth distribution function over a range of strike prices.

Let us suppose there are  $M$  strike times  $T_1 < T_2 < \dots < T_M$  (typically options expire in three-monthly cycles, so it is natural to model these dates discretely). Then, if  $\zeta_i(K)$ ,  $i = 1, \dots, M$ , are smooth density functions describing the distribution of strike prices for options maturing at  $T_i$ , the price  $C_i(x, t; K)$  of the option expiring at  $T_i$  with strike price  $K$  will satisfy (2.27) with  $C_i(x, T_i; K) = (x - K)^+$ ,  $C_i(0, t; K) = Ke^{-r(T_i-t)}$  and  $C^*(x, t) := \sum_{i=1}^M \mathcal{H}(T_i - t)W_i(x, t)$ , where

$$W_i(x, t) := \int_0^\infty C_i(x, t; K)\zeta_i(K)dK.$$

The denominator of the diffusion coefficient is now controlled because  $C^*(x, t)$  satisfies

$$\frac{\partial C^*}{\partial t} + \frac{1}{2} \left[ \frac{\left(1 - S_0^{-1} \frac{\partial C^*}{\partial x}\right)}{1 - S_0^{-1} \frac{\partial C^*}{\partial x} - S_0^{-1} x \frac{\partial^2 C^*}{\partial x^2}} \right]^2 \sigma^2 x^2 \frac{\partial^2 C^*}{\partial x^2} + r \left( x \frac{\partial C^*}{\partial x} - C^* \right) = 0,$$

in each interval  $T_{i-1} < t < T_i$ , with the terminal conditions

$$C^*(x, T_i) = \sum_{p=i+1}^M W_p(x, T_i) + \int_0^\infty \zeta_i(K) (x - K)^+ dK,$$

for  $i = M, M-1, \dots, 1$  and  $T_0 := 0$ . The boundary condition at any  $t < T_M$  is clearly

$$C^*(0, t) = \sum_{i=1}^M \mathcal{H}(T_i - t) e^{-r(T_i - t)} \int_0^\infty K \zeta_i(K) dK.$$

So, in the ‘top’ interval  $(T_{M-1}, T_M)$ , the equation for  $C^*(x, t)$  is well-posed (in the sense of the denominator remaining positive) provided that

$$1 - S_0^{-1} C_x^*(x, T_M) - S_0^{-1} x C_{xx}^*(x, T_M) = 1 - S_0^{-1} \left[ \int_0^x \zeta_M(K) dK - x \zeta_M(x) \right] > 0,$$

for all  $x > 0$ . This is a mild requirement on the distribution function  $\zeta_M(\cdot)$ . With this established, the equation for  $C_M(x, t; K)$  is well-posed in  $(T_{M-1}, T_M)$ , and we can proceed similarly to establish well-posedness for  $C^*(x, t)$ , and hence for  $C_i(x, t; K)$ , in each  $(T_{i-1}, T_i)$  under a regularity condition on  $\zeta_i(\cdot)$ .

### 2.7.3 The Full Model

The following equations summarize the feedback pricing model for a European call option that we shall study in detail in the next sections:

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2} \left[ \frac{\left(1 - \rho \frac{\partial C}{\partial x}\right)}{1 - \rho \frac{\partial C}{\partial x} - \rho x \frac{\partial^2 C}{\partial x^2}} \right]^2 \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + r \left( x \frac{\partial C}{\partial x} - C \right) &= 0, \quad t < T - \varepsilon, \\ C(x, T - \varepsilon) &= C_{BS}(x, T - \varepsilon), \\ C(0, t) &= 0, \\ \lim_{x \rightarrow \infty} |C(x, t) - (x - K e^{-r(T-t)})| &= 0, \end{aligned} \quad (2.32)$$

with  $C(x, t) = C_{BS}(x, t)$  for  $T - \varepsilon \leq t \leq T$ .

That is, we study in detail, in sections 3, 4 and 5, the model from the demand function  $D(x, y) = U(y^\gamma/x)$  with  $U(z) = \beta z$ .

## 3 Asymptotic Results for small $\rho$

In this section, we obtain results that are valid as  $\rho$ , the proportion of the total volume of the asset traded by the program traders, tends to zero. This means that we assume  $\rho$  is small enough that (2.32) can be considered a small perturbation to the classical Black-Scholes partial differential equation (2.4). Rubinstein [21] assesses alternative pricing formulas by calculating their implied Black-Scholes volatilities for various strike prices and times-to-maturity and comparing with observed historical implied Black-Scholes volatilities. We present the feedback effects from our model both directly in terms of prices and as measured in this way.



### 3.1 Regular Perturbation Series Solution

We calculate the first-order correction to the Black-Scholes pricing formula for a European option under the effects of feedback when  $\rho \ll 1$ . The full problem for  $C(x, t)$ , the price of the option when the underlying stock price is  $x > 0$  at time  $t < T$ , is given at the end of section (2.7.3).

The  $\rho = 0$  solution is the Black-Scholes formula (2.29). Then constructing a regular perturbation series

$$C(x, t) = C_{BS}(x, t) + \rho \bar{C}(x, t) + \mathcal{O}(\rho^2), \quad (3.1)$$

and defining

$$L_{BS}C := C_t + \frac{1}{2}\sigma^2 x^2 C_{xx} + r(xC_x - C), \quad (3.2)$$

we expand (2.24) for small  $\rho$  to obtain

$$L_{BS}C + \rho\sigma^2 x^3 C_{xx}^2 = \mathcal{O}(\rho^2), \quad (3.3)$$

so that, substituting from (3.1) and equating terms of magnitude  $\mathcal{O}(\rho)$ ,  $\bar{C}(x, t)$  satisfies

$$L_{BS}\bar{C} = -\sigma^2 x^3 \left( \frac{\partial^2 C_{BS}}{\partial x^2} \right)^2.$$

Differentiating the expression (2.29) for  $C_{BS}$  gives

$$\begin{aligned} L_{BS}\bar{C} &= -\frac{x e^{-d_1^2}}{2\pi(T-t)}, \quad t < T - \varepsilon \\ \bar{C}(x, T - \varepsilon) &= 0, \\ \bar{C}(0, t) &= 0, \\ \lim_{x \rightarrow \infty} |\bar{C}(x, t)| &= 0. \end{aligned}$$

We next transform the problem for  $\bar{C}$  to an inhomogeneous heat equation. Under the transformation

$$x = K e^y, \quad (3.4)$$

$$t = T - \frac{2\tau}{\sigma^2}, \quad (3.5)$$

$$\bar{C}(x, t) = K e^{(-\frac{1}{2}(k-1)y - \frac{1}{4}(k+1)^2\tau)} u(y, \tau), \quad (3.6)$$

where  $k = 2r/\sigma^2$ , we obtain the following problem for  $u(y, \tau)$  in  $-\infty < y < \infty$ ,  $\tau > \varepsilon\sigma^2/2$

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial y^2} &= \frac{1}{2\pi\tau} \exp \left[ -\frac{y^2}{2\tau} - \frac{1}{4}(k+1)^2\tau - \frac{y}{2}(k+1) \right], \\ u(y, \frac{\varepsilon}{2}\sigma^2) &= 0, \\ e^{-\frac{1}{2}(k-1)y} u(y, \tau) &\rightarrow 0 \text{ as } y \rightarrow -\infty, \end{aligned} \quad (3.7)$$

and  $u$  is bounded as  $y \rightarrow +\infty$ . (See Wilmott *et al* [27] for further details.)

If the right-hand side of (3.7) is denoted by  $f(y, \tau)$ , the solution can be expressed

$$u(y, \tau) = \int_{\frac{\varepsilon}{2}\sigma^2}^{\tau} \int_{-\infty}^{\infty} B(\xi, s; y, \tau) f(\xi, s) d\xi ds,$$

where

$$B(\xi, s; y, \tau) = \frac{1}{\sqrt{4\pi(\tau-s)}} \exp\left(-\frac{(\xi-y)^2}{4(\tau-s)}\right).$$

Therefore,

$$\begin{aligned} u(y, \tau) &= \int_{\frac{\xi}{2}\sigma^2}^{\tau} \int_{-\infty}^{\infty} \frac{1}{2\pi s \sqrt{4\pi(\tau-s)}} \exp\left[-\frac{(\xi-y)^2}{4(\tau-s)} - \frac{\xi^2}{2s} - \frac{1}{4}(k+1)^2 s - \frac{\xi}{2}(k+1)\right] d\xi ds \\ &= \int_{\frac{\xi}{2}\sigma^2}^{\tau} \frac{e^{-\frac{1}{4}(k+1)^2 s - \frac{y^2}{4(\tau-s)}}}{2\pi s \sqrt{4\pi(\tau-s)}} \int_{-\infty}^{\infty} e^{-\alpha\xi^2 - \beta\xi} d\xi ds, \end{aligned}$$

where  $\alpha = \frac{1}{2s} + \frac{1}{4(\tau-s)}$  and  $\beta = \frac{-y}{2(\tau-s)} + \frac{1}{2}(k+1)$ . Evaluating the inner integral as  $\sqrt{\pi/\alpha} \exp(\beta^2/4\alpha)$ , and using

$$\frac{\beta^2}{4\alpha} = \frac{s[y - (k+1)(\tau-s)]^2}{4(2\tau-s)(\tau-s)},$$

we obtain the solution

$$u(y, \tau) = \frac{1}{2\pi} \int_{\frac{\xi}{2}\sigma^2}^{\tau} \frac{\exp\left[-\frac{1}{4}(k+1)^2 s - \frac{y^2}{4(\tau-s)} + \frac{s[y - (k+1)(\tau-s)]^2}{4(2\tau-s)(\tau-s)}\right]}{\sqrt{2\tau s - s^2}} ds. \quad (3.8)$$

Now to remove the  $s = 0$  singularity which causes the integrand to become large near the lower limit, and make the integral amenable to quadrature methods, we make the transformation  $v = \sqrt{s/2\tau}$  to give

$$u(y, \tau) = \int_{\sigma\sqrt{\varepsilon/4\tau}}^{\sqrt{1/2}} M(y, \tau, v) dv, \quad (3.9)$$

where

$$M(y, \tau, v) = \frac{1}{\pi\sqrt{1-v^2}} \exp\left[-\frac{1}{2}(k+1)^2 \tau v^2 - \frac{y^2}{4\tau(1-2v^2)} + \frac{v^2[y - \tau(k+1)(1-2v^2)]^2}{4\tau(1-v^2)(1-2v^2)}\right],$$

which has a well-defined limit as  $1 - 2v^2 \rightarrow 0$

$$\lim_{v \rightarrow \sqrt{1/2}} M(y, \tau, v) = \frac{\sqrt{2}}{\pi} \exp\left[-\frac{1}{4}(k+1)^2 \tau - \frac{y^2}{2\tau} - \frac{1}{2}y(k+1)\right].$$

Clearly,  $M > 0$  in the interval of integration, and so the first-order correction,  $\bar{C}$  given by (3.6) and (3.9), is positive in  $x > 0$ ,  $t < T$ . The perturbation of the idealized Black-Scholes reference model due to the presence of the program traders therefore has the effect of increasing the no-arbitrage price of the European option. Since the Black-Scholes formula (2.29) is an increasing function of the volatility parameter  $\sigma$ , ( $\partial C_{BS}/\partial \sigma = x e^{-d_1^2/2} \sqrt{(T-t)/2\pi} > 0$ ), this result gives us initial confirmation that program traders cause market volatility to increase, and moreover, from the form of the perturbation series constructed (3.1), it is linearly increasing in the parameter  $\rho$ . A more direct indication of feedback volatility can be calculated from (2.22). This tells us that, according to the program traders, the underlying asset price under feedback follows the random walk described by (2.9), where the diffusion term in this case is

$$\frac{\sigma x (1 - \rho C_x)}{1 - \rho C_x - \rho x C_{xx}} = \sigma x \left[1 + \rho x \frac{\partial^2 C_{BS}}{\partial x^2}\right] + \mathcal{O}(\rho^2).$$

Since  $\partial^2 C_{BS}/\partial x^2 = e^{-d_1^2/2}/x\sigma\sqrt{2\pi(T-t)} > 0$ , feedback market ‘spot’ volatility is always greater than that used in the reference Black-Scholes model. See also the comments at the end of section 4.

Throughout this section, we shall illustrate results with the example of a six-month European option ( $T = 0.5$  years) with a strike price  $K = 5$  when the constant spot interest rate is  $r = 0.04$  and the reference volatility is  $\sigma = 0.4$ . Plots of  $C_{BS}(x, t)$ ,  $\bar{C}(x, t)$  and the perturbed prices for fixed  $x$  and for fixed  $t$  are given in Figure 3.1. In the latter, we take  $\rho = 0.05$  with the corresponding smoothing parameter  $\varepsilon = 0.003$ .

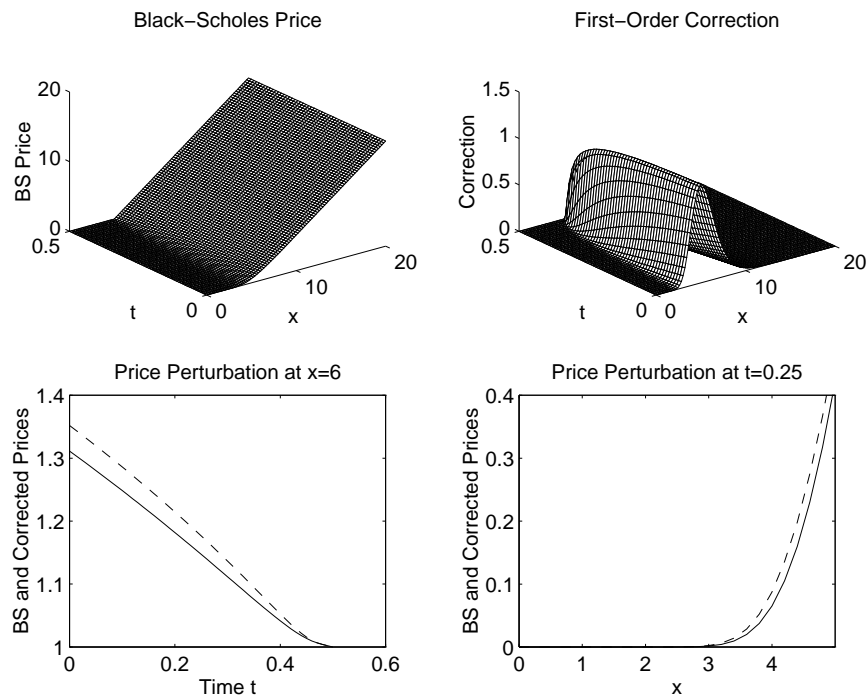


Figure 3.1: First-Order Perturbation:  $\rho = 0.05$ ,  $\sigma = 0.4$ ,  $K = 5$ ,  $T = 0.5$

In Figure 3.2, we show the perturbed dynamic hedging strategy

$$\phi(x, t) = \frac{\partial C_{BS}}{\partial x} + \rho \frac{\partial \bar{C}}{\partial x},$$

compared with the original Black-Scholes hedging strategy. Recall that  $\phi(x, t)$  tell us how much of the underlying asset the program traders should hold at time  $t$  to insure against the risk of the call option, and as the graphs show, this amount can increase or decrease under feedback perturbation, with effects most noticeable around the strike price  $x = K$ .

### 3.2 Least Squares Approximation by the Black-Scholes Formula

We now further exploit the view that (2.24) is a small perturbation to (2.4) by considering the situation where program traders wish to use the Black-Scholes formula to price a European call option, accounting for the feedback effects by using an adjusted volatility estimate. Effectively, we are estimating how the presence of the program traders changes Black-Scholes volatility; that

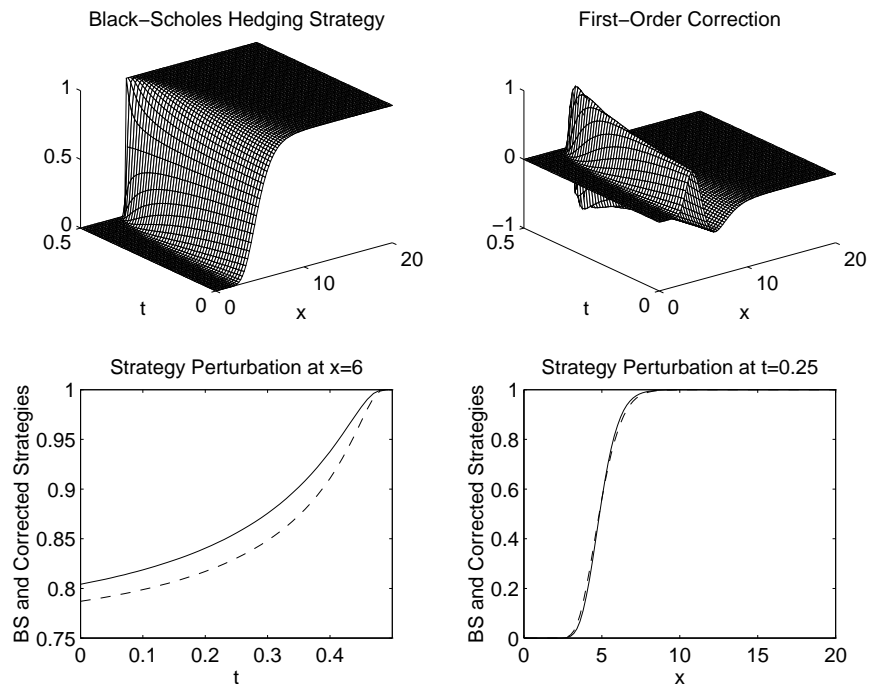


Figure 3.2: First-Order Perturbation to Hedging Strategy :  $\rho = 0.05$ ,  $\sigma = 0.4$ ,  $K = 5$ ,  $T = 0.5$

is, what volatility parameter should be used in (2.29) to best approximate the solution to (2.24)? Our motivation behind this is to relate feedback effects to a commonly quoted synoptic variable, namely implied volatility.

### 3.2.1 Adjusted Volatility as a function of time

We calculate the Black-Scholes volatility that should be used in (2.29) to minimize the mean-square approximation error in  $x$  at each time. Thus, if  $C(x, t; \sigma)$  is the feedback price satisfying (2.32), where  $\sigma$  is the reference traders' (given) constant estimate of the underlying asset's volatility, then at each fixed  $t$ , we calculate  $\hat{\sigma}^*(t)$  which solves

$$\min_{\hat{\sigma}(t)} \int_0^\infty [C(x, t; \sigma) - C_{BS}(x, t, \hat{\sigma}(t))]^2 dx. \quad (3.10)$$

Here  $C_{BS}(x, t; \hat{\sigma}(t))$  denotes the Black-Scholes formula evaluated at asset price  $x$ , time  $t$  with volatility parameter  $\hat{\sigma}(t)$ , and the minimization is over values of  $\hat{\sigma}(t)$  for which the integral is well-defined.

We have the regular perturbation solution (3.1), and we also linearize  $\hat{\sigma}(t) = \sigma + \rho\gamma(t) + \mathcal{O}(\rho^2)$ , assuming the adjusted volatility  $\hat{\sigma}^*(t)$  will be close to  $\sigma$  for small  $\rho$ . Then we find

$$C_{BS}(x, t, \hat{\sigma}(t)) = C_{BS}(x, t, \sigma) + \rho\gamma(t) R(x, t; \sigma) + \mathcal{O}(\rho^2),$$

where

$$R(x, t; \sigma) = \frac{\partial C_{BS}}{\partial \sigma}(x, t; \sigma) = \frac{x e^{-\frac{1}{2}d_1^2(\sigma)\sqrt{T-t}}}{\sqrt{2\pi}}. \quad (3.11)$$

The linearized minimization problem is now

$$\min_{\gamma(t)} \int_0^\infty [\bar{C}(x, t; \sigma) - \gamma(t) R(x, t; \sigma)]^2 dx.$$

The minimizing correction term is given by

$$\gamma^*(t) = \frac{\int_0^\infty \overline{C}(x, t; \sigma) R(x, t; \sigma) dx}{\int_0^\infty R(x, t; \sigma)^2 dx}, \quad (3.12)$$

and the adjusted volatility at time  $t$  is  $\hat{\sigma}^*(t) = \sigma + \rho\gamma^*(t) + \mathcal{O}(\rho^2)$ .

We next consider how  $\gamma^*(t)$  behaves as  $t \rightarrow T$  in order to obtain an indication of the validity of the asymptotic estimate to  $\hat{\sigma}^*(t)$ . The integral in the denominator is given by

$$\int_0^\infty R(x, t; \sigma)^2 dx = \frac{K^3 \sigma (T-t)^{3/2}}{2\sqrt{\pi}} \exp\left[-3(T-t)\left(r - \frac{1}{4}\sigma^2\right)\right],$$

which behaves like  $\text{const.}(T-t)^{3/2}$  as  $t \rightarrow T$ . We also find

$$\int_0^\infty \overline{C}(x, t; \sigma) R(x, t; \sigma) dx \sim D_0(T - \varepsilon - t),$$

for some constant  $D_0$ . Hence, as  $t \rightarrow T - \varepsilon$ ,

$$\gamma^*(t) \sim D_1(T - \varepsilon - t)^{-1/2},$$

for some constant  $D_1$ , which suggests the asymptotic estimate to the adjusted Black-Scholes volatility is valid as long as  $T - t > \mathcal{O}(\rho^2)$ , since  $\varepsilon = \mathcal{O}(\rho^2)$  when  $\rho \ll 1$ .

Finally we note that  $\overline{C}(x, t)$  and  $R(x, t)$  are positive, by inspection, so that  $\gamma^*(t) > 0$  for all  $t < T$  which implies that the first-order correction to the reference volatility is always positive: Black-Scholes volatility, calculated in this manner, always increases as a result of the presence of program traders in this  $\rho \ll 1$  setting.

In Figure 3.3,  $\sigma$  and  $\hat{\sigma}^*(t)$  are plotted for our standard example, using Simpson's Rule to evaluate the integral in the numerator of (3.12) numerically. With  $\rho = 0.05$ , base volatility is seen to increase by between 10 – 18% over time in the region of validity  $t \leq 0.37$ .

### 3.2.2 Adjusted Volatility as a function of Asset Price

Similarly we can find a perturbation to the base volatility  $\sigma$  which varies with the stock price:

$$\hat{\sigma}^*(x) = \sigma + \rho\gamma^*(x) + \mathcal{O}(\rho^2),$$

where, for each fixed  $x > 0$ ,  $\hat{\sigma}^*(x)$  solves

$$\min_{\hat{\sigma}(x)} \int_0^T [C(x, t; \sigma) - C_{BS}(x, t, \hat{\sigma}(x))]^2 dt. \quad (3.13)$$

The first-order correction  $\gamma^*(x)$  is given by

$$\gamma^*(x) = \frac{\int_0^T \overline{C}(x, t; \sigma) R(x, t; \sigma) dt}{\int_0^T R(x, t; \sigma)^2 dt}, \quad (3.14)$$

and, as for the time-dependent volatility correction, it is strictly positive. A plot for the example option appears in Figure 3.3, and volatility appears to increase by up to 12%.

We could similarly minimize the mean-square error in both  $x$  and  $t$  and look at the adjustment to volatility as a function of the strike price  $K$  or the reference volatility  $\sigma$ . It turns out that the adjusted volatility increases near-linearly with  $\sigma$ , and is relatively constant as  $K$  varies, provided that  $K$  is not near zero.

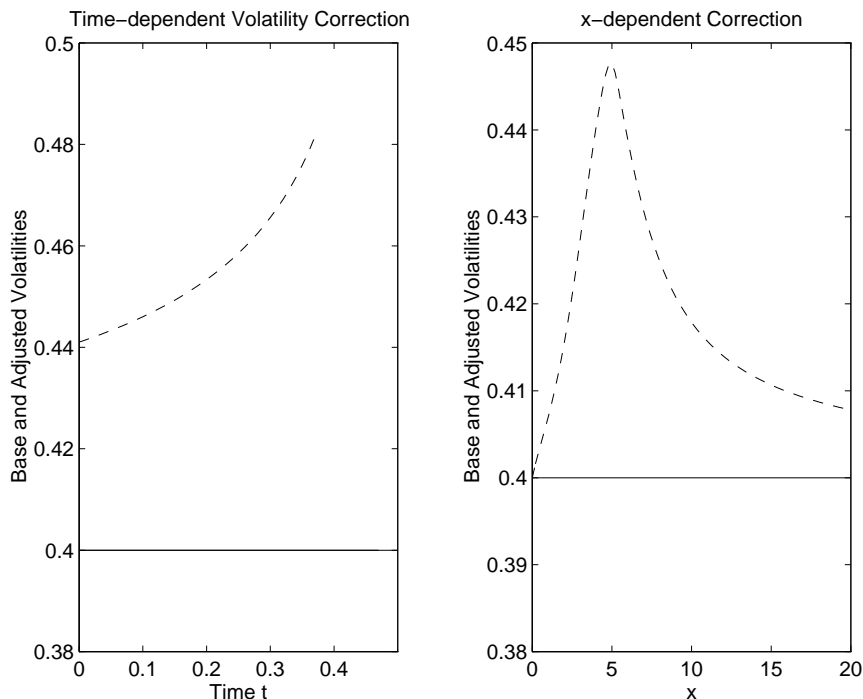


Figure 3.3: First-Order Corrected Black-Scholes Volatilities:  $\rho = 0.05$ ,  $\sigma = 0.4$ ,  $K = 5$ ,  $T = 0.5$

### 3.3 Extension to Multiple Options

Many authors (see [1, 13, 16, 21]) have attempted to construct pricing models that capture the observed non-constant variation of implied Black-Scholes volatility with strike price. That is, given observed prices for European options on the same underlying asset, but with different strike prices, inverting the Black-Scholes formula (2.29) for  $\sigma$ , and plotting the resulting implied volatilities against the strike prices reveals a non-constant graph. There has been some success with stochastic volatility models as proposed by Hull and White [14] for example, and reviewed in [8, Chapter 8]. An approach to stochastic volatility effects based on separation of time scales and asymptotic analysis is given in [26].

To address this issue, we return to the special case of section (2.6) in which the program traders are trading to insure against  $n$  call options with strike prices  $K_1 < K_2 < \dots < K_n$ . For simplicity, we assume they all have the same expiration date  $T$  and uniform frequency of occurrence. Then, the smoothing parameter  $\varepsilon$  is the same as for the scalar case. Their prices  $C_i(x, t)$  satisfy (2.27)-(2.28) in  $t < T - \varepsilon$ , with  $h_i(x) = (x - K_i)^+$ ,  $T_i \equiv T$  and  $\zeta_i \equiv \zeta/n$ , and the smoothing correction  $C_i(x, T - \varepsilon) = C_{BS}(x, T - \varepsilon; K_i)$ .

We now generalize the calculations of this section for the single option to obtain a least-squares adjusted Black-Scholes volatility for the feedback price of each option and thereby the variation of this implied volatility with the discretely distributed strike prices.

First we construct regular perturbation series

$$C_i(x, t) = C_{BS}(x, t; K_i) + \frac{\rho}{n} \bar{C}_i(x, t) + \mathcal{O}(\rho^2), \quad (3.15)$$

where  $\rho = \zeta/S_0$  as before. Expanding for small  $\rho$  and using (2.29) gives

$$L_{BS}\overline{C}_i = -\frac{x e^{-\frac{1}{2}d_1^2(K_i)}}{2\pi(T-t)} \sum_{j=1}^n e^{-\frac{1}{2}d_1^2(K_j)},$$

where  $d_1(K_j)$  is defined by (2.30) with  $K$  replaced by  $K_j$ , and boundary and terminal conditions are zero. Solving for each  $i$ ,

$$\overline{C}_i(x, t) = K_i e^{(-\frac{1}{2}(k-1)y_i - \frac{1}{4}(k+1)^2\tau)} u_i(y_i, \tau), \quad (3.16)$$

where

$$\begin{aligned} y_i &= \log(x/K_i), \\ \tau &= \sigma^2(T-t)/2, \\ u_i(y_i, \tau) &= \sum_{j=1}^n e^{-\frac{a_{ij}}{2}(k+1)\tau} \int_{\sigma\sqrt{\varepsilon/4\tau}}^{\sqrt{1/2}} M_j(y_i + a_{ij}, \tau, v) dv, \\ a_{ij} &= \log(K_i/K_j), \end{aligned}$$

and

$$\begin{aligned} M(y, \tau, v) &= \frac{1}{\pi\sqrt{1-v^2}} \exp \left[ -\frac{1}{2}(k+1)^2\tau v^2 - \frac{y^2}{4\tau(1-2v^2)} - \frac{a_{ij}^2}{8\tau v^2} \right. \\ &\quad \left. + \frac{v^2 \left[ y - \tau(k+1)(1-2v^2) + \frac{a_{ij}(1-2v^2)}{2v^2} \right]^2}{4\tau(1-v^2)(1-2v^2)} \right]. \end{aligned}$$

Then we can calculate  $\hat{\sigma}_i^*$  which solves

$$\min_{\hat{\sigma}_i} \int_0^T \int_0^\infty [C_i(x, t; \sigma) - C_{BS}(x, t; \hat{\sigma}_i, K_i)]^2 dx dt,$$

whose linearized expansion is given by  $\hat{\sigma}_i^* = \sigma + (\rho/n)\gamma_i^* + \mathcal{O}(\rho^2)$ , where

$$\gamma_i^* = \frac{\int_0^T \int_0^\infty \overline{C}_i(x, t; \sigma) R_i(x, t; \sigma) dx dt}{\int_0^T \int_0^\infty R_i(x, t; \sigma)^2 dx dt}, \quad (3.17)$$

where

$$R_i(x, t; \sigma) = \frac{\partial C_{BS}}{\partial \sigma}(x, t; \sigma, K_i).$$

An interpolated plot of  $\sigma_i^*$  against  $K_i$  is shown in Figure 3.4 for evenly distributed strike prices. We see that volatility increases by up to 8%, and that the peak volatility rise is at the arithmetic average of the strike prices,  $K = 5$ . This reflects qualitatively that observed volatility patterns might reveal information about the distribution of strike prices of options being hedged. However, this computation (and others where, for example  $x$  and/or  $t$  are fixed rather than integrated over) suggests that feedback cannot by itself explain observed smile patterns of implied volatility. Since it is known (see [20, 26]) that these patterns can be produced by stochastic volatility models, it will be interesting to build feedback from hedging strategies into these models and see how the resulting smile curves are flattened.

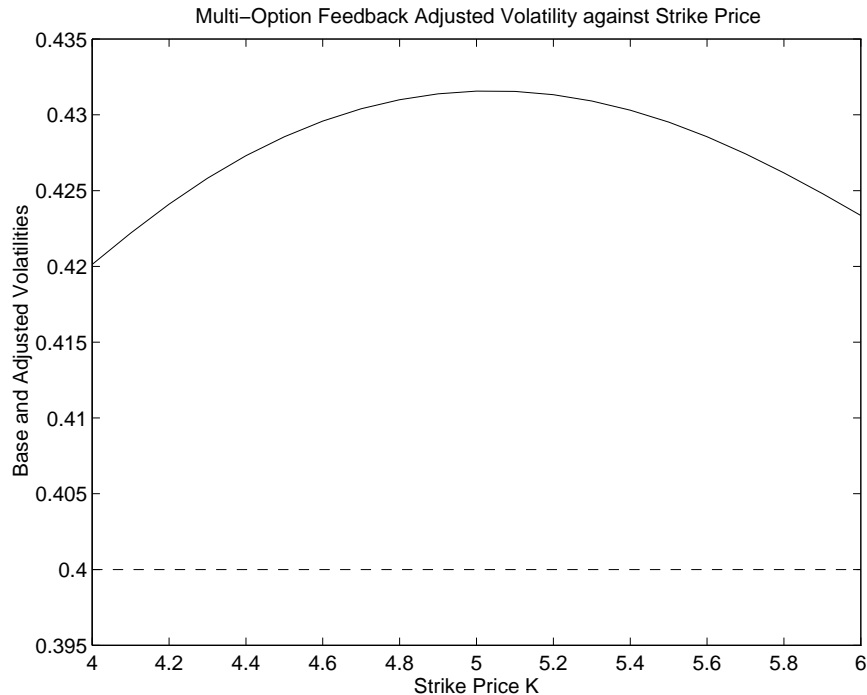


Figure 3.4: First-Order Corrected Black-Scholes Volatilities for 21 Options with strike prices at even intervals of 0.1 between  $K_{min} = 4$  and  $K_{max} = 6$  and  $\rho = 0.05$

## 4 General Asymptotic Results

We generalize the results of Section 3 to obtain analogous conclusions about the impact on market volatility of hedging strategies for an arbitrary European derivative security with payoff function  $h(x)$ . Our main tool is the Minimum Principle for the Black-Scholes partial differential operator, and with this and assumptions of almost everywhere smoothness and convexity on  $h(x)$ , we find that feedback always causes derivative prices and market volatility to increase, as in the particular case of the European call option.

**Theorem 4.1 (Minimum Principle)** *Suppose  $u(x, t)$  is a sufficiently smooth function satisfying, for some  $t_0 < T$*

$$L_{BS}u(x, t) \leq 0, \text{ for } x > 0, t_0 \leq t < T, \quad (4.1)$$

$$u(x, T) \geq 0, \text{ for all } x > 0, \quad (4.2)$$

$$u(0, t) \geq 0, \text{ for all } t_0 \leq t < T, \quad (4.3)$$

$$u(x, t) \sim U(x, t) \geq 0, \text{ as } x \rightarrow \infty, \quad (4.4)$$

where  $L_{BS}$  is defined by (3.2). Then for  $t_0 \leq t \leq T$ ,  $u(x, t) \geq 0$ .

*Proof.* See, for example Protter & Weinberger [19].

As in Section (3.1), we now look at the first-order correction to the feedback price of a derivative security  $C^h(x, t)$  satisfying (2.24) with non-negative terminal payoff  $C^h(x, T) = h(x) \geq 0$ . We shall also require that  $h$  is twice continuously differentiable almost everywhere. In the following, we shall not explicitly refer to the smoothing detailed in section (2.7.1), but the results are clearly applicable if  $T$  is replaced by  $T - \varepsilon$  and  $h(x)$  by  $C_{BS}^h(x, T - \varepsilon)$ .



**Proposition 2** For  $\rho \ll 1$ , the first-order correction to the Black-Scholes price for the derivative is non-negative.

*Proof.* We construct a regular perturbation series

$$C^h(x, t) = C_{BS}^h(x, t) + \rho \bar{C}^h(x, t) + \mathcal{O}(\rho^2),$$

where  $C_{BS}^h(x, t)$  satisfies the Black-Scholes equation and the terminal, boundary and far-field conditions. Expanding for small  $\rho$  and comparing  $\mathcal{O}(\rho)$  terms,  $\bar{C}^h(x, t)$  satisfies

$$\begin{aligned} L_{BS} \bar{C}^h &= -\sigma^2 x^3 \left( \frac{\partial^2 C_{BS}^h}{\partial x^2} \right)^2 \leq 0, \\ \bar{C}^h(x, T) &= 0, \\ \bar{C}^h(0, t) &= 0, \\ \lim_{x \rightarrow \infty} |\bar{C}^h(x, t)| &= 0. \end{aligned}$$

Then by the Minimum Principle,  $\bar{C}^h(x, t) \geq 0$ .

Thus we know that feedback causes derivative prices to increase. Next we consider how best to approximate the feedback price with the solution of the Black-Scholes equation for this security in the least-squares sense of Section (3.2). We shall require the following results.

**Lemma 1** If  $h''(x) \geq 0$ ,

$$\frac{\partial^2 C_{BS}^h}{\partial x^2} \geq 0,$$

in  $x > 0, t \leq T$ .

*Proof.* Let  $W(x, t) = \partial^2 C_{BS}^h / \partial x^2$ . Then differentiating  $L_{BS} C_{BS}^h = 0$  twice with respect to  $x$  gives

$$\begin{aligned} W_t + \frac{1}{2} \sigma^2 x^2 W_{xx} + (r + 2\sigma^2) W_x + (\sigma^2 - r) W &= 0, \\ W(x, T) &= h''(x). \end{aligned}$$

(That  $C_{BS}^h$  is twice differentiable follows from the Green's function solution and the smoothness assumptions on  $h$ .) Then we can use the Minimum Principle, since only the constant coefficients are different from the Black-Scholes operator, to deduce that  $W(x, t) \geq 0$ .

**Lemma 2** The Black-Scholes price is a non-decreasing function of volatility.

*Proof.* Let  $R^h(x, t; \sigma) = \partial C_{BS}^h / \partial \sigma$ . Differentiating the Black-Scholes equation for  $C_{BS}^h$  with respect to  $\sigma$  gives

$$\begin{aligned} L_{BS} R^h &= -\sigma x^2 \frac{\partial^2 C_{BS}^h}{\partial x^2} \leq 0, \\ W(x, T) &= 0. \end{aligned}$$

By the Minimum Principle,  $R^h \geq 0$  in  $x > 0$  and  $t < T$ .

**Proposition 3** *To first-order in  $\rho$ , the adjusted Black-Scholes volatilities  $\hat{\sigma}^*(t)$  which solves (3.10), and  $\hat{\sigma}^*(x)$  which solves (3.13) (with  $C$  and  $C_{BS}$  replaced by  $C^h$  and  $C_{BS}^h$ ) are not less than the reference volatility  $\sigma$ , provided the terminal payoff function is convex.*

*Proof.* Following Section (3.2.1),  $\hat{\sigma}^*(t)$  is given by  $\hat{\sigma}^*(t) = \sigma + \rho\gamma^*(t) + \mathcal{O}(\rho^2)$  where

$$\gamma^*(t) = \frac{\int_0^\infty \overline{C}^h(x, t; \sigma) R^h(x, t; \sigma) dx}{\int_0^\infty R^h(x, t; \sigma)^2 dx}. \quad (4.5)$$

Since  $\overline{C}^h \geq 0$ , it is sufficient to prove  $R^h \geq 0$ , for then  $\gamma^*(t) \geq 0$  and  $\hat{\sigma}^*(t) \geq \sigma$ . This follows from the previous Lemma. The same is true for  $\hat{\sigma}^*(x)$  from equation (3.14).

Furthermore, all these results generalize to the multi-option case because the small  $\rho$  expansion effectively decouples the system:

$$L_{BS} C_i^h = -\sigma^2 x^3 \left( \frac{\partial^2 C_{BS}^{h*}}{\partial x^2} \right) \left( \frac{\partial^2 C_{(i)BS}^h}{\partial x^2} \right) \leq 0.$$

In general, the increased volatility due to program trading can be observed both in the sense of implied volatilities and realised ‘spot’ volatility, which refers to the coefficient in (2.10) times  $\sigma$ . From (2.22), this can be written

$$\left[ 1 + \frac{\rho x C_{xx}}{V(1 - \rho\Phi)U'(V(1 - \rho\Phi)) - \rho x \Phi_x} \right],$$

so that, assuming the denominator stays positive, this factor is greater than the no-feedback volatility  $\sigma$  whenever the  $\Gamma(C_{xx})$  is positive. We are grateful to an anonymous referee for pointing out that volatility only increases when the  $\Gamma$  of the replicated portfolio is positive (which is true for the convex payoffs of call and put options). The destabilising feedback effect of positive  $\Gamma$  replicating strategies is discussed further by Schönbucher [22].

## 5 Numerical Solutions and Data Simulation

We now return to the setting where  $\rho$ , the market share of the program traders, is not necessarily small, and equation (2.32) for the feedback price of a European option must be solved numerically. A finite-difference scheme to do so for both scalar and multi-option cases is outlined in [25].

Using these solutions, we simulate market data for the underlying and consider typical discrepancies between historical estimates of volatility (from fitting the data to a geometric Brownian Motion) and Black-Scholes implied volatility from feedback options prices given by our model.

To construct a numerical solution to (2.32), we first transform the call option price into  $P(x, t)$  where

$$P(x, t) = C(x, t) - (x - Ke^{-r(T-t)}) \quad (5.1)$$

to make the behavior of the unknown function zero as  $x$  becomes large<sup>8</sup>. Then  $P(x, t)$  satisfies

$$P_t + \frac{1}{2} \left[ \frac{1 - \rho - \rho P_x}{1 - \rho - \rho(P_x + x P_{xx})} \right]^2 \sigma^2 x^2 P_{xx} + r(x P_x - P) = 0, \quad t < T - \varepsilon \quad (5.2)$$

---

<sup>8</sup>Although this looks like the put-call parity relationship, the nonlinear PDE is not invariant under this transformation, so we have not assumed that put-call parity holds; this transformation is only a tool to make the equations more tractable to numerical methods.

$$\begin{aligned}
P(x, T - \varepsilon) &= P_{BS}(x, T - \varepsilon), \\
P(0, t) &= Ke^{-r(T-t)}, \\
\lim_{x \rightarrow \infty} |P(x, t)| &= 0,
\end{aligned}$$

and  $P(x, t) = P_{BS}(x, t) := C_{BS}(x, t) - (x - Ke^{-r(T-t)})$  for  $T - \varepsilon \leq t \leq T$ .

## 5.1 Data Simulation

In this section, we illustrate typical pricing and volatility discrepancies that might arise if feedback effects from dynamic hedging strategies are not accounted for and the classical Black-Scholes model is used instead. We simulate typical price trajectories of the underlying asset using the feedback stochastic differential equation (2.9) in  $0 \leq t \leq T = 0.5$  with our numerical solution for the feedback option pricing equation (2.32) and  $\rho = 0.1$ , and we use the parameter values  $\mu_1 = 0.15$ , and  $\eta_1 = 0.3$  for the drift and diffusion coefficients in the geometric Brownian motion model (2.21) for the reference traders' income which are needed for the drift term  $\alpha(X_t, Y_t, t)$  in (2.9).<sup>9</sup> The paths are started at  $x = 4.5$ , just lower than the strike price  $K = 5$ , and  $\sigma$ , the best estimate of the reference traders' volatility parameter using historical data up to  $t = 0$ , is taken to be 0.4 as in our previous examples. The simulation is done by forward Euler: let  $\hat{X}^{(i)}$  be the numerically generated representation of  $X_{i\Delta t}$  along a particular path, and  $\hat{Y}^{(i)}$  the same for  $Y_{i\Delta t}$ . Then,

$$\hat{X}^{(i+1)} = \hat{X}^{(i)} + \alpha(\hat{X}^{(i)}, \hat{Y}^{(i)}, i\Delta t)\Delta t + v(\hat{X}^{(i)}, \hat{Y}^{(i)}, i\Delta t)\eta(\hat{Y}^{(i)}, i\Delta t)\theta^{(i)}\sqrt{\Delta t},$$

where  $\theta^{(i)}$  are independent  $\mathcal{N}(0, 1)$  random variables, and  $\hat{Y}^{(i)} = \left\{ \hat{X}^{(i)} \left[ 1 - \rho C_x(\hat{X}^{(i)}, i\Delta t) \right] \right\}^{\eta/\sigma}$ , from (2.7), and  $i = 0, 1, 2, \dots$ .

Then we calculate the constant historical volatility parameter  $\sigma^{est}$  that would be inferred by trying to fit  $X_t$  (here regarded as historical data) to a lognormal distribution:

$$\sigma^{est} = \frac{1}{\sqrt{\Delta t}} \sqrt{\frac{1}{N-1} \sum_{i=1}^N \left[ \log \left( \frac{\hat{X}^{(i)}}{\hat{X}^{(i-1)}} \right) \right]^2 - \frac{1}{N(N-1)} \left[ \sum_{i=1}^N \log \left( \frac{\hat{X}^{(i)}}{\hat{X}^{(i-1)}} \right) \right]^2},$$

using the asset prices for  $0 \leq t \leq 0.2 - \Delta t$ , so that  $N = 0.2/\Delta t - 1$ . This is the volatility parameter the program traders would use if they were to continually calibrate the Black-Scholes formula with up-to-the-minute volatility estimates from the latest data. Now we look at how this method of pricing compares with using the numerical solution of the feedback equation in the time interval  $0.2 \leq t \leq 0.4$ . The top graph of Figures (5.1)-(5.3) shows the simulated  $X_t$  in this period.

Next we compute  $C(\hat{X}^{(i)}, i\Delta t)$  using the numerical solution and  $C_{BS}(\hat{X}^{(i)}, i\Delta t; \sigma^{est})$  from the Black-Scholes formula, and these are plotted in the middle graphs. Frey and Stremme [11] asked the question whether the Black-Scholes model was still valid given the market inelasticity induced by the program traders: these pictures show that the answer is yes, as expected since the feedback model was constructed to be in the neighborhood of the classical theory, but also that significant mispricings can occur. The average percentage price discrepancy over each path,

$$\chi = \sum_{i \geq N} \frac{C(\hat{X}^{(i)}, i\Delta t) - C_{BS}(\hat{X}^{(i)}, i\Delta t; \sigma^{est})}{C(\hat{X}^{(i)}, i\Delta t)}$$

<sup>9</sup>Note that  $\mu_1$  and  $\eta_1$  are not needed in the feedback option pricing model.

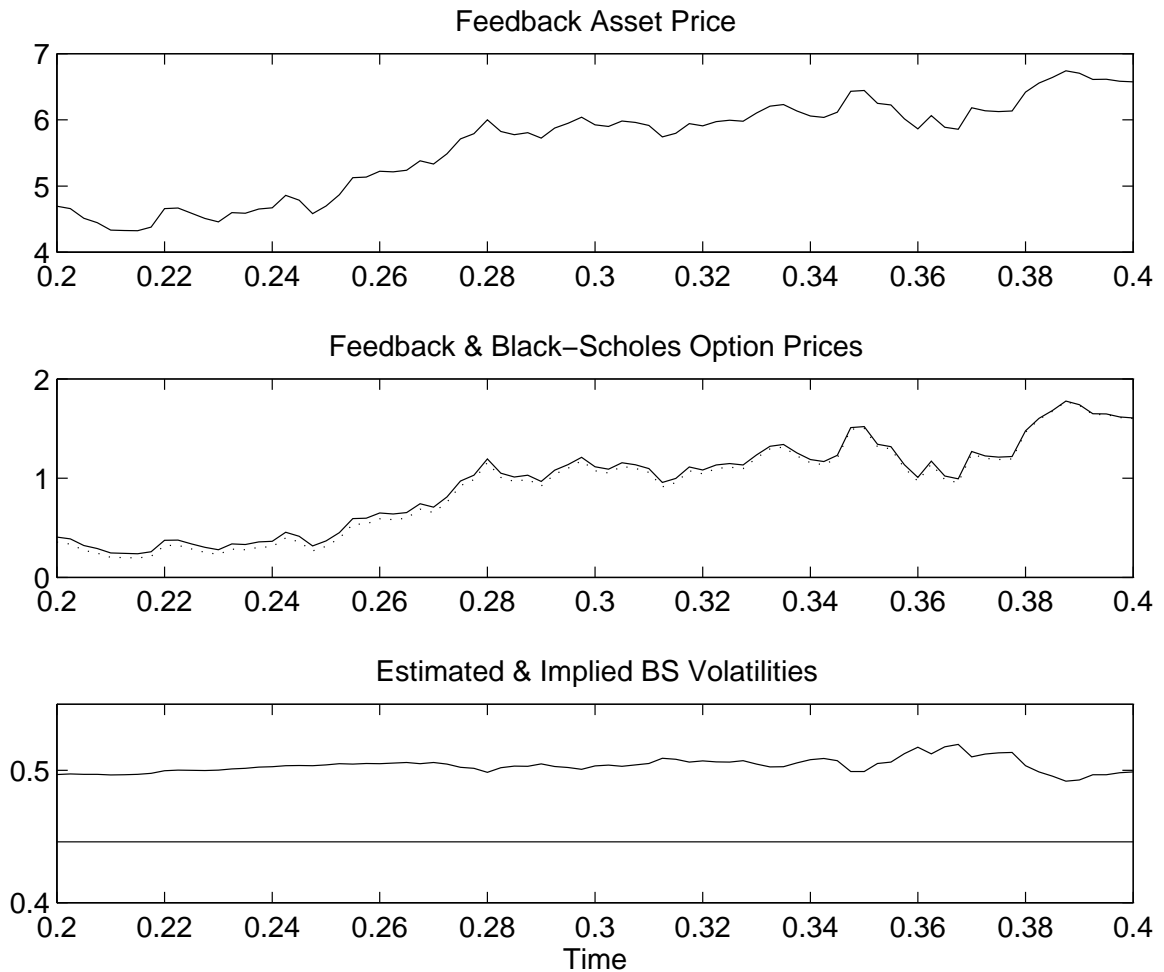


Figure 5.1: Data Simulation:  $\rho = 0.1$ ,  $\sigma = 0.4$ ,  $K = 5$ ,  $T = 0.5$ . The top graph shows the feedback asset price  $X_t$  in  $0.2 \leq t \leq 0.4$ . In the middle graph, the solid line is the feedback price and the dotted line the Black-Scholes price using  $\sigma^{est}$  from  $0 \leq t < 0.2$ . Here effects are shown for an asset price path rising above the strike price of the option, and  $\sigma^{est} = 0.446$ . The average relative mispricing by the Black-Scholes formula is  $\chi = 6.6\%$ .

is given with each of the pictures, and has been observed to vary between 1% and 28% for various paths. An average value of  $\chi$  over 100 simulated paths was found to be  $\bar{\chi} = 8.2\%$ .

Finally, we plot in the bottom graphs of Figures (5.1)-(5.3) the implied Black-Scholes volatility from the feedback option price  $C(\hat{X}^{(i)}, i\Delta t)$  in comparison with the constant  $\sigma^{est}$ . Experiments reveal that in nearly all cases simulated, this calibration to the Black-Scholes formula underestimates the implied Black-Scholes volatility. This means that, although  $\sigma^{est} > \sigma$ , the full feedback effect on market volatility and option prices is not captured by the classical model with a constant volatility parameter. In practice, users of the Black-Scholes formula would update  $\sigma^{est}$  more often than once in the lifetime of the six-month option, but similar discrepancies will result.

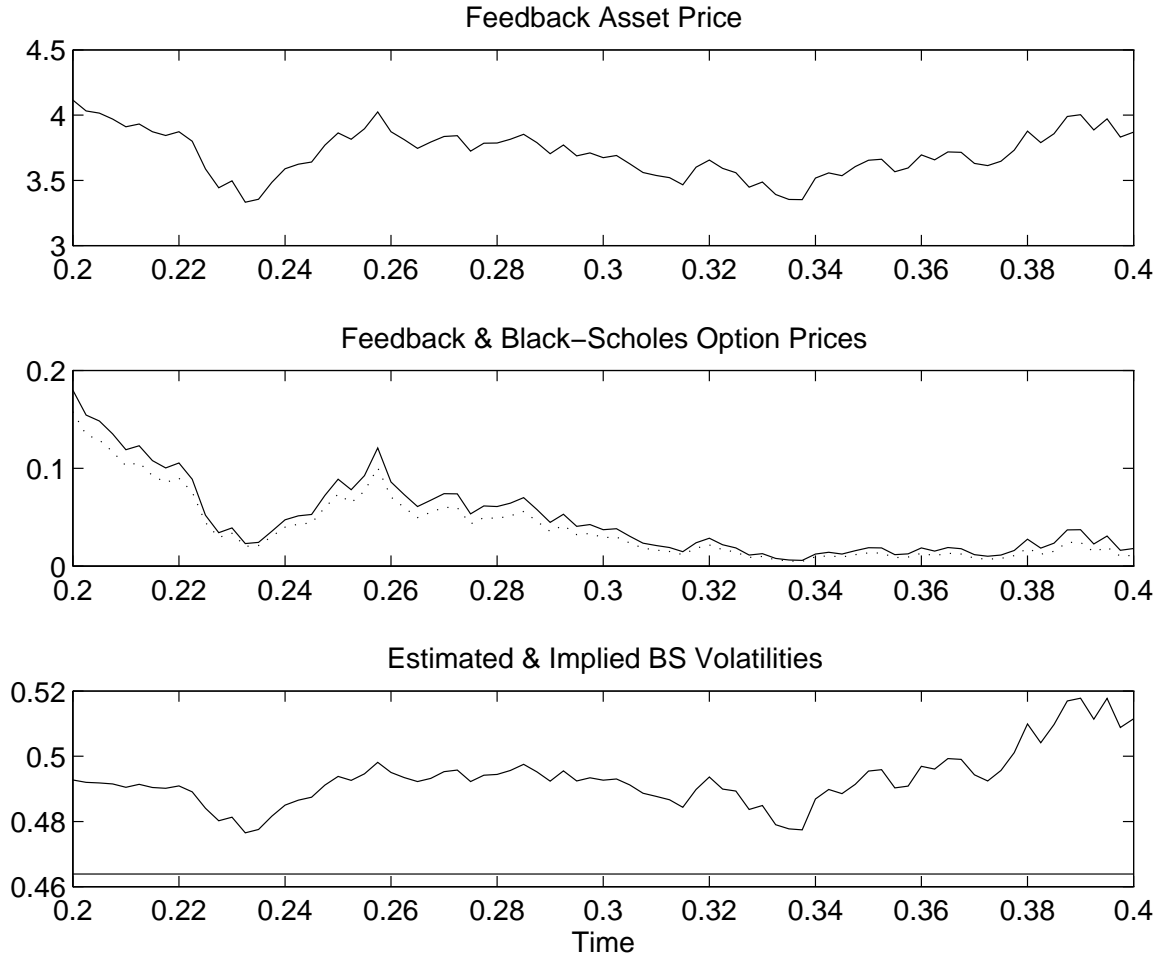


Figure 5.2: Data Simulation: Here effects are shown for an asset price path falling below the strike price of the option, and  $\sigma^{est} = 0.464$ ,  $\chi = 22.0\%$ .

## 5.2 Numerical Solution for many options

We briefly outline the procedure to obtain the numerical solution to the system (2.27), initially when all the strike times are equal:  $T_i \equiv T$ , but the strike prices vary:  $h_i(x) = (x - K_i)^+$ . This was considered for small  $\rho$  in section (3.3). For simplicity, we also assume a uniform distribution  $\zeta_i \equiv \zeta$ , for all  $i = 1, \dots, n$ .

Analogous to (5.1), we transform to put option prices

$$P_i(x, t) = C_i(x, t) - (x - K_i e^{-r(T-t)}), \quad (5.3)$$

and obtain a system of  $n$  equations analogous to (5.2):

$$\begin{aligned} \frac{\partial P_i}{\partial t} + \frac{1}{2} \left[ \frac{1 - \tilde{\rho} - \tilde{\rho} P_x^*}{1 - \tilde{\rho} - \tilde{\rho} (P_x^* + x P_{xx}^*)} \right]^2 \sigma^2 x^2 \frac{\partial^2 P_i}{\partial x^2} + r \left( x \frac{\partial P_i}{\partial x} - P_i \right) &= 0, \quad t < T - \varepsilon \\ P_i(x, T - \varepsilon) &= P_{BS}(x, T - \varepsilon; K_i), \\ P_i(0, t) &= K_i e^{-r(T-t)}, \\ \lim_{x \rightarrow \infty} |P_i(x, t)| &= 0, \end{aligned} \quad (5.4)$$

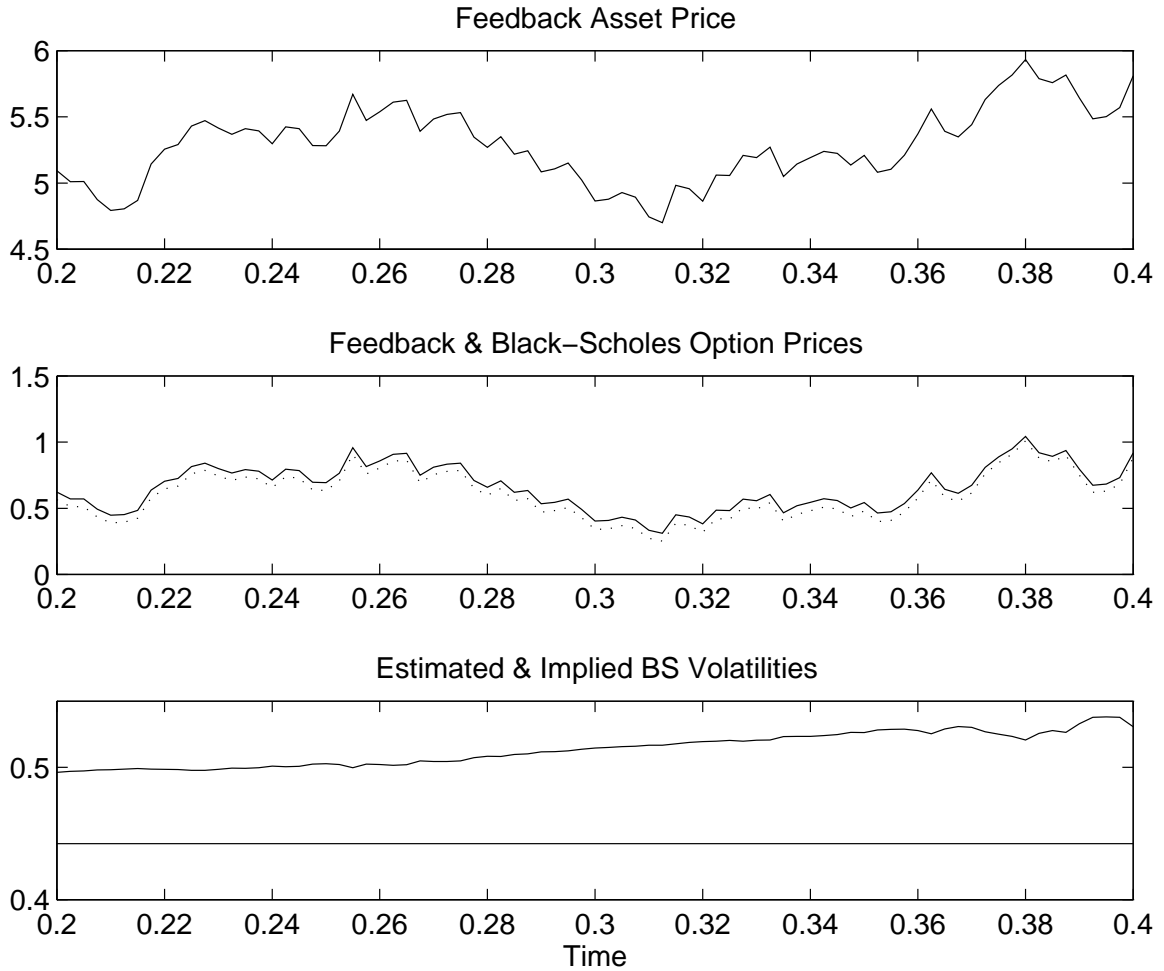


Figure 5.3: Data Simulation: Asset price path both above and below the strike price, and  $\sigma^{\varepsilon st} = 0.442$ ,  $\chi = 9.6\%$ .

where  $P^*(x, t) := \sum_{i=1}^n P_i(x, t)$ ,  $\tilde{\rho} := \zeta S_0^{-1}/n = \rho/n$ , and  $\varepsilon$  is the smoothing parameter corresponding to  $\tilde{\rho}$  from section (2.7.1). Now a finite-difference scheme is readily applicable.

### 5.2.1 Volatility Implications

We calculate the average implied Black-Scholes volatility under feedback effects from many call options spread around the strike price  $K = 5$  that we used as our example in the scalar case. That is, we solve for  $C_1(x, 0), \dots, C_n(x, 0)$  in (2.27) at time  $t = 0$ , when the strike prices are evenly distributed between  $K_{min} = 4$  and  $K_{max} = 6$  at intervals of 0.1. Then we calculate the implied Black-Scholes volatilities  $\sigma_1(x), \dots, \sigma_n(x)$  at  $t = 0$  and average these to obtain  $\sigma^{av\varepsilon}(x)$ . This is shown in figure (5.4) with  $\rho = 0.1$ .

The graph shows that the biggest increase in Black-Scholes volatility is in the neighborhood of the strike prices, and there is an averaging of the peaks over this price range. This is consistent with our results from the scalar case. In addition, there is an overall volatility increase at all prices as a consequence of the program trading.

We also compare the effect of the spreading of strike prices on the actual volatility increase. Frey

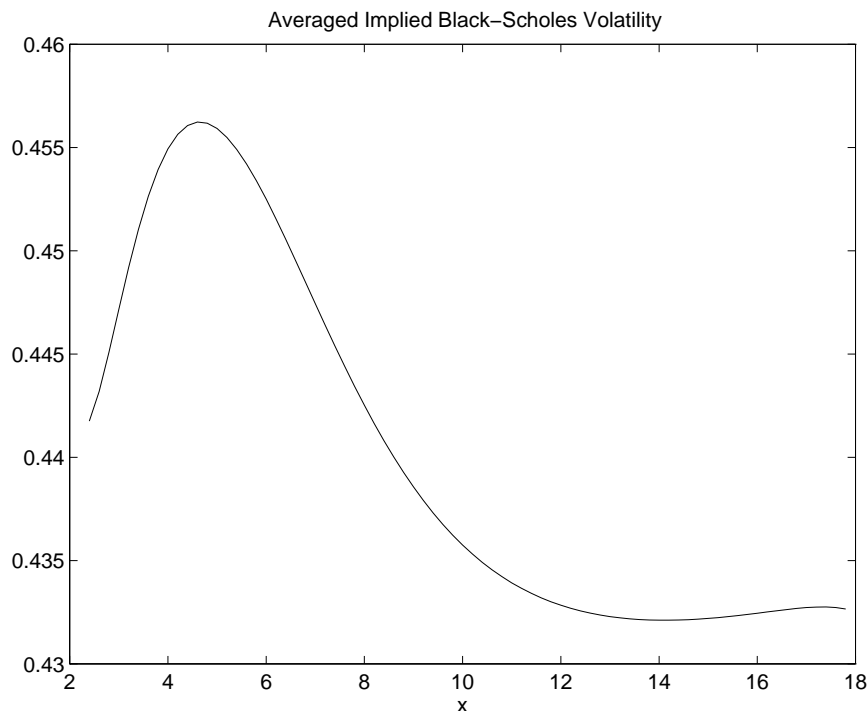


Figure 5.4: Averaged Black-Scholes implied volatility  $\sigma^{ave}(x)$  at  $t = 0$  from options with strike prices 4, 4.1,  $\dots$ , 6, and  $\rho = 0.1$ ,  $\sigma = 0.4$ .

and Stremme [11] found that increasing heterogeneity of the payoff function distribution reduced the level of the volatility increase in their model. This is qualitatively confirmed for our model in figure (5.5) in which we plot the volatility coefficient

$$V(x, t) := \frac{1 - \rho C_x}{1 - \rho C_x - \rho x C_{xx}} \sigma$$

at  $t = 0$  when there is only one strike price  $K = 5$ , and the equivalent

$$V^*(x, t) := \frac{1 - (\rho/n)C_x^*}{1 - (\rho/n)C_x^* - (\rho/n)x C_{xx}^*} \sigma$$

for the  $n = 21$  option types with strike prices between  $K_{min} = 3$  and  $K_{max} = 7$  at intervals of 0.2 apart. The interpretation is that the sensitivity of program trading to price fluctuations around one particular strike price is spread over a larger range of prices: as  $x$  increases from \$4 to \$5, for example, the program traders buy more stock to insure against likely losses on the options with strikes less than \$5, but the ones with strikes close to \$7 are still relatively ‘safe’ and insurance for these options does not, at this stage, induce as significant a volatility rise as if they too had been struck at \$5.

### 5.2.2 Varying strike times

To complete the picture, the numerical solution algorithm can be adapted to the system of many options with various strike times  $T_1 < T_2 < \dots < T_M$ . The only amendment is to keep track of which options are still active (unexpired) at each time and contribute to the feedback through  $C^*$ . That is, the number of options defining  $C^*$  is a function of time:  $n = n(t)$ .

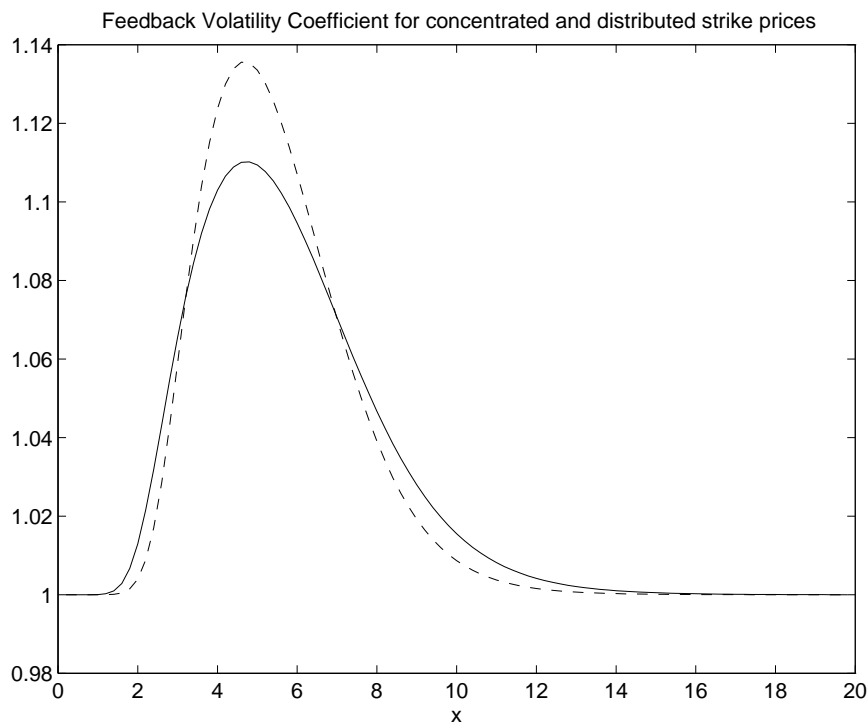


Figure 5.5: The dashed curve shows the feedback volatility coefficient  $V(x, 0)$  for a single option type  $K = 5$ , and the solid curve shows the equivalent  $V^*(x, 0)$  for strike prices evenly spread between 3 and 7.

Since we are interested in the volatility coefficient  $V^*(x, t)$ , we can add up the equations for the option prices  $C_i$  (in the correct proportion if the strike prices are not uniformly distributed for each strike time) and solve a scalar equation for  $C^*(x, t)$  in  $t < T_M$ . We do this for the case when there are three strike times  $T_1 = 0.25, T_2 = 0.5$  and  $T_3 = 0.75$ , corresponding to 3-, 6- and 9-month contracts, and 21 options of each maturity with strike prices evenly spread between  $K_{min} = 4$  and  $K_{max} = 6$  (and  $\zeta_i \equiv \zeta$  as before).

The volatility coefficient  $V^*(x, t)$  is shown in figure (5.6), and a cross-section at  $x = 5$  in the top graph of figure (5.7). The volatility rises to a sharp peak as each strike time is approached and then falls as some of the options expire, creating less feedback. We note that the height of these spikes is controlled by our smoothing requirements: hedging very close to the strike times of the options is not accurately modelled by a Black-Scholes-type dynamic hedging strategy (it is likely to be very individual, and it is doubtful that the program traders follow a particular program at such times), and the soon-to-expire options are assumed not to contribute to  $V^*(x, t)$  once they are within  $\varepsilon$  of their strike time.

The bottom graph of figure (5.7) shows typical volatility behavior  $V^*(X_t, t)$  where  $X_t$  is a typical feedback trajectory (with constant drift  $\bar{\mu} = 0.15$  for simplicity):

$$dX_t = \bar{\mu}X_t dt + V^*(X_t, t)X_t dW_t.$$

Again, the presence of peaks close to the strike times is apparent.



Volatility Coefficient for multiple strike times and prices

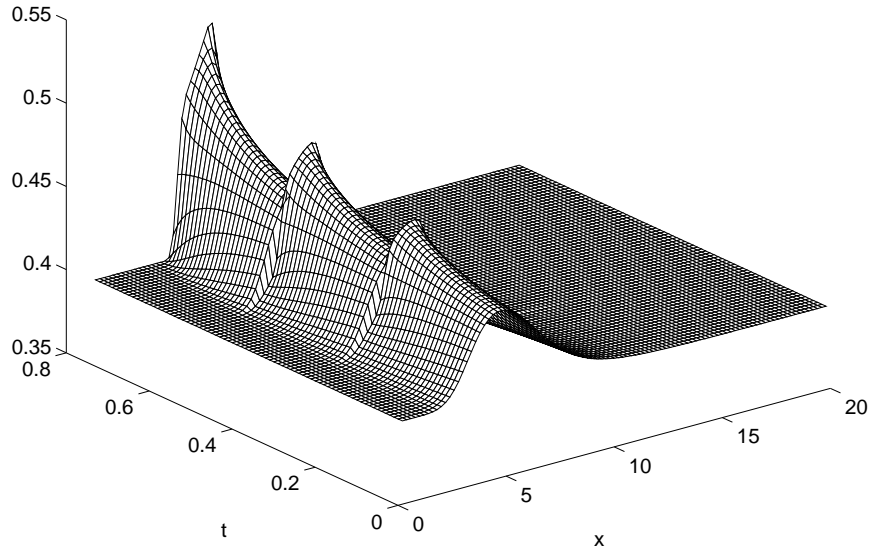


Figure 5.6: Volatility Coefficient  $V^*(x, t)$  for strike times 0.25, 0.5, 0.75 and strike prices 4, 4.1,  $\dots$ , 6.

## 6 Summary and conclusions

We have presented and analyzed an options pricing model that accounts self-consistently for the effect of dynamic hedging on the volatility of the underlying asset. Following Frey and Stremme [11], we consider two classes of traders, reference and program traders. We then incorporate the influence of the program traders into the pricing of the option. This leads to a new family of non-linear partial differential equations that reduce to the classical Black-Scholes model in the absence of portfolio insurance trading. The solutions give the feedback adjusted price of the option and the program traders' hedging strategy. Moreover, our model provides a quantitative estimate of the observed increase in volatility due to program trading.

One way in which the self-consistent option pricing models can be used is for *estimating* the fraction of trading that is done for portfolio insurance. We are studying this at present. Another issue that requires attention is incorporation of feedback effect into stochastic volatility models that are becoming more and more popular in the industry.

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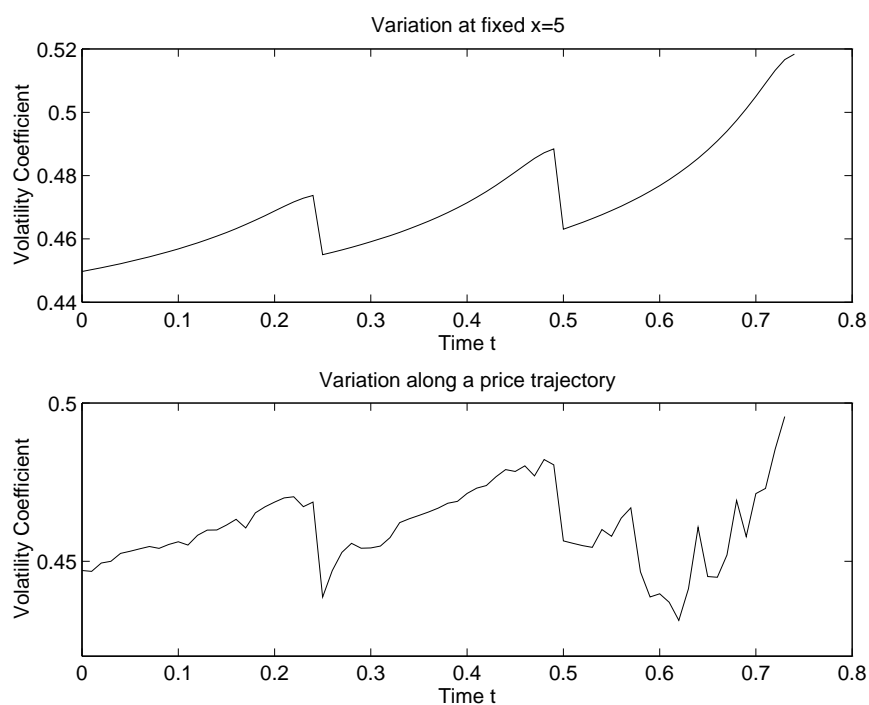


Figure 5.7: Volatility Coefficient for strike times 0.25, 0.5, 0.75 and strike prices 4, 4.1,  $\dots$ , 6 at  $x = 5$ , and along a price path  $X_t$ .

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