Optimal Static-Dynamic Hedges for Barrier Options

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Abstract

We study optimal hedging of barrier options using a combination of a static position in vanilla options and dynamic trading of the underlying asset. The problem reduces to computing the Fenchel-Legendre transform of the utility-indifference price as a function of the number of vanilla options used to hedge. Using the well-known duality between exponential utility and relative entropy, we provide a new characterization of the indifference price in terms of the minimal entropy measure, and give conditions guaranteeing differentiability and strict convexity in the hedging quantity, and hence a unique solution to the hedging problem. We discuss computational approaches within the context of Markovian stochastic volatility models.

Keywords: Hedging, derivative securities, stochastic control, indifference pricing, stochastic volatility.

1 Introduction

Exotic options are variations of standard calls and puts, tailored according to traders' needs. These options are mainly traded in over-the-counter (OTC) markets. As of December 2000, the outstanding notional amount in OTC derivatives markets was \$95 trillion compared with \$14 trillion on exchanges. (See [29]). In this paper, we focus on barrier options which are among the most popular exotic options. According to a research report [28], barrier option trading accounts for 50% of the volume of all exotic traded options and 10% of the volume of all traded securities.

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Barrier options are contingent claims that have certain aspects triggered if the underlying asset reaches a certain barrier level during the life of the claim. The main advantage of barrier options is that they are cheaper alternatives of their vanilla counterparts. However, since the option payoff depends on whether the barrier has been hit or not as well as the terminal stock price, the pricing and hedging problems for these options are more complicated. The pricing formula for a barrier option under the Black-Scholes model first appeared in a paper by Merton [30].

Barrier options have different *flavors*: an *out* option expires worthless if the stock price hits the barrier where it is *knocked-out*. *In* options on the other hand do not pay unless the barrier has been triggered. According to the relative value of the initial stock price and the barrier level, these options are called *down* or *up*. As an example, a down and in call pays like a regular call option provided that the stock prices goes below the barrier level before the maturity of the option.

The Black-Scholes methodology for hedging options, so called dynamic hedging, eliminates the risk of the option position by trading *continuously* the underlying stock and bonds. Assuming that the stock price satisfies

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \quad S_0 = S \tag{1.1}$$

for constant $\mu, \sigma > 0$ and W a standard Brownian motion, it is well known that the risk in any options position can be totally eliminated. The amount of stock to hold at each instant depends on the sensitivity of the option price to the stock price, known as the Delta of the option. The Delta of barrier claims can be quite high when the stock price is close to the barrier level. Given that continuous trading is not possible, a discretization of the continuous model gives poor results when the Delta of the option is large, even if the underlying model is right. With this motivation, there has been extensive research on alternative ways of hedging barrier options. Bowie and Carr [4] introduced the idea of *static* hedging through a portfolio of vanilla options formed at initiation where no trading occurs afterwards. Using a binomial tree model for the stock price, Derman *et al.* [11] formed a static hedging portfolio by using a finite number of puts and calls such that the value of the portfolio matches the value of the barrier option on the nodes that are on the barrier and at maturity. By their construction, the number of vanilla options that are being used is related to the number of periods in the tree and the performance of the strategy improves as the number of periods is increased. Assuming that the stock price satisfies (1.1), Carr and Chou ([7], [6]) showed that any barrier claim is replicable by holding a portfolio of vanilla calls and puts statically until the stock price hits the barrier. In their approach, the necessity of continuous trading of the underlying is replaced by the necessity of trading options with a continuum of different strikes. In [1], Bardos *et al.* extended this idea to the case where μ and σ in (1.1) are deterministic functions of time and stock price. A strategy using other options which is applicable in the case of incomplete markets and which is robust to model misspecification was given in [5] by Brown *et al.*. They found upper and lower bounds for the price of barrier claims in terms of vanilla options which are interpreted as hedging strategies. Their calculations assumed that interest rates are zero and the extension to non-zero interest rates is not trivial. Examining the static hedging portfolio of a down and in call proposed by Carr and Chou, we notice that this portfolio is not equally weighted among the continuum of strike prices and we propose using only the option with the greatest weight in the hedge, combined with dynamic trading in the underlying.

In this paper we address the question of hedging barrier options in incomplete markets, that is we assume that all the uncertainties in the market are not hedgeable through trading the available assets. There is no straightforward way to extend the static hedging ideas proposed previously to this case, and it is an open question how the static hedging approaches perform in realistic incomplete markets. The idea of combining dynamic trading in the stock (for which transactions costs are relatively small) with buy-and-hold (static) positions in liquidly traded vanilla options (where transaction costs are much higher and dynamic trading is not feasible) was used in the context of portfolio optimization in [24]. The key issue is how much capital to allocate to the derivatives, and how much to the stock and bonds. That is, we optimize over possible derivatives positions the value function of the dynamic hedging problem that is the solution of a stochastic control problem with random endowment.

We model an investor with an exponential utility function given as

$$U(z) = -e^{-\gamma z} \tag{1.2}$$

where $\gamma > 0$ is the coefficient of absolute risk aversion. Having bought the down and in call, we try to maximize her expected final utility by choosing an optimal number of vanilla options to sell. As the strategy proposes only a partial hedge, we want to take one step further, and discuss an optimal dynamic trading strategy in the underlying in addition to the static vanilla option position. Our main motivation is to find a compromise between the dynamic and static hedging strategies allowing a wider range of possible hedging scenarios. The extension of our ideas to the other types of barrier options is straightforward.

The rest of the paper is organized as follows: In Section 2, we summarize the derivation of static hedging and give the static hedging portfolio for a down and in call option for completeness. In Section 3, we use the connection between exponential utility maximization and entropy minimization to deduce that the optimization problem reduces to finding the convex dual of the utility indifference price of a particular type of barrier option, and we examine properties of the indifference price, in particular strict convexity with respect to the number of options. In Section 4, we give an application of the problem within a stochastic volatility model for the stock price. In this case the price satisfies a second order quasilinear PDE which does not have an explicit solution, and we study the optimal number of put options to trade numerically. In Section 5, we conclude.

2 Static Hedging of Barrier Options in the Black-Scholes Model

Assuming constant interest rates and a frictionless market where the stock price process $(S_t)_{t\geq 0}$ is given by (1.1), Carr and Chou in [7] and [6] showed that there exists a portfolio of European calls, puts and forwards replicating any barrier claim. It is worth noting that their assumptions do not extend beyond the usual Black Scholes assumptions used to recover dynamic hedging strategies. However, being more robust to misspecified models and transaction costs associated with dynamic hedging, static hedging might perform better in real life applications.

Here, we summarize their arguments for a down and in call option which pays the difference between the stock price S and the strike price K at the maturity T of the option given that this difference is positive and the barrier level B has been hit at any time before T. In this discussion we include only the case where $B < K$. The price of this option is given by

$$f(t,S) = e^{-r(T-t)}\mathbb{E}^{Q}_{t,S}\left\{(S_T-K)^{+}\mathbf{1}_{\{\tau^B\leq T\}}\right\}$$

where $\tau^B = \inf\{u \geq t : S_u \leq B\}$, and the expectation is taken under the unique measure Q which is equivalent to P, and under which the discounted stock price is a martingale. We use $\mathbb{E}_{t,S}$ to denote expectation conditional on $\{S_t = S\}$. By an iterated expectation argument the price is given by

$$f(t,S) = e^{-r(T-t)}\mathbb{E}^{Q}_{t,S}\left\{\mathbf{1}_{\{\tau^B\leq T\}}\mathbb{E}^{Q}_{\tau^B,B}\left\{(S_T-K)^{+}\right\}\right\} \tag{2.1}$$

and the inner expectation can be written as

$$\mathbb{E}^{Q}_{\tau^B,B}\left\{(S_T-K)^{+}\right\} = \int_K^{\infty}(x-K)p(B,x,\tau^B,T)\,\mathrm{d}x$$

where $p(B,x,\tau^B,T)$ is the Q-probability of the stock price density at time T given that it is equal to B at time τ^B. The stock price process is log-normal under Q, hence defining $\bar{x} = B^2/x$ we obtain

$$\mathbb{E}^{Q}_{\tau^B,B}\left\{(S_T-K)^{+}\right\} = \int_0^{B^2/K}\left(\frac{\bar{x}}{B}\right)^{k}\left(\frac{B^2}{\bar{x}}-K\right)p(B,\bar{x},\tau^B,T)\,\mathrm{d}\bar{x}$$

where $k = 1 - \frac{2r}{\sigma^2}$. Notice that, at this step the log-normality of the stock price under Q is crucial. The expression above is, up to the discounting factor, the price of a European option with payoff

$$\left(\frac{S_T}{B}\right)^k \left(\frac{B^2}{S_T} - K\right)^+ \tag{2.2}$$

when the stock price is at the barrier. By construction, this value is equal to the value of our down and in call option when the stock price is on the barrier.

Any twice differentiable European payoff $F(S)$ can be written as

$$F(S_T) = F(B) + (S_T - B)F'(B) + \int_0^B F''(K)(K - S_T)^+ \,\mathrm{d}K + \int_B^\infty F''(K)(S_T - K)^+ \,\mathrm{d}K,$$

which gives a replicating portfolio for the option with payoff $F(S_T)$ in terms of puts, calls, bonds and forwards. Applying this to the option with payoff (2.2) and using that the second derivative of the hockey stick call payoff is a δ-function, (2.2) can be replicated by a portfolio of put options given as below

$$\left(\frac{B}{K}\right)^{k-2} \text{ puts at strike } K' = \frac{B^2}{K}, \tag{2.3}$$

$$\left(\frac{\bar{K}}{B}\right)^{k-2} (k-1)\left(\frac{k-2}{\bar{K}} - \frac{kK}{B^2}\right) \mathrm{d}\bar{K} \text{ puts at strike } \bar{K} \text{ for } \bar{K} < K'. \tag{2.4}$$

Any investor who wants to hedge her long position in a down and in call can short this portfolio. If the stock price does not hit the barrier, the portfolio will expire worthless like the barrier option. On the other hand if the stock price hits the barrier, the investor can liquidate the hedging portfolio which is constructed to have the same value as the barrier option when the stock is on the barrier. We refer the reader to [6] for further details.

To have a better understanding of this replicating portfolio, in Figure 2 we plot the number of put options included in the replicating portfolio found by this method for a specific down and in call option. In the figure, it is assumed that only integer strike prices are available and the minimum available strike is 50. A striking feature of the figure is that the put option with strike $K' = B^2/K$ has the greatest weight in the portfolio. A natural question arising from this conclusion is what happens when we use this put option alone, namely the put option with strike price K'.

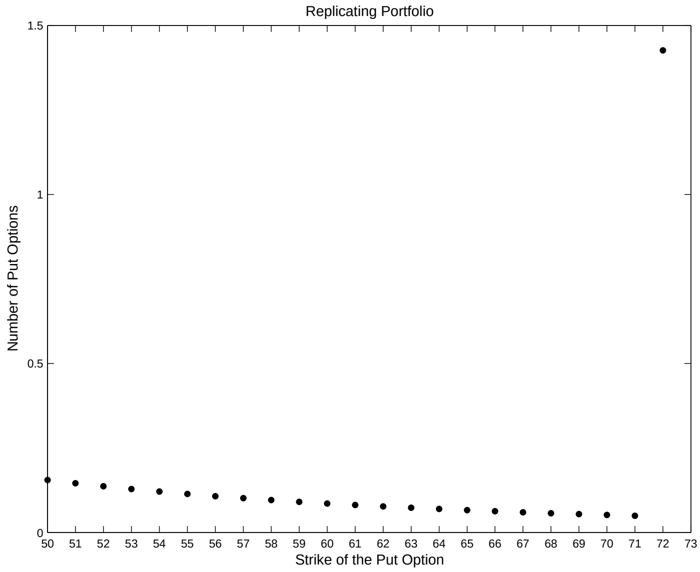

Figure 1: Number of each put in the replicating portfolio. Parameters used are $S_0 = 98.5112$ (such that $S = 100 \times e^{-rT}$), $K = 100$, $B = 85$, $T = 0.5$, $r = 3\%$, $\sigma = 0.225$.

3 Static-Dynamic Hedging of Barrier Options in Incomplete Markets

We study the problem of an investor who wants to hedge a long position in a down and in call option. We assume that the market is incomplete, in that not all the risks are hedgeable through trading the underlying stock and bonds. Guided by static hedging, the initial component of the hedge is to sell a number (to be determined) of put options with strike price K'. We assume that the investor maximizes her expected utility of wealth at time T, when both the barrier option and the put options expire, by choosing the number of put options to short optimally. Our goal is to find this optimal quantity, which of course depends on the (given) market price of the puts.

Trading the put options supplies only a partial hedge, and the investor also trades the underlying stock continuously during the life of the options. If markets were complete, all claims could be replicated by trading the stock dynamically, given sufficient initial capital, and any position in the put option could be synthesized with such a trading strategy. Therefore, static derivatives hedges are redundant in complete markets, but of course they are very valuable tools in realistic markets, for example to hedge volatility risk.

The preferences of the investor are described by an exponential utility function given in (1.2). Since we need to consider the expected utility of a portfolio taking negative values, a utility function defined on $\mathbb{R}$ rather than $\mathbb{R}_+$ is needed. Due to its simplicity and algebraic convenience, exponential utility is the most common example of such utility functions in the literature.

Throughout, the interest rate r is constant and all the processes are defined on $[0, T]$. We assume that the probability space in the background is $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ that satisfies the usual conditions of right continuity and completeness. Hence, without loss of generality, we assume all the processes have paths that are right-continuous with left-limits. Instead of working with stock prices, we introduce the forward price $X_t = e^{r(T-t)} S_t$, and, for this section, we only assume that $(X_t)_{0 \le t \le T}$ is a positive semimartingale adapted to $\mathbb{F}$, and is locally bounded.

3.1 Statement of Problem

We model an investor with a long position in a down and in call option, with barrier level B and strike price $K > B$, and an initial wealth of $\tilde{v}_0$ dollars. In general, $\tilde{v}_0$ is negative due to the long position. The investor sells an additional α put options for the market price $\tilde{p}$ dollars at time zero, and follows a self-financing trading strategy in the underlying stock and bonds with the (available) initial wealth $\tilde{v}_0 + \alpha \tilde{p}$. Let π_t be the number of stocks held at time t. We assume that the trading strategy π is adapted to the given filtration.

Throughout the paper, except for the stock price, today's value of a random variable is denoted by a letter with tilde while its T-forward value is denoted by the corresponding plain letter. We define $v_0 = \tilde{v}_0 e^{rT}$ and $p = \tilde{p} e^{rT}$. Let $(V_t^\pi)_{0 \le t \le T}$ denote the *forward wealth* process that starts at $v_0 + \alpha p$ and is generated by the trading strategy π. It is given by

$$V_t^\pi = v_0 + \alpha p + \int_0^t \pi_s \, \mathrm{d}X_s. \tag{3.1}$$

To formalize the problem, let us introduce

$$u(v_0 + \alpha p, B^\alpha) = \sup_{\pi \in \Theta(P)} \mathbb{E} \left\{ -e^{-\gamma (V_T^\pi - B^\alpha)} \right\}, \tag{3.2}$$

the maximum (over stock-bond trading strategies) expected utility of the investor given that α put options were sold. In (3.2), $\Theta(P)$ is a suitable set of strategies to be defined below, and B^α is the payoff of the combined option position formed by α puts with strike K' minus our down and in call option.

Its payoff is

$$B^\alpha = \alpha (K' - S_T)^+ - (S_T - K)^+ \mathbf{1}_{\{\min_{0 \le t \le T} S_t \le B\}}.$$

Since we are working with forward prices instead of stock prices, we write the payoff as

$$B^\alpha = \alpha P' - C \mathbf{1}_{\{\min_{0 \le t \le T} e^{-r(T-t)} X_t \le B\}},$$

where P' denotes the payoff of the put option and C denotes the payoff of the call option

$$P' = (K' - X_T)^+, \tag{3.3}$$

$$C = (X_T - K)^+. \tag{3.4}$$

We also write B^0 as the payoff of the short barrier position

$$B^0 = -C\,\mathbf{1}_{\{\min_{0\le t\le T} e^{-r(T-t)} X_t \le B\}}. \tag{3.5}$$

The objective is to find

$$\alpha^* = \arg\max_{\alpha} u(v_0 + \alpha p, B^\alpha), \tag{3.6}$$

the optimal static hedging position, assuming (as we shall give conditions for below) such a maximum exists and is unique. As this definition shows, it is the optimizer of a function u which is itself the value function for a stochastic control problem.

We will also see that the supremum in the definition of $u(v_0 + \alpha p, B^\alpha)$ is achieved. The dynamic part of the optimal hedging strategy then comes from the optimizer in (3.2) with α replaced by α^*.

3.2 The Dual Problem

For an arbitrary $\mathcal{F}_T$-measurable payoff D, $u(z, D)$ is the maximum expected utility of an agent who has a short position in the claim D, initial wealth \$$z$, and trades the underlying stock with the optimal strategy. The dual of this maximization problem can be defined, and in the literature there are numerous results referenced below which show that there is no duality gap between the solutions of the primal and the dual problem in the case of exponential utility as well as others, with or without the claim.

We begin with some definitions.

Definition 1 *1. $\mathbb{P}_a(P)$ denotes the set of absolutely continuous (with respect to P) local martingale measures*

$$\mathbb{P}_a(P) = \{Q \ll P \,|\, X \text{ is a local } (Q, \mathbb{F})\text{-martingale}\}. \tag{3.7}$$

2. $\mathbb{P}_f(P)$ denotes the set of absolutely continuous local martingale measures with finite entropy relative to P:

$$\mathbb{P}_f(P) = \{Q \in \mathbb{P}_a(P) \,|\, H(Q|P) < \infty\}, \tag{3.8}$$

where the relative entropy H is defined by

$$H(Q|P) = \begin{cases} \mathbb{E}\left\{\frac{\mathrm{d}Q}{\mathrm{d}P}\log\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right\}, & Q \ll P, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.9}$$

3. *The set of allowable trading strategies,* $\Theta(P)$ *is*

$$\Theta(P) = \{\pi \in L(X) \,|\, \int \pi \mathrm{d}X \textit{ is a } (Q,\mathbb{F})\textit{-martingale for all } Q \in \mathbb{P}_f(P)\} \tag{3.10}$$

where $L(X)$ *is the set of* $(\mathcal{F}_t)_{0\le t\le T}$*-predictable* X*-integrable* $\mathbb{R}$*-valued processes.*

4. *We also define* $\mathbb{P}_e(P)$*, the set of equivalent local martingale measures:*

$$\mathbb{P}_e(P) = \{Q \in \mathbb{P}_a(P) \,|\, Q \sim P\}\,. \tag{3.11}$$

The primal (investment) problem is to find $u(z, D)$:

$$u(z,D) \quad = \quad \sup_{\pi\in\Theta(P)} \mathbb{E}\left\{-e^{-\gamma\,(V_T^\pi - D)}\right\}, \tag{3.12}$$

and we make the following assumption on the claim D.

Assumption 1

$$D \in L^1(Q) \textit{ for all } Q \in \mathbb{P}_a(P), \tag{3.13}$$

and

$$\mathbb{E}\left\{e^{\gamma D}\right\} < \infty. \tag{3.14}$$

Throughout the paper, we assume there exists an equivalent local martingale measure with finite relative entropy with respect to P:

Assumption 2

$$\mathbb{P}_f(P) \cap \mathbb{P}_e(P) \neq \emptyset. \tag{3.15}$$

For exponential utility, a duality result including a contingent claim in a general semi-martingale setting was first shown in Delbaen *et al.* [10]. Assuming that D is bounded below and $\mathbb{E}\,\left\{e^{(\gamma+\epsilon)D}\right\} < \infty$ for some $\epsilon > 0$, in addition to Assumption 2, they show that $u(z, D)$ is given by

$$u(z,D) = -\exp\left(\gamma \sup_{Q\in\mathbb{P}_f(P)} \left(\mathbb{E}^Q\{D\} - \frac{1}{\gamma}H(Q|P)\right) - \gamma z\right). \tag{3.16}$$

Notice that our regularity assumptions (3.13) and (3.14) on D are different than those in [10]. That their duality result holds under the assumption we make on D follows from their Lemma 3.5 and the discussion before their main theorems. In fact, their assumptions on D were made to guarantee that

$$D \in L^1(Q) \text{ for all } Q \in \mathbb{P}_f(P) \cup \mathbb{P}_f(P^D) \tag{3.17}$$

where P^D is defined by

$$\frac{\mathrm{d}P^D}{\mathrm{d}P} = c^D e^{\gamma D}, \text{ with } (c^D)^{-1} = \mathbb{E}\left\{e^{\gamma D}\right\},$$

and $\mathbb{P}_f(P^D)$ is defined in a similar way as $\mathbb{P}_f(P)$ with the reference measure changed to P^D. The claim-dependent prior P^D is well-defined because of (3.14). Under assumption (3.13), (3.17) follows trivially.

Delbaen *et al.* [10] give three different theorems for three different choices of allowable trading strategies. Later, Becherer [2] gives slightly modified versions of these sets using the extensions of [25]. Our choice for the feasible set of strategies, $\Theta(P)$ defined in (3.10) is the Θ_2 in [2].

In addition to the equality of the solutions of the primal and dual problems, these results also show that the suprema in both problems are attained in their corresponding feasible sets. Moreover, for the dual problem, the supremum is achieved by a measure in $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$.

The duality result for the case of Brownian filtration also appeared in Rouge and El Karoui [35]. A similar duality relation was also established by Schachermayer [36] with a general class of utility functions defined on $\mathbb{R}$, but without a contingent claim. He gives necessary and sufficient conditions on the utility functions for the duality result to hold. His results were extended to include a claim by Owen [32].

3.3 Indifference Prices

In this section, we recast our barrier hedging problem in terms of the utility indifference pricing mechanism.

3.3.1 General Expression for the Indifference Price of a Claim

Recall that, in the general setting of the previous section, our investor has initial wealth z and has to pay the claim that yields the random amount D at time T. We define her utility indifference price of the contingent claim at time zero as the largest amount $\tilde{h}(z, D)$ she would be willing to pay to be free from her obligation for the claim judged by exponential utility:

$$u(z, D) = u(z - e^{rT}\tilde{h}(z, D), 0). \tag{3.18}$$

Let us introduce $h(z, D) = e^{rT}\tilde{h}(z, D)$, the T-forward value of indifference price.

Using the duality result, we deduce that

$$h(z, D) = \sup_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}^Q\{D\} - \frac{1}{\gamma} H(Q|P) \right) - \sup_{Q \in \mathbb{P}_f(P)} \left(-\frac{1}{\gamma} H(Q|P) \right), \tag{3.19}$$

or, equivalently,

$$h(z, D) = \frac{1}{\gamma} \log \frac{u(0, D)}{u(0, 0)}. \tag{3.20}$$

As these formulas suggest, the indifference price does not depend on the initial wealth level which is computationally advantageous. From here on we omit the dependence in the notation. Following the definition of [21], [10] also studies (3.19) in Markovian models. Additionally, (3.19) appeared in the paper by Rouge and El Karoui [35] where they study the indifference price using backward stochastic differential equations.

3.3.2 Hedging Problem Solution given by the Fenchel-Legendre Transform of the Indifference Price

Returning to our original problem, we see that (3.6) is equivalent to finding

$$\alpha^* = \arg\max_{\alpha} u(v_0 + \alpha p - h(B^\alpha), 0). \tag{3.21}$$

To apply the duality result to B^α, we need to validate (3.13) and (3.14).

Remark 1 *For the case of B^α, (3.14) is trivially satisfied since it is bounded above by $\alpha K'$. Assuming that the forward price X is a* positive *$(P, \mathbb{F})$-semimartingale, we guarantee that for all $Q \in \mathbb{P}_a(P)$ X is a $(Q, \mathbb{F})$-supermartingale, and conclude that B^α is in $L^1(Q)$ by noting that*

$$\mathbb{E}^Q \{|B^\alpha|\} \leq \mathbb{E}^Q\{X_T\} + \alpha K' \leq X_0 + \alpha K' < \infty.$$

Since X is the forward price, positivity is a natural assumption.

From (3.20), it follows that

$$\alpha^* = \arg\max_{\alpha} (\alpha p - h(B^\alpha)) = \arg\max_{\alpha} \left(\alpha\tilde{p} - \tilde{h}(B^\alpha)\right). \tag{3.22}$$

Our hedging problem is thus recast as finding the Fenchel-Legendre transform of the indifference price $\tilde{h}$ of B^α as a function of α, evaluated at the market price $\tilde{p}$. From (3.19), the indifference price depends on α only through the supremum over $Q \in \mathbb{P}_f(P)$ of the *affine* function of α:

$$\alpha \mathbb{E}^Q\{P'\} + \mathbb{E}^Q\{B^0\} - \frac{1}{\gamma} H(Q|P).$$

Therefore the indifference price is convex in α. We discuss conditions that guarantee the existence and the strict convexity of $\tilde{h}(B^\alpha)$ (and thereby uniqueness of the solution α^* to the hedging problem) in Section 3.3.5.

3.3.3 Indifference Price via Relative Entropy Penalization

The indifference price given as the difference of two separate optimization problems in (3.19) can also be written as one optimization problem in a similar form if we are willing to change our reference measure from the real life measure to the minimal entropy martingale measure. The minimal entropy martingale measure, Q^E is defined as the measure in $\mathbb{P}_f(P)$ that minimizes the entropy with respect to the real life measure

$$Q^E = \arg\min_{Q\in\mathbb{P}_f(P)} H(Q|P).$$

Key results about the minimal entropy martingale measure can be found in Fritelli [17] and Grandits and Rheinländer [18]. We shall need the following lemma, an extended form of the Theorems 2.2-5 in [17] due to [25], re-written here in our notation.

Lemma 1 *(Theorem 2.2-5 of Fritelli [17] and Theorem 2.1 of [25]) Under the assumption (3.15), Q^E exists, is unique, is in $\mathbb{P}_f(P)\cap\mathbb{P}_e(P)$ and its density has the form*

$$\frac{\mathrm{d}Q^E}{\mathrm{d}P} = c^E e^{-\gamma V_T^{\pi^E}}, \tag{3.23}$$

where

$$\pi^E = \arg\max_{\pi\in\Theta(P)} \mathbb{E}\left\{-e^{-\gamma V_T^{\pi}}\right\}, \tag{3.24}$$

and

$$\log c^E = H(Q^E|P) < \infty.$$

We start by introducing the sets $\mathbb{P}_a(Q^E)$, $\mathbb{P}_e(Q^E)$, $\mathbb{P}_f(Q^E)$, $\Theta(Q^E)$ as the sets defined in similar ways to $\mathbb{P}_a(P)$, $\mathbb{P}_e(P)$, $\mathbb{P}_f(P)$, $\Theta(P)$ in (3.7), (3.11), (3.8), (3.10) respectively, with the reference measure changed to Q^E instead of P. We want to show that the indifference price $h(D)$ given by (3.19) can equivalently be found by (3.26) below. As the initial wealth does not play a role in the discussion of the indifference price, we take it to be zero. The modification for a nonzero wealth follows by subtracting the initial wealth from $V_T^{\pi^E}$. The result given in Lemma 2 allows us to specify the indifference price in terms of the extreme expected payout over a space of risk-neutral measures, but penalized by the entropy distance from a particular *prior* risk neutral measure, namely Q^E. For the proofs below we assume that P is not Q^E; in that case the results would follow trivially.

Lemma 2 *Assume (3.15) and*

$$\frac{\mathrm{d}Q^E}{\mathrm{d}P} \in L^2(P). \tag{3.25}$$

The indifference price of a contingent claim D that satisfies Assumption 1 is given by

$$h(D) = \sup_{Q\in\mathbb{P}_f(Q^E)} \left(\mathbb{E}^Q\{D\} - \frac{1}{\gamma} H(Q|Q^E) \right). \tag{3.26}$$

Further, if the following holds

$$\mathbb{E}\left\{e^{2\gamma D}\right\} < \infty, \tag{3.27}$$

then the indifference price of the claim D is given by

$$h(D) = \frac{1}{\gamma} \log \left(- \sup_{\pi\in\Theta(Q^E)} \mathbb{E}^{Q^E} \left\{ -e^{-\gamma(V_T^\pi - D)} \right\} \right). \tag{3.28}$$

Proof: From Lemma 1, Q^E exists, is unique, and is equivalent to P, under our assumption (3.15). For a measure $Q \ll P$, the simple equality

$$\mathbb{E}^Q \left\{ \log \frac{\mathrm{d}Q}{\mathrm{d}P} \right\} = \mathbb{E}^Q \left\{ \log \frac{\mathrm{d}Q}{\mathrm{d}Q^E} \right\} + \mathbb{E}^Q \left\{ \log \frac{\mathrm{d}Q^E}{\mathrm{d}P} \right\}$$

and (3.23) allow us to write the entropy of Q with respect to Q^E in terms of its entropy with respect to P

$$H(Q|P) = H(Q|Q^E) + H(Q^E|P) - \gamma \mathbb{E}^Q \left\{ V_T^{\pi^E} \right\}. \tag{3.29}$$

We first want to show that $\mathbb{P}_f(P) = \mathbb{P}_f(Q^E)$. If we choose Q in $\mathbb{P}_f(P)$, the last term on the right hand side of (3.29) is zero since π^E is in $\Theta(P)$. $H(Q^E|P)$ is finite since $Q^E \in \mathbb{P}_f(P)$ and $H(Q|P)$ is finite with our choice, therefore we conclude that $\mathbb{P}_f(P) \subset \mathbb{P}_f(Q^E)$ by noting $Q \ll P \sim Q^E$.

To observe the reverse equality, we fix Q in $\mathbb{P}_f(Q^E)$. Assuming for the moment that $e^{\gamma\left|V_T^{\pi^E}\right|}$ is in $L^1(Q^E)$, and using Lemma 3.5 of Delbaen *et al.* [10] for the random variable $\left|V_T^{\pi^E}\right|$, we write

$$\mathbb{E}^Q \left\{ \left| V_T^{\pi^E} \right| \right\} \leq H(Q|Q^E) + e^{-1} \mathbb{E}^{Q^E} \left\{ e^{\gamma\left|V_T^{\pi^E}\right|} \right\}.$$

Both of the terms on the right hand side are finite, therefore we have that $V_T^{\pi^E}$ is in $L^1(Q)$. Since Q is arbitrary in $\mathbb{P}_f(Q^E)$, and as we guarantee the finiteness of the last term on the right hand side of (3.29), with similar arguments to the first inclusion, we conclude that $\mathbb{P}_f(P) \supset \mathbb{P}_f(Q^E)$. But then the last term on the right hand side of (3.29) is zero for all $Q \in \mathbb{P}_f(Q^E)$. The assumption (3.25) with

$$\mathbb{E}^{Q^E} \left\{ e^{\gamma V_T^{\pi^E}} \right\} \;=\; \mathbb{E} \left\{ \frac{\mathrm{d}Q^E}{\mathrm{d}P} \, e^{\gamma V_T^{\pi^E}} \right\} = c^E < \infty$$

guarantees that $e^{\gamma\left|V_T^{\pi^E}\right|}$ is in $L^1(Q^E)$ because

$$\begin{aligned}
\mathbb{E}^{Q^E}\left\{e^{\gamma\left|V_T^{\pi^E}\right|}\right\} &= \mathbb{E}^{Q^E}\left\{e^{\gamma V_T^{\pi^E}}\mathbf{1}_{\{V_T^{\pi^E}>0\}}\right\} + \mathbb{E}^{Q^E}\left\{e^{-\gamma V_T^{\pi^E}}\mathbf{1}_{\{V_T^{\pi^E}\leq 0\}}\right\} \\
&\leq \mathbb{E}^{Q^E}\left\{e^{\gamma V_T^{\pi^E}}\right\} + \mathbb{E}^{Q^E}\left\{e^{-\gamma V_T^{\pi^E}}\right\} \\
&= c^E + \left(c^E\right)^{-1}\mathbb{E}\left\{\left(\frac{\mathrm{d}Q^E}{\mathrm{d}P}\right)^2\right\} < \infty,
\end{aligned}$$

where we have used (3.23) in the last step. Using

$$H(Q|P) = H(Q|Q^E) + H(Q^E|P) \tag{3.30}$$

in (3.19) we achieve (3.26).

If we can verify Assumption 1 for the prior Q^E, (3.28) follows from (3.26) by the duality result. As $Q^E \sim P$, (3.13) is trivial. Using (3.25), (3.27), and the Cauchy-Schwarz inequality, we have

$$\mathbb{E}^{Q^E}\left\{e^{\gamma D}\right\} \leq \sqrt{\mathbb{E}\left\{e^{2\gamma D}\right\}\mathbb{E}\left\{\left(\frac{\mathrm{d}Q^E}{\mathrm{d}P}\right)^2\right\}} < \infty.$$

□

We can take one more step in characterizing the indifference price and write it as a problem of minimizing entropy with respect to a certain (prior or reference) measure. First, let us introduce

$$\frac{\mathrm{d}P^{D,E}}{\mathrm{d}Q^E} = c^{D,E}e^{\gamma D}, \quad \text{with } (c^{D,E})^{-1} = \mathbb{E}^{Q^E}\left\{e^{\gamma D}\right\} \tag{3.31}$$

and the set $\mathbb{P}_f(P^{D,E})$ with its obvious definition.

Corollary 1 *Assume (3.15), (3.25), and (3.27). The indifference price of the claim D is given by*

$$h(D) = -\frac{1}{\gamma}\left(\inf_{Q\in\mathbb{P}_f(P^{D,E})} H(Q|P^{D,E}) + \log c^{D,E}\right). \tag{3.32}$$

PROOF: By (3.25) and (3.27), $c^{D,E}$ is in $(0,\infty)$ and $P^{D,E}$ is well-defined. For a measure $Q \ll Q^E$, the following holds

$$H(Q|Q^E) = H(Q|P^{D,E}) + \log c^{D,E} + \mathbb{E}^Q\{\gamma D\}. \tag{3.33}$$

For $Q \in \mathbb{P}_f(Q^E)$ the last term is finite by Assumption 1 as $Q^E \sim P$, which implies that $Q \in \mathbb{P}_f(P^{D,E})$. As $P^{D,E} \sim Q^E$ the converse implication follows similarly. Using (3.33) in (3.26) we conclude. □

Delbaen *et al.* [10] use this approach to reduce the problem of proving duality with a contingent claim to the simpler case without a claim. In their case the prior measure is P rather than Q^E as here and introducing P^D they work with the simplified version of the problem.

3.3.4 Strict Convexity of the Indifference Price

The Fenchel-Legendre transform of the indifference price that we aim to find is given by direct differentiation if the indifference price is strictly convex in α. We start by showing that $h(\alpha D)$ is differentiable in α for a bounded payoff D and combine this result with the known properties of the indifference pricing mechanism to find a sufficient condition that guarantees strict convexity.

Proposition 1 *Assume that D is bounded. The indifference price $h(\alpha D)$ is differentiable in $\alpha \in \mathbb{R}$.*

Proof: We need to show that

$$\lim_{\epsilon \downarrow 0} \frac{h((\alpha+\epsilon)D) - h(\alpha D)}{\epsilon} = \lim_{\epsilon \uparrow 0} \frac{h((\alpha+\epsilon)D) - h(\alpha D)}{\epsilon} = c. \tag{3.34}$$

We start by calculating the first term which is equal to

$$\lim_{\epsilon \downarrow 0} \frac{1}{\gamma \epsilon} \log \left(\frac{\sup_{\pi \in \Theta(P)} \mathbb{E}\{-e^{-\gamma(V_T^\pi - (\alpha+\epsilon)D)}\}}{\sup_{\pi \in \Theta(P)} \mathbb{E}\{-e^{-\gamma(V_T^\pi - \alpha D)}\}} \right) \tag{3.35}$$

by (3.20).

We introduce

$$\frac{\mathrm{d}P^{\alpha D}}{\mathrm{d}P} = c^{\alpha D} e^{\gamma \alpha D}, \quad \text{with } (c^{\alpha D})^{-1} = \mathbb{E}\{e^{\gamma \alpha D}\} \in (0, \infty). \tag{3.36}$$

With the obvious definitions of $\mathbb{P}_f(P^{\alpha D})$ and $\Theta(P^{\alpha D})$, we note that $\mathbb{P}_f(P^{\alpha D})$ and $\mathbb{P}_f(P)$ are equal. This follows from the boundedness of D and

$$H(Q|P) = H(Q|P^{\alpha D}) + \log c^{\alpha D} + \mathbb{E}^Q\{\gamma \alpha D\}, \tag{3.37}$$

which also implies that $\Theta(P^{\alpha D})$ and $\Theta(P)$ are equal.

In terms of $P^{\alpha D}$ (3.35) can be expressed as

$$\lim_{\epsilon \downarrow 0} \frac{1}{\gamma \epsilon} \log \left(\frac{\sup_{\pi \in \Theta(P^{\alpha D})} \mathbb{E}^{P^{\alpha D}}\{-e^{-\gamma(V_T^\pi - \epsilon D)}\}}{\sup_{\pi \in \Theta(P^{\alpha D})} \mathbb{E}^{P^{\alpha D}}\{-e^{-\gamma V_T^\pi}\}} \right). \tag{3.38}$$

This expression is the limit as ϵ goes to zero of the indifference price of ϵD options judged by an investor with subjective measure $P^{\alpha D}$ (compare with (3.20)). Since Assumption 1 and Assumption 2 are satisfied with the prior $P^{\alpha D}$ for $\alpha \in \mathbb{R}$, using (3.16) we re-write (3.38) as

$$\lim_{\epsilon \downarrow 0} \sup_{Q \in \mathbb{P}_f(P^{\alpha D})} \left(\mathbb{E}^Q\{D\} - \frac{1}{\gamma \epsilon} H(Q|P^{\alpha D}) \right) - \sup_{Q \in \mathbb{P}_f(P^{\alpha D})} \left(\frac{1}{\gamma \epsilon} H(Q|P^{\alpha D}) \right). \tag{3.39}$$

Taking the limit as ϵ goes to zero is equivalent to taking the limit as the risk aversion parameter goes to zero with the prior $P^{\alpha D}$ fixed. For a bounded payoff, Proposition 1.3.4 in [2] proves that as the risk aversion parameter goes to zero, the indifference price goes to the expectation of the payoff under the minimal entropy martingale measure. In other words, the limit in (3.39) is equal to

$$\mathbb{E}^{Q^{\alpha D, E}}\{D\}$$

where $Q^{\alpha D, E}$ is the measure in $\mathbb{P}_f(P^{\alpha D})$ minimizing the relative entropy with respect to $P^{\alpha D}$. The existence and uniqueness of this measure follows from Lemma 1 and Assumption 2. Paying special attention to the direction of the limit, it is straightforward to see that the second term in (3.34) is given by

$$-\lim_{\epsilon \downarrow 0} \sup_{Q \in \mathbb{P}_f(P^{\alpha D})} \left(\mathbb{E}^Q\{-D\} - \frac{1}{\gamma \epsilon} H(Q|P^{\alpha D}) \right) - \sup_{Q \in \mathbb{P}_f(P^{\alpha D})} \left(-\frac{1}{\gamma \epsilon} H(Q|P^{\alpha D}) \right) = \mathbb{E}^{Q^{\alpha D, E}}\{D\}.$$

□

We have seen that $h(\alpha D)$ is a convex and differentiable function of α. In the next proposition, we give a sufficient condition, which guarantees that the indifference price is in fact a strictly convex function of α.

Proposition 2 *Assume D is bounded. The indifference price $h(\alpha D)$ is a strictly convex function of α on $\mathbb{R}$ if the following holds*

$$\inf_{Q \in \mathbb{P}_e(P)} \mathbb{E}^Q\{D\} < \sup_{Q \in \mathbb{P}_e(P)} \mathbb{E}^Q\{D\}. \tag{3.40}$$

PROOF: Pick α_1, $\alpha_2 \in \mathbb{R}$ such that $\alpha_2 > \alpha_1$ and assume that the indifference price of αD is a linear function of α on the line segment between the two. We define $P^1 = P^{\alpha_1 D}$ and $P^2 = P^{\alpha_2 D}$ as in (3.36), and Q^1 and Q^2 as the measures that minimize entropy with respect to P^1 and P^2, respectively. From (3.37), we obtain

$$(H(Q^2|P^1) - H(Q^1|P^1)) + ((H(Q^1|P^2) - H(Q^2|P^2)) = \gamma(\alpha_2 - \alpha_1)(\mathbb{E}^{Q^1}\{D\} - \mathbb{E}^{Q^2}\{D\}).$$

As $\mathbb{E}^{Q^i}\{D\}$ is the slope of the the indifference price at α_i, the right hand side is equal to zero by our linearity assumption. The measure Q^1 is the minimizer of $H(Q|P^1)$ over $\mathbb{P}_f(P)$

which includes Q^2, so the first term in the left hand side is nonnegative. The same conclusion applies to the second term, therefore both terms are zero. Then the uniqueness of the minimal entropy martingale measure implies that $Q^1 = Q^2$.

Using (3.36) with α_i, and Lemma 1, the density of Q^i can be specified as follows

$$\frac{\mathrm{d}Q^i}{\mathrm{d}P} = e^{c_i - \gamma(V_T^{\pi_i} - \alpha_i D)}, \text{ for } i = 1, 2,$$

where π_1 and π_2 are two trading strategies in $\Theta(P^i) = \Theta(P)$. Combining this density representation with the equality of Q^1 and Q^2, we find

$$(\alpha_1 - \alpha_2)D = \text{const} + V_T^{\pi_1} - V_T^{\pi_2}.$$

Then for all $Q \in \mathbb{P}_f(P)$, $\mathbb{E}^Q\{D\}$ is a constant. For bounded D, Corollary 5.1 in [10] states that the supremum of $\mathbb{E}^Q\{D\}$ over $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$ is equal to the supremum over the enlarged set $\mathbb{P}_e(P)$. A similar result follows for the infimum as D is bounded, contradicting the assumption in (3.40). □

The solution of the minimization problem on the left hand side of (3.40) is the sub-hedging price of the claim D, and the solution of the maximization problem on the right hand side is the super-hedging price of the claim D.

3.3.5 Indifference Price of the Put-Barrier Position

To see that the result of Proposition 1 carries over to the case of B^α, namely to conclude that $h(\alpha P' + B^0)$ is differentiable in α, it is enough to change the definition of $P^{\alpha D}$ introduced in (3.36) to

$$\frac{\mathrm{d}P^{B^\alpha}}{\mathrm{d}P} = c^{B^\alpha} e^{\gamma B^\alpha}, \quad \text{with } (c^{B^\alpha})^{-1} = \mathbb{E}\{e^{\gamma B^\alpha}\}.$$

From Remark 1, this measure is well-defined and B^α satisfies Assumption 1. The rest of the proof of Proposition 1 follows and the derivative of the indifference price is given by the expectation of P' under the measure that minimizes the entropy with respect to P^{B^α}.

The modification of Proposition 2 does not follow trivially unlike in the previous case.

Proposition 3 *The indifference price $h(B^\alpha)$ is a strictly convex function of α on $\mathbb{R}$ if the following holds*

$$\inf_{Q \in \mathbb{P}_e(P)} \mathbb{E}^Q\{P'\} < \sup_{Q \in \mathbb{P}_e(P)} \mathbb{E}^Q\{P'\}. \tag{3.41}$$

PROOF: From Remark 1 and (3.16), the indifference price of B^α can be written as

$$\begin{aligned} h(B^\alpha) &= \sup_{Q \in \mathbb{P}_f(P^{B^0})} \left(\alpha\, \mathbb{E}^Q\{P'\} - \frac{1}{\gamma} H\left(Q | P^{B^0}\right) + \frac{1}{\gamma} H\left(Q^{B^0,E} | P^{B^0}\right) \right) \\ &\quad - \frac{1}{\gamma}\left(H\left(Q^{B^0,E} | P^{B^0}\right) + \log c^{B^0} - H(Q^E|P) \right). \end{aligned} \tag{3.42}$$

The supremum in the right hand side of (3.42) is the indifference price of α puts judged with subjective measure P^{B^0} and, from Proposition 2, is a strictly convex function of α if

$$\inf_{Q\in\mathbb{P}_e(P^{B^0})} \mathbb{E}^Q\{P'\} < \sup_{Q\in\mathbb{P}_e(P^{B^0})} \mathbb{E}^Q\{P'\}.$$

As P^{B^0} is equivalent to P and the rest of the terms in (3.42) are independent of α, the result follows. □

Proposition 4 *The optimal number of put options to trade, α^*, defined in (3.22) exists if the market price p is between the super-hedging price and the sub-hedging price of the put option:*

$$\inf_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\left\{P'\right\} < p < \sup_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\left\{P'\right\}. \tag{3.43}$$

Proof: Corollary 5.1 in [10] shows that for a bounded payoff D

$$\lim_{\gamma\uparrow\infty} h(\alpha D,\gamma) = \sup_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\{D\}.$$

From (3.19), it is easy to see that for $\alpha > 0$, $h(\alpha D,\gamma) = \alpha h(D,\alpha\gamma)$ and as in Corollary 1.3.5 of Becherer [2] we conclude

$$\lim_{\alpha\uparrow\infty} \frac{1}{\alpha} h(\alpha D) = \sup_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\{D\}. \tag{3.44}$$

Applying this result with the bounded put payoff, P', and with the prior P^{B^0}, from (3.42) we get

$$\lim_{\alpha\uparrow\infty} \frac{1}{\alpha} h(B^\alpha) = \sup_{Q\in\mathbb{P}_e(P^{B^0})} \mathbb{E}^Q\{P'\} = \sup_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\{P'\}$$

as $P \sim P^{B^0}$.

A similar result for the reverse limit and the sub-hedging price follows from

$$\lim_{\alpha\downarrow-\infty} \frac{1}{\alpha} h(B^\alpha) = -\lim_{\alpha\uparrow\infty} \frac{1}{\alpha} h(B^{-\alpha}) = -\sup_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\left\{-P'\right\} = \inf_{Q\in\mathbb{P}_e(P)} \mathbb{E}^Q\left\{P'\right\}.$$

As $h(B^\alpha)$ is a convex function of α, (3.43) is enough to guarantee the existence of α^* in (3.22).
□

4 Stochastic Volatility Models

It is now widely accepted that the Black-Scholes model given in (1.1) is inadequate to capture properties observed in returns and option prices empirically. One of the natural extensions is relaxing the assumption of deterministic coefficients. Following [20, 22], for example, we model the volatility as another stochastic process having an arbitrary correlation with the stock price process. For details on how these models better describe the market, we refer the reader to Fouque *et al.* [15].

In the context of diffusion models and exponential utility, indifference pricing has been studied by Davis in [9]. He modelled the prices of two highly correlated assets as geometric Brownian motions and covered the case of claims written on the non-traded asset where the other correlated asset is available for hedging. In a similar model to Davis [9], in the case of power utility Zariphopoulou [38] studied the prices of contingent claims using PDE methods. Henderson [19] and Musiela and Zariphopoulou [31] established utility indifference price as an expectation under the minimal entropy martingale measure within the same model in the case of exponential utility. In the case of power utility, Pham [33] proved the existence of a smooth solution to the optimal investment (Merton) problem within a stochastic volatility model. His case does not involve a claim. In [37], Sircar and Zariphopoulou studied the utility indifference price of European options in a stochastic volatility framework. They give the price as a solution to a second order quasilinear PDE, they propose bounds for the price and analyze the problem by asymptotic methods in the limit of the volatility being a fast mean reverting process. The *no arbitrage* pricing of barrier options under stochastic volatility models, as well as lookback and passport options, which can also be characterized by boundary value problems, is studied in [23].

We introduce a volatility driving process $(Y_t)_{0\leq t\leq T}$ and leave the dependence of volatility on this process generic up to regularity conditions. The stock price process and the volatility driving process are solutions of the following stochastic differential equations

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma(t,Y_t)S_t\,\mathrm{d}W_t^1 \qquad S_0 = xe^{-rT}, \tag{4.1}$$

$$\mathrm{d}Y_t = b(t,Y_t)\,\mathrm{d}t + a(t,Y_t)\left(\rho\,\mathrm{d}W_t^1 + \rho'\,\mathrm{d}W_t^2\right) \qquad Y_0 = y, \tag{4.2}$$

where W^1 and W^2 are two independent Brownian motions on the given space and the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is assumed to be the augmented filtration generated by these two processes. The parameter ρ controls the instantaneous correlation between shocks to S and Y, and $\rho' = \sqrt{1-\rho^2}$. We assume that $a(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are bounded above and below away from zero, and smooth with bounded derivatives. We also assume that $b(\cdot,\cdot)$ is smooth with bounded derivatives. The forward stock price process is the unique solution of

$$\mathrm{d}X_t = (\mu - r)X_t\,\mathrm{d}t + \sigma(t,Y_t)X_t\,\mathrm{d}W_t^1, \qquad X_0 = x,$$

We now derive the PDE ((4.13) below) that the indifference pricing function $\phi(t,x,y)$ solves. The indifference price at time zero is given by $h(B^\alpha) = \phi(0,x,y)$. We start by finding the minimal entropy martingale measure, Q^E.

4.1 Minimal Entropy Martingale Measure

The well-known minimal martingale measure P^0 which is defined by

$$\frac{\mathrm{d}P^0}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu - r}{\sigma(s, Y_s)} \mathrm{d}W_s^1 - \frac{1}{2}\int_0^T \frac{(\mu - r)^2}{\sigma^2(s, Y_s)} \mathrm{d}s\right)$$

is equivalent to P and the forward price X is a P^0-local martingale. The relative entropy of P^0 with respect to P is given by

$$H(P^0|P) = \mathbb{E}^{P^0}\left\{\frac{1}{2}\int_0^T \frac{(\mu - r)^2}{\sigma^2(s, Y_s)} \mathrm{d}s\right\}$$

and is finite by the assumptions on σ. Therefore, $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$ is non-empty and we know that Q^E exists and is equivalent to P. Without loss of generality, we consider the set over which the optimization takes place as $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$.

We denote by $\Lambda(P)$ the set of adapted processes λ such that $\int_0^T \lambda_t^2 \,\mathrm{d}t < \infty$ P-a.s. For any $P^\lambda \in \mathbb{P}_e(P)$, X is a P^λ-local martingale hence its drift is zero under P^λ. By the Cameron-Martin-Girsanov theorem, we conclude that the density of P^λ has the form

$$\frac{\mathrm{d}P^\lambda}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu - r}{\sigma(s, Y_s)} \mathrm{d}W_s^1 + \int_0^T \lambda_s \,\mathrm{d}W_s^2 - \frac{1}{2}\int_0^T \left(\frac{(\mu - r)^2}{\sigma^2(s, Y_s)} + \lambda_s^2\right) \mathrm{d}s\right)$$

for some $\lambda \in \Lambda(P)$.

Since Q^E is in $\mathbb{P}_f(P) \cap \mathbb{P}_e(P)$, there exists $\lambda^E \in \Lambda(P)$ such that Q^E is equal to P^{λ^E}. Under Q^E, X and Y satisfy

$$\begin{aligned}
\mathrm{d}X_t &= \sigma(t, Y_t) X_t \,\mathrm{d}W_t^{E,1}, \\
\mathrm{d}Y_t &= \left(b(t, Y_t) - \rho a(t, Y_t)\frac{\mu - r}{\sigma(t, Y_t)} + \rho' a(t, Y_t)\lambda_t^E\right) \mathrm{d}t + a(t, Y_t)\left(\rho \,\mathrm{d}W_t^{E,1} + \rho' \,\mathrm{d}W_t^{E,2}\right),
\end{aligned}$$

where $W^{E,1}$, and $W^{E,2}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, Q^E)$ defined by

$$\begin{aligned}
\mathrm{d}W_t^{E,1} &= \mathrm{d}W_t^1 + \frac{\mu - r}{\sigma(t, Y_t)} \mathrm{d}t, \\
\mathrm{d}W_t^{E,2} &= \mathrm{d}W_t^2 - \lambda_t^E \,\mathrm{d}t.
\end{aligned}$$

Next we construct a candidate for the minimal martingale measure, which we call $P^{c,E} = P^{\lambda^{c,E}}$. For λ in $\mathcal{H}^2(P^\lambda)$, where $\mathcal{H}^2(Q)$ consists of all adapted processes u that satisfy the integrability constraint $\mathbb{E}^Q\left\{\int_0^T u_t^2 \,\mathrm{d}t\right\} < \infty$, the relative entropy $H(P^\lambda|P)$ is given by

$$H(P^\lambda|P) = \mathbb{E}^{P^\lambda}\left\{\frac{1}{2}\int_0^T \left(\frac{(\mu - r)^2}{\sigma^2(s, Y_s)} + \lambda_s^2\right) \mathrm{d}s\right\}.$$

Let

$$\psi(t, y) = \sup_{\lambda \in \mathcal{H}^2(P^\lambda)} \mathbb{E}^{P^\lambda}\left\{-\frac{1}{2}\int_t^T \left(\frac{(\mu - r)^2}{\sigma^2(s, Y_s)} + \lambda_s^2\right) \mathrm{d}s \,\Big|\, Y_t = y\right\}. \tag{4.3}$$

The associated Hamilton-Jacobi-Bellman (HJB) equation for $\psi(t,y)$ is

$$\begin{aligned}\psi_t + \mathcal{L}_y^0\psi + \max_\lambda\left(\rho' a(t,y)\lambda\psi_y - \frac{1}{2}\lambda^2\right) &= \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(t,y)}, \quad t<T,\\ \psi(T,y) &= 0,\end{aligned}$$

where $\mathcal{L}_y^0$ is the infinitesimal generator of the process (Y_t) under P^0 and is given by

$$\mathcal{L}_y^0 = \frac{1}{2}a^2(t,y)\frac{\partial^2}{\partial y^2} + \left(b(t,y) - \rho a(t,y)\frac{\mu-r}{\sigma(t,y)}\right)\frac{\partial}{\partial y}.$$

Evaluating the maximum in the HJB equation, we have

$$\begin{aligned}\psi_t + \mathcal{L}_y^0\psi + \frac{1}{2}\rho'^2 a^2(t,y)(\psi_y)^2 &= \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(t,y)}, \quad t<T,\\ \psi(T,y) &= 0,\end{aligned} \tag{4.4}$$

with the corresponding optimal control

$$\lambda_t^{c,E} = \rho'\, a(t,Y_t)\psi_y(t,Y_t). \tag{4.5}$$

The quasilinear PDE (4.4) can be linearized by the Hopf-Cole transformation (see [13]):

$$\psi(t,y) = \frac{1}{(1-\rho^2)}\log f(t,y).$$

Then f satisfies

$$\begin{aligned}f_t + \mathcal{L}_y^0 f &= (1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2(t,y)}f, \quad t<T,\\ f(T,y) &= 1.\end{aligned} \tag{4.6}$$

Using Theorem II.9.10 in [27], we have

$$f(t,y) = \mathbb{E}^{P^0}\left\{\exp\left(-\int_t^T \frac{(\mu-r)^2(1-\rho^2)}{2\sigma^2(s,Y_s)}\,\mathrm{d}s\right)\Big|Y_t=y\right\}$$

as the unique solution to (4.6) which is continuously differentiable once with respect to t and twice with respect to y. From this probabilistic representation, we see that f is bounded above and away from zero under our assumptions on σ.

As ψ is given by logarithmic transformation of f

$$\psi(t,y) = \frac{1}{(1-\rho^2)}\log\mathbb{E}^{P^0}\left\{\exp\left(-\int_t^T \frac{(\mu-r)^2(1-\rho^2)}{2\sigma^2(s,Y_s)}\,\mathrm{d}s\right)\Big|Y_t=y\right\}, \tag{4.7}$$

it is bounded and satisfies the same differentiability conclusions as f, and its optimality can be concluded by the verification Theorem IV.3.1 in Fleming and Soner [14]: the value function (4.3) is given by (4.7).

Taking the derivative of (4.6) with respect to y and using the probabilistic representation of the solution in a similar way, under the conditions on the coefficients, we conclude that $\psi_y(t,y)$ and hence $\lambda^{c,E}(t,y)$ are bounded. Therefore, $\lambda^{c,E}$ defined in (4.5) is an optimizer by the verification Theorem IV.3.1 in [14]. The Novikov condition is satisfied, hence $\frac{\mathrm{d}P^{c,E}}{\mathrm{d}P}$ is a P-martingale, and $\frac{\mathrm{d}P^{c,E}}{\mathrm{d}P}$ is the density of an equivalent martingale measure. The entropy of $P^{c,E}$ can be recovered from $H(P^{c,E}|P) = -\psi(0,y)$.

We next verify that $\lambda^{c,E}$ is equal to λ^E, or equivalently our candidate measure $P^{c,E}$ is the minimal entropy martingale measure. To prove the result, we use Proposition 3.2 in [18]. A similar argument appears in [3] using the results in [34], but for stochastic volatility models where the volatility process may be unbounded above and may become zero.

Proposition 5 *(Proposition 3.2 of Grandits and Rheinländer [18]) Assume there exists* $\widetilde{Q} \in \mathbb{P}_e(P) \cap \mathbb{P}_f(P)$. *Then* $\widetilde{Q} = Q^E$ *if and only if the following hold:*

(i)

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}P} = e^{c+\int_0^T \nu_t \mathrm{d}X_t}, \tag{4.8}$$

for a constant c *and* X*-integrable* ν*,*

(ii) $\mathbb{E}^Q\left\{\int_0^T \nu_t\,\mathrm{d}X_t\right\} = 0$ *for* $Q = \widetilde{Q}, Q^E$.

Applying Itô's formula to ψ, which has the necessary smoothness properties, and using the fact that $\psi(T,y)$ is equal to zero for all $y \in \mathbb{R}$ and satisfies the PDE (4.4), we deduce that $\frac{\mathrm{d}P^{c,E}}{\mathrm{d}P}$ has the form given in (4.8) with

$$\nu_t = -\frac{1}{\sigma(t,Y_t)X_t}\left(\rho a(t,Y_t)\psi_y(t,Y_t) + \frac{\mu - r}{\sigma(t,Y_t)}\right) \tag{4.9}$$

and $c = -\psi(0,y)$. We refer the reader to the proof of Theorem 3.3 in [3] for the detailed calculations. For $P^\lambda \in \mathbb{P}_e(P)$, recall that

$$\mathrm{d}X_t = \sigma(t,Y_t)X_t\,\mathrm{d}W_t^\lambda,$$

where W^λ is a Brownian motion on $(\Omega, \mathbb{F}, P^\lambda)$. For ν given in (4.9),

$$\mathbb{E}^{P^\lambda}\left\{\int_0^T \nu_t\,\mathrm{d}X_t\right\} = -\mathbb{E}^{P^\lambda}\left\{\int_0^T \left(\rho a(t,Y_t)\psi_y(t,Y_t) + \frac{\mu - r}{\sigma(t,Y_t)}\right)\mathrm{d}W_t^\lambda\right\} = 0$$

under the assumptions on the diffusion coefficients. As $Q^E \in \mathbb{P}_e(P)$, condition *(ii)* in Proposition 5 is satisfied and we conclude that $\lambda^{c,E}$ is equal to λ^E and $P^{c,E}$ is the minimal entropy martingale measure.

4.2 Indifference Price

Our second step is finding the indifference price of $h(B^\alpha)$ as defined in (3.26). We start by noting that Q^E as we found in the previous section satisfies (3.25) and (3.27) as σ^{-1} and λ^E are bounded in addition to B^α being bounded above. From Corollary 1, we know that the maximizing measure in (3.26), which we call P^α, exists and is unique. Similar to the previous section, we aim to characterize this measure. We follow the steps in the previous section now with an option included and the prior measure changed to Q^E. For any P^λ in $\mathbb{P}_e(Q^E)$, there is a $\lambda \in \Lambda(Q^E)$ such that the Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}P^\lambda}{\mathrm{d}Q^E} = \exp\left(\int_0^T \lambda_s \,\mathrm{d}W_s^{E,2} - \frac{1}{2}\int_0^T \lambda_s^2 \,\mathrm{d}s\right). \tag{4.10}$$

Since P^α is in $\mathbb{P}_f(Q^E) \cap \mathbb{P}_e(Q^E)$, there exists $\lambda^\alpha \in \Lambda(Q^E)$ such that P^α is equal to P^{λ^α}. Under the new measure P^α, X_t and Y_t satisfy

$$\begin{aligned}
\mathrm{d}X_t &= \sigma(t,Y_t)X_t \,\mathrm{d}W_t^{\alpha,1}, \\
\mathrm{d}Y_t &= \left(b(t,Y_t) - \rho a(t,Y_t)\frac{\mu - r}{\sigma(t,Y_t)} + \rho'^2 a^2(t,Y_t)\psi_y(t,Y_t) + \rho' a(t,Y_t)\lambda_t^\alpha\right)\mathrm{d}t \\
&\quad + a(t,Y_t)\left(\rho \,\mathrm{d}W_t^{\alpha,1} + \rho' \,\mathrm{d}W_t^{\alpha,2}\right),
\end{aligned}$$

where $W^{\alpha,1}$, and $W^{\alpha,2}$ are two independent Brownian motions on $(\Omega, \mathcal{F}, P^\alpha)$ defined by

$$\begin{aligned}
\mathrm{d}W_t^{\alpha,1} &= \mathrm{d}W_t^{E,1}, \\
\mathrm{d}W_t^{\alpha,2} &= \mathrm{d}W_t^{E,2} - \lambda_t^\alpha \,\mathrm{d}t.
\end{aligned}$$

We first find a candidate measure, $P^{c,\alpha}$, in the set of equivalent martingale measures with $\lambda \in \mathcal{H}^2(P^\lambda)$. For such λ,

$$H(P^\lambda|Q^E) = \mathbb{E}^{P^\lambda}\left\{\frac{1}{2}\int_0^T \lambda_s^2 \,\mathrm{d}s\right\}.$$

Let us introduce

$$\phi(t,x,y) = \sup_{\lambda \in \mathcal{H}^2(P^\lambda)} \mathbb{E}^{P^\lambda}\left\{\alpha P' - C\mathbf{1}_{\{\tau_t < T\}} - \frac{1}{2\gamma}\int_t^T \lambda_s^2 \,\mathrm{d}s \Big| X_t = x, Y_t = y\right\}, \tag{4.11}$$

where

$$\tau_t = \min\left\{\inf\left\{u \geq t : e^{-r(T-u)}X_u \leq B\right\}, T\right\}.$$

Recall that P' is the payoff of the put option with strike K' and C is the payoff of the call option with strike K. For $x \leq Be^{r(T-t)}$, the option is 'knocked in', and $\phi(t,x,y) = \Phi(t,x,y)$, solution of the following HJB PDE problem on the full $x > 0$ domain:

$$\begin{aligned}
\Phi_t + \mathcal{L}_{x,y}^E \Phi + \frac{1}{2}\gamma\rho'^2 a^2(t,y)(\Phi_y)^2 &= 0, \quad t < T, \quad x > 0, \\
\Phi(T,x,y) &= \alpha(K' - x)^+ - (x - K)^+.
\end{aligned} \tag{4.12}$$

For $x > Be^{r(T-t)}$, the corresponding HJB problem for ϕ is:

$$\begin{aligned} \phi_t + \mathcal{L}^E_{x,y}\phi + \frac{1}{2}\gamma\rho'^2 a^2(t,y)(\phi_y)^2 &= 0, \quad t < T, \quad x > Be^{r(T-t)}, \qquad (4.13) \\ \phi(T,x,y) &= \alpha(K'-x)^+, \\ \phi(t, Be^{r(T-t)}, y) &= \Phi(t, Be^{r(T-t)}, y). \qquad (4.14) \end{aligned}$$

In (4.12) and (4.13), $\mathcal{L}^E_{x,y}$ is the generator of (X_t, Y_t) under Q^E,

$$\mathcal{L}^E_{x,y} = \mathcal{L}^0_y + \rho'^2 a^2(t,y)\psi_y(t,y)\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2(t,y)x^2\frac{\partial^2}{\partial x^2} + \rho\sigma(t,y)a(t,y)x\frac{\partial^2}{\partial x \partial y}.$$

Intuitively, the barrier boundary condition (4.14) arises because when the stock price hits the barrier, the barrier crossing proviso is fulfilled, and the problem reduces to finding the indifference price of the vanilla options. Once the maturity is reached, and there is no time left to trade, the holder is left with her payoff from the put options, and the barrier option does not contribute.

We will denote by R the value process at time $t \geq 0$ for the problem initiated at time zero:

$$R_t = \begin{cases} \phi(t, X_t, Y_t), & t \leq \tau_0, \\ \Phi(t, X_t, Y_t), & \text{otherwise.} \end{cases}$$

Our candidate measure that solves (3.26) in the stochastic volatility model with D replaced by B^α is $P^{c,\alpha} = P^{\lambda^{c,\alpha}}$, where

$$\lambda^{c,\alpha}_t = \begin{cases} \gamma\rho'\, a(t,Y_t)\phi_y(t,X_t,Y_t), & t \leq \tau_0, \\ \gamma\rho'\, a(t,Y_t)\Phi_y(t,X_t,Y_t), & \text{otherwise.} \end{cases} \qquad (4.15)$$

Unlike the previous case with no claims, there are no explicit solutions of (4.13) and (4.12). Existence of unique classical solutions to these equations in the class of functions that are continuously differentiable once with respect to t and twice with respect to x and y follows by adapting the analysis of the classical quadratic cost control problem [13] to the case of unbounded controls (see [33] for example). We will assume that the partial derivatives Φ_y and ϕ_y are bounded. By differentiating the PDEs (4.12) and (4.13), along with their respective boundary conditions, with respect to x, it is straightforward to derive *linear* PDE problems for Φ_x and ϕ_x with bounded boundary conditions. Our assumptions on the coefficients and the y-derivatives of Φ and ϕ, and the probabilistic representation of the solutions of these PDEs then imply that ϕ_x and Φ_x are also bounded. Consequently, $\lambda^{c,\alpha}$ is bounded and defines an equivalent martingale measure with finite entropy as the Novikov condition guarantees that $\frac{\mathrm{d}P^{c,\alpha}}{\mathrm{d}P}$ is a P-martingale.

When there is no claim, Proposition 5 was useful in stating the optimality of $P^{c,E}$. In the present case, to be able to use the same proposition, we recall that finding the indifference price is equivalent to minimizing the entropy with respect to a claim dependent prior. In particular, Corollary 1 implies that P^{α} minimizes the entropy with respect to the prior $P^{B^{\alpha},E}$, where $P^{B^{\alpha},E}$ is defined as in (3.31) with B^{α} replacing D. Therefore, we show that our candidate measure, $P^{c,\alpha}$, minimizes entropy with respect to $P^{B^{\alpha},E}$ and conclude by the uniqueness of the minimal entropy martingale measure.

We would like to write the density $\frac{\mathrm{d}P^{c,\alpha}}{\mathrm{d}P^{B^{\alpha},E}}$ in the form of (4.8). In a slight abuse of notation, let us define

$$R_{y,t} = \begin{cases} \phi_y(t, X_t, Y_t), & t \leq \tau_0, \\ \Phi_y(t, X_t, Y_t), & \text{otherwise}, \end{cases}$$

with $R_{x,t}$ defined analogously with x-derivatives of ϕ and Φ. Therefore, $\lambda_t^{c,\alpha} = \gamma\rho' a(t, Y_t) R_{y,t}$ and

$$\frac{\mathrm{d}P^{c,\alpha}}{\mathrm{d}Q^E} = \exp\left(\int_0^T \gamma\rho'\, a(t, Y_t) R_{y,t}\, \mathrm{d}W_t^{E,2} - \frac{1}{2}\int_0^T \gamma^2\rho'^2\, a^2(t, Y_t) R_{y,t}^2\, \mathrm{d}t\right). \tag{4.16}$$

We first apply Itô's formula to ϕ, which has the necessary smoothness, and also substitute from (4.13) to obtain

$$\begin{aligned} \phi(\tau_0, X_{\tau_0}, Y_{\tau_0}) \;=\; & \phi(0, x, y) - \int_0^{\tau_0} \frac{1}{2}\gamma\rho'^2 a^2(t, Y_t)(\phi_y(t, X_t, Y_t))^2 \mathrm{d}t \\ & + \int_0^{\tau_0} \rho' a(t, Y_t)\phi_y(t, X_t, Y_t) \mathrm{d}W_t^{E,2} \\ & + \int_0^{\tau_0} \left[\sigma(t, Y_t) X_t \phi_x(t, X_t, Y_t) + \rho a(t, Y_t)\phi_y(t, X_t, Y_t)\right] \mathrm{d}W_t^{E,1}, \end{aligned} \tag{4.17}$$

and similarly for Φ from τ_0 to T using its PDE (4.12). From these, we obtain

$$\begin{aligned} \int_0^T \frac{1}{2}\gamma\rho'^2 a^2(t, Y_t) R_{y,t}^2 \mathrm{d}t = R_0 - R_T + & \int_0^T \rho' a(t, Y_t) R_{y,t} \mathrm{d}W_t^{E,2} \\ + & \int_0^T \left[\sigma(t, Y_t) X_t R_{x,t} + \rho a(t, Y_t) R_{y,t}\right] \mathrm{d}W_t^{E,1}. \end{aligned} \tag{4.18}$$

Since R_T is equal to $\alpha P' - C$ if $\tau_0 < T$, and is equal to $\alpha P'$ otherwise, it is equal to the barrier option payoff B^{α}. Substituting from (4.18) for the second integral in (4.16), and performing the further measure change to prior $P^{B^{\alpha},E}$, we write $\frac{\mathrm{d}P^{c,\alpha}}{\mathrm{d}P^{B^{\alpha},E}}$ in the form (4.8), with

$$\nu_t = -\gamma\left(R_{x,t} + \frac{\rho a(t, Y_t) R_{y,t}}{\sigma(t, Y_t) X_t}\right) \tag{4.19}$$

and $c = -\log\left(\mathbb{E}^{Q^E}\{\gamma B^{\alpha}\}\right) - \gamma R_0 = -\log\left(\mathbb{E}^{Q^E}\{\gamma B^{\alpha}\}\right) - \gamma\phi(0, x, y)$.

For $P^{\lambda} \in \mathbb{P}_e(Q^E)$, recall that

$$\mathrm{d}X_t = \sigma(t, Y_t) X_t\, \mathrm{d}W_t^{\lambda},$$

where W^λ is a Brownian motion on $(\Omega, \mathbb{F}, P^\lambda)$. For ν given in (4.19),

$$\mathbb{E}^{P^\lambda}\left\{\int_0^T \nu_t \,\mathrm{d}X_t\right\} \quad = \quad -\gamma\mathbb{E}^{P^\lambda}\left\{\int_0^T R_{x,t}\mathrm{d}X_t + \int_0^T \rho a(t,Y_t)R_{y,t}\mathrm{d}W_t^\lambda\right\} = 0$$

under the assumptions on the diffusion coefficients and the boundedness of Φ_y and ϕ_y, because $\mathbb{E}^{P^\lambda}\left\{\int_0^T X_t^2\,\mathrm{d}t\right\} < \infty$.

As $P^\alpha \in \mathbb{P}_e(P^{B^\alpha,E}) = \mathbb{P}_e(Q^E)$, condition *(ii)* in Proposition 5 is satisfied and we conclude that $\lambda^{c,\alpha}$ is equal to λ^α.

4.3 Optimal Dynamic Trading Strategy

Having characterized the static hedging part of our formulation, we next find the optimal trading strategy in the underlying stock.

For the pure investment problem, finding the optimal trading strategy is called Merton's problem in the literature and we start by solving this problem. We utilize the connection between the density of Q^E and the optimal Merton trading strategy given in Lemma 1. In Section 4.1, an alternative representation of the density of Q^E is given as the exponential of the stochastic integral of ν_t in (4.9) with respect to the forward price. The uniqueness of the minimal entropy martingale measure yields that $\nu_t = -\gamma\pi_t^E$, and

$$\pi_t^E = \frac{1}{\gamma\sigma(t,Y_t)X_t}\left(\rho a(t,Y_t)\psi_y(t,Y_t) + \frac{\mu - r}{\sigma(t,Y_t)}\right).$$

For the case with the claim, we use the definition of $P^{B^\alpha,E}$ given in (3.31), and Lemma 1 to connect the density of P^α to $\pi^{\alpha,E}$, where $\pi^{\alpha,E}$ is the optimal Merton strategy with prior $P^{B^\alpha,E}$:

$$\frac{\mathrm{d}P^\alpha}{\mathrm{d}Q^E} = c^{\alpha,E}e^{-\gamma(V_T^{\pi^{\alpha,E}} - B^\alpha)}.$$

As in the no claim case, we conclude that ν_t given in (4.19) is equal to $-\gamma\pi_t^{\alpha,E}$. The density of P^α can be connected to π^α, where π^α solves the primal hedging problem in (3.2):

$$\frac{\mathrm{d}P^\alpha}{\mathrm{d}P} = c^\alpha e^{-\gamma(V_T^{\pi^\alpha} - B^\alpha)}.$$

The reader is referred to Proposition 1.2.3 in [2] for the proof of this relation. Therefore, $\pi_t^\alpha = \pi_t^{\alpha,E} + \pi_t^E$, and

$$\pi_t^\alpha = \frac{\mu - r}{\gamma\sigma^2(t,Y_t)X_t} + \frac{\rho a(t,Y_t)\psi_y(t,Y_t)}{\gamma\sigma(t,Y_t)X_t} + \frac{\rho a(t,Y_t)R_{y,t}}{\sigma(t,Y_t)X_t} + R_{x,t}. \tag{4.20}$$

In other words, the dynamic strategy is simply computed from the indifference pricing functions Φ and ϕ and their derivatives, which are found numerically, and the Merton value function ψ which is given by (4.7). The individual terms in (4.20) can be interpreted, respectively, as follows: the Merton ratio, the volatility hedging term for the Merton stock-bond portfolio, the volatility hedging term for the barrier-put basket, and the Delta hedging term for the barrier-put basket.

An alternative for finding (4.20) in the stochastic volatility model is introducing the HJB equations associated with (4.11). In this Markovian set-up the analog of the relation (3.20) holds for all times $t < T$, which allows to characterize the optimal trading strategy in terms of the dual variables, R and ψ. We refer the reader to [37] for the detailed calculations.

4.4 Numerical Solutions

The solution for (4.13) cannot be found explicitly in general. However, as the nonlinearity is mild (the PDE is only quasilinear, or semi-linear), numerical solution work extremely well, as we demonstrate here. We present an example where we model Y as the following Ornstein-Uhlenbeck process

$$\mathrm{d}Y_t = -5Y_t\,\mathrm{d}t + \sqrt{10}\left(\rho\,\mathrm{d}W_t^1 + \rho'\,\mathrm{d}W_t^2\right),$$

with $\rho = -0.5$. The unique invariant distribution of Y is Gaussian with mean 0 and variance 1. We take $\sigma(t,y)$ to be the bounded function

$$\sigma(y) = 0.7\arctan(y-1)/\pi + 0.4.$$

At the mean level of Y, $\sigma(0)$ is .225 and σ takes values in $(.15, .4)$ while Y takes values in one standard deviation confidence interval. In the case of two standard deviations, this interval is $(.12, .58)$.

To find the solution to (4.13), we use a finite difference method where we approximate the first and second order derivatives in x and y by central differences. In other words, we use an approximation

$$\phi(t_n, x_i, y_j) \cong A^n_{i,j}, \text{ for } i = 0,..,I, j = 0,..,J, \text{ and } n = 0,..,N$$

initialized for at time T and iterated through the finite difference approximation of (4.13). As an example, the partial derivative with respect to y for x_i at time t_n is approximated as

$$\frac{\partial\phi}{\partial y}(t_n, x_i, y_j) \cong \frac{A^n_{i,j+1} - A^n_{i,j-1}}{2\,\mathrm{d}y}.$$

We evaluate the algorithm for x in $[0, 150]$, and y in $[-3, 3]$. We use 25,000 time steps corresponding to $\mathrm{d}t = 0.2 \times 10^{-4}$ while $\mathrm{d}x = 0.5$ and $\mathrm{d}y = .1$. We first solve for $\Phi(t,x,y)$ and use this solution as the barrier condition of $\phi(t,x,y)$. For $\Phi(t,x,y)$, the value at the boundary

$x = 150$ is taken as the minus call payoff, and at $x = 0$, it is taken as the put payoff. At $x = 150$, $\phi(t, x, y)$ is forced to be zero. For the variable y, Neumann boundary conditions are used at both boundaries. The indifference price at time zero is simply $\tilde{h}(B^\alpha) = e^{-rT}\phi(0, x, y)$.

In Figure 2, we show the indifference price of B^α as a function of x for fixed $\alpha = 1$, and $\gamma = 1$ and y as shown in the legend.

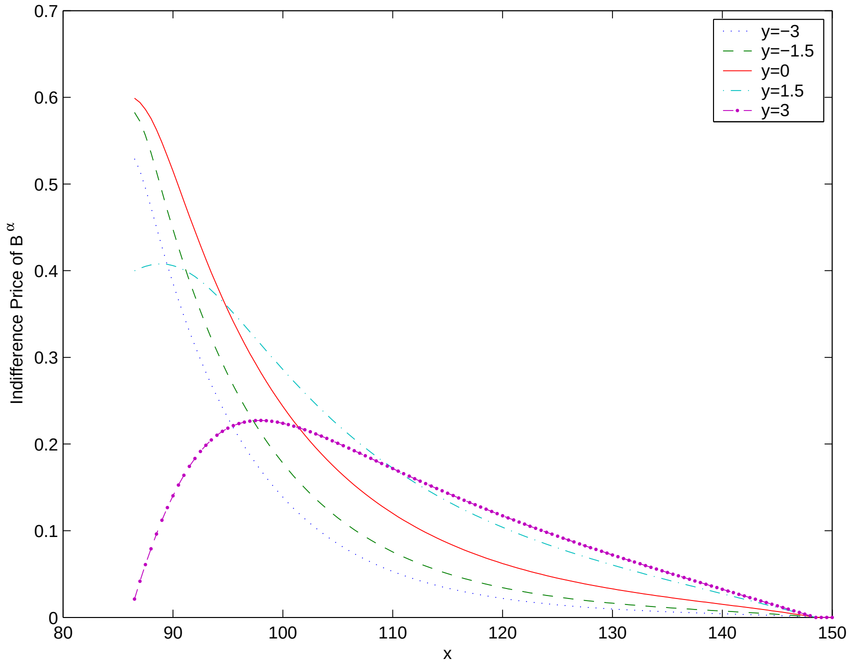

Figure 2: Indifference Price as a function of x and y. $K = 100$, $B = 85$, $T = 0.5$, $r = 3\%$, $\mu = 0.15$.

4.4.1 Strict Convexity and Optimal Hedging Position

In [16], Frey and Sin showed that the super-hedging price of the put option in the stochastic volatility model is the Black-Scholes price evaluated at the maximum volatility, hence it is easy to conclude that $h(B^\alpha)$ is strictly convex in α on $\mathbb{R}_+$. Then the optimal number of puts to short α^* is satisfies

$$\tilde{h}'(B^{\alpha^*}) = \tilde{p}.$$

We approximate the derivative in α (denoted by $'$) by central differences, as plotted in Figure 3. The optimal number of put options is then the value at the x-axis corresponding to the market price on the y-axis.

The price of a vanilla option is in one-to-one correspondence with its implied volatility. Implied volatility of an option is the volatility parameter that matches the market price of the

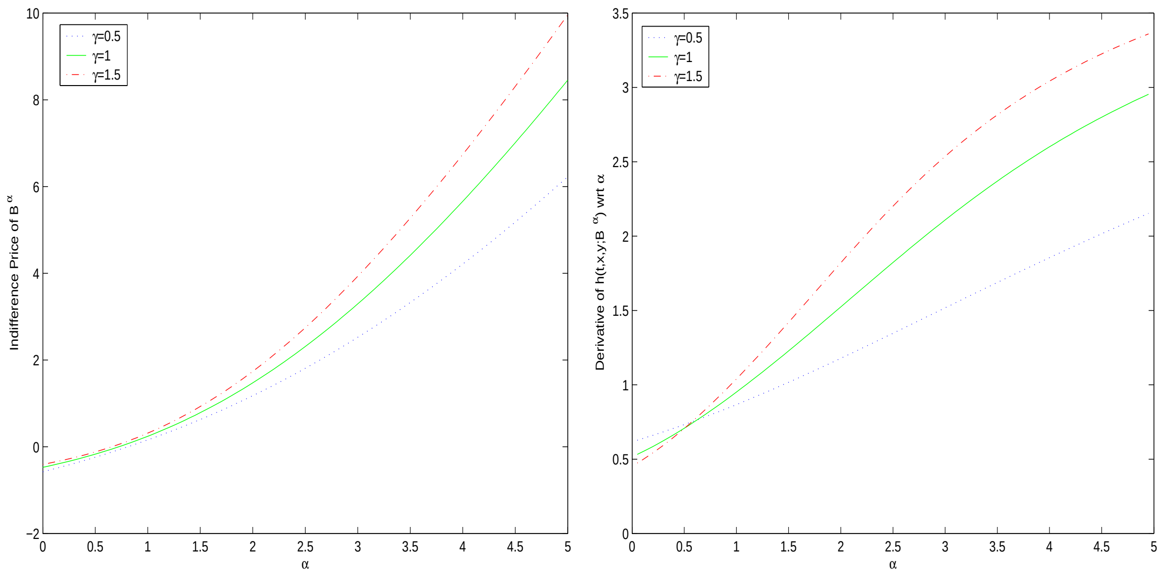

Figure 3: (Left) Indifference Price as a function of α. (Right) Derivative of the Indifference Price with respect to α. $x = 100$, $y = 0$, γ as shown in the legend and other parameters are fixed as in Figure 2.

option when plugged into the Black-Scholes formula. Instead of the market price of the put option, in Figure 4, we give the optimal number of put options α^* to sell given the implied volatility of the put option. As a benchmark, we see from Figure 4 that the strategy of discarding other strikes and shorting 1.41 put options with strike K' is optimal if the implied volatility of the put option is around 0.36, 0.375, 0.39 when γ is 0.5, 1, 1.5 respectively.

5 Conclusion

The preceding analysis proposes an approach for hedging barrier options combining a static position in a certain type of put option with a dynamic hedging strategy in the underlying. The optimal number of options in the static position is characterized by the Fenchel-Legendre transform of the indifference price, and the optimal dynamic strategy is given in terms of the indifference price of the combined options position. The indifference price is characterized here as the solution of an entropy penalization problem where the prior measure is the minimal entropy martingale measure. We give an example using a stochastic volatility model for the underlying.

The main direction for future work is computational issues. In practice, there will be a variety of vanilla options available for hedging and efficient computation of the indifference

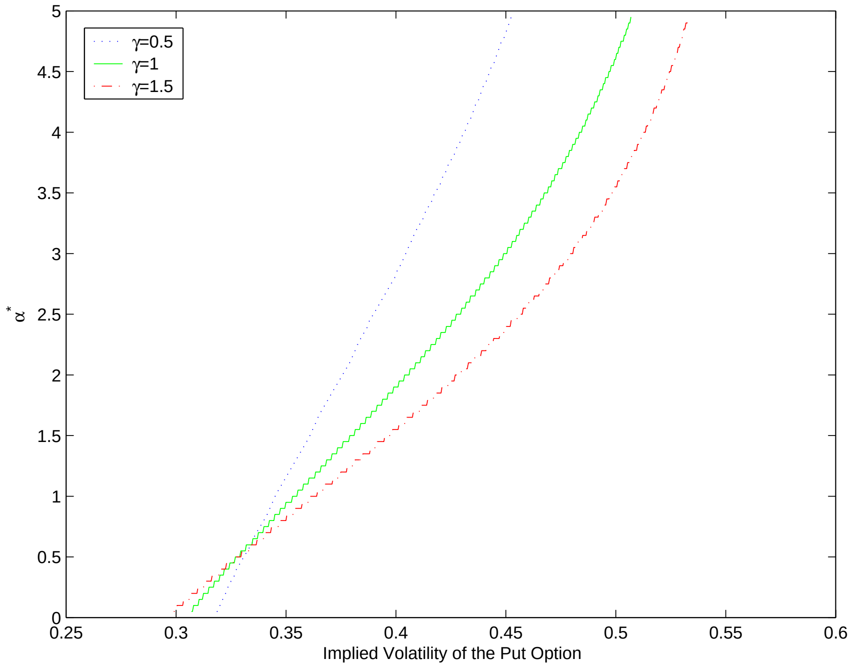

Figure 4: The optimal number of put options to sell given the implied volatility of the put option and the derivative of the indifference price as in Figure 3. $x = 100$, $y = 0$, γ as shown in the legend and other parameters are fixed as in Figure 2.

price of the basket of exotic and hedging options, and then of the Fenchel-Legendre transform in the vanilla weights is important. Extension to hedging of other types of exotic options is straightforward if the corresponding pricing problem is well-understood. However, for strongly path-dependent contracts like lookbacks and Asian options, for example, there is typically an increase in dimensionality and simulation methods or series expansion approximations, extended to handle the nonlinear utility-indifference pricing mechanism, may be required.

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